GMT 19/20

Lecture 5, 26/3/20

Hausdorff dimension

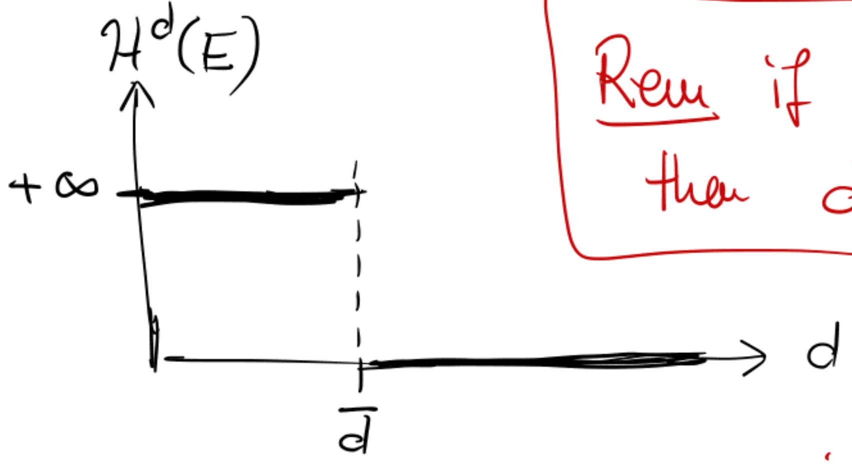
Rem Take $0 \leq d < d'$. Then

$$\mathcal{H}^d(E) < +\infty \implies \mathcal{H}^{d'}(E) = 0$$

$$\mathcal{H}^{d'}(E) > 0 \implies \mathcal{H}^{d}(E) = +\infty$$

Proof is an exercise

Then if we fix E then



Rem if $0 < \mathcal{H}^d(E) < +\infty$ then $d = \dim_H(C)$

$$\bar{d} := \sup \{ d \mid \mathcal{H}^d(E) + \infty \}$$

$$= \inf \{ d \mid \mathcal{H}^d(E) = 0 \}$$

$= \dim_H(E)$

is called Hausdorff dimension of E!

Long exercise: Cantor set!

$$\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$$

where $\mathcal{C}_0 := [0,1]$

$\mathcal{C}_1 := [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$\mathcal{C}_2 := [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup \dots\dots$

$\mathcal{C}_n = \bigcup_{i=1}^{2^n} I_{n,i}$ with

$I_{n,i}$ closed interval with length $1/3^n$.

Q1 What is

$d := \dim_H(C)$? $= \frac{\log 2}{\log 3}$

Q2 What is

$\mathcal{H}^d(C)$? $= 1$

$\dim_H(C)$ is EASY to guess!

Indeed

$$\mathcal{C} = \underbrace{(\mathcal{C} \cap [0,\tfrac{1}{3}])}_{= \frac{1}{3}C} \,\dot\cup\, \underbrace{(\mathcal{C} \cap [\tfrac{2}{3},1])}_{= \frac{2}{3} + \frac{1}{3}C}$$

Then $\forall d$

$$\mathcal{H}^d(C) = \cancel{2\mathcal{H}^d(\tfrac{1}{3}C)} = \frac{2}{3^d}\mathcal{H}^d(C)$$

If $0 < \mathcal{H}^d(C) < +\infty$ then $1 = \frac{2}{3^d} \Rightarrow d = \frac{\log 2}{\log 3}$

Guess: $\dim_H(C) = d = \frac{\log 2}{\log 3}$

To prove the guess we show that

$$0 < \mathcal{H}^d(C) < +\infty$$

(with $d = \frac{\log 2}{\log 3}$, solution of $\frac{2}{3^d} = 1$)

1) Finding an upper bound for $\mathcal{H}^d(C)$ is EASY.

It suffices to exhibit a cover of C.

We fix n and consider the cover $\{ I_{n,i} \mid i = 1, \dots, 2^n \}$ (c.c. of C_n)

$I_{n,i}$ interval with length = diam = $\frac{1}{3^n}$

so for $\delta \geq \frac{1}{3^n}$

$$\mathcal{H}^d_\delta(C) \leq \sum_i \left(\mathrm{diam}(I_{n,i})\right)^d$$

$$= \frac{2^n}{3^{nd}} = \left(\frac{2}{3^d}\right)^n = 1$$

no renorm. constant!!

$$\Longrightarrow \mathcal{H}^d(C) \leq 1$$

2) Lower bound for $\mathcal{H}^d(C)$.

More difficult, because we should start with an arbitrary cover $\{E_i\}$ of C.

Step 1 We can assume that E_i are convex and open (i.e., open intervals)

Since C is compact, we can also assume that $\{E_i\}$ is FINITE

Step 2 Recall that $\mathcal{I} = \left\{ I_{n,i} \;\middle|\; \begin{matrix} n=0,1,\dots \\ i=1,\dots,2^n \end{matrix} \right\}$ is a basis of the topology of C

Then $\forall i$ we can find $I_{i,j} \in \mathcal{I}$ s.t.

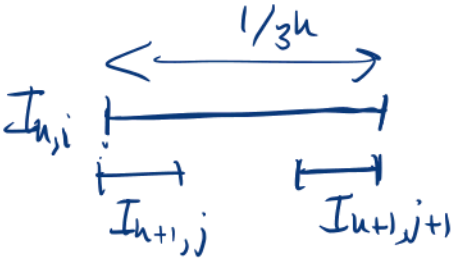
$$E_i \supset \bigcup_j I_{i,j} \supset E_i \cap C$$

Since $I_{i,j}$ are open in C

Then a finite subcoll. of $\{I_{i,j}\}$ covers C.

In particular, since $\{I_{i,j}\}$ is finite then there exists $\bar{n}$ s.t. each $I_{i,j}$ is a c.c. of C_n for some $n \le \bar{n}$.

3) Key step: Replacing $I_{n,i}$ with $I_{n+1,j}$ and $I_{n+1,j}$ is "good" in the computation of $\mathcal{H}^d_\delta$



in the sense that

$$(\operatorname{diam}(I_{n,i}))^d = (\operatorname{diam}(I_{n+1,j}))^d + (\operatorname{diam}(I_{n+1,j+1}))^d$$

This suggests that

$$\sum_i (\operatorname{diam}(E_i))^d \geq \sum_{i=1}^{2^{\bar{n}}} (\operatorname{diam}(I_{\bar{n},i}))^d = 1$$

$\boxed{\text{Conclusion } \mathcal{H}^d(C) \geq 1.}$

A complete proof is more involved!

Try yourself!

Covering Theorems

Rem Essential tools in "Hard GMT", but can be avoided in dealing with currents and fin. per. sets

Spirit of of covering theorems?

Given a family $\mathcal{F}$ of balls that cover a certain set E, you want to extract a "better" subcover $\mathcal{F}'$

Meaning of "better" varies.
It could be

- $\mathcal{F}'$ is disjoint
- balls in $\mathcal{F}'$ do not overlap "too much"
- given some measure μ,
$$\sum_{B\in\mathcal{F}'} \mu(B) \leq \mu(E)+\varepsilon$$

There are two families of results.

Those that hold in every metric space X but require something about the measure μ (doubling property)

— Vitali-type covering th. —

Those that work only in $\mathbb{R}^n$ but for every measure μ

— Besicovitch type —

Lemma (Vitali). Given

- X metric space
- $\mathcal{F}$ family of (closed) balls $B_i = \overline{B(x_i, r_i)}$ with $r_i \le R < +\infty \ \forall i$

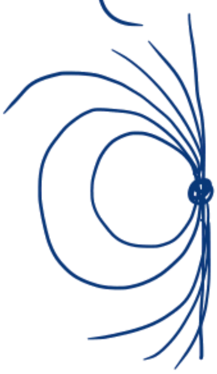
Then $\exists\, \mathcal{G} \subset \mathcal{F}$ s.t.

- $\mathcal{G}$ is disjoint ($B, B' \in \mathcal{G} \Rightarrow B \cap B' = \emptyset$)
- $\widehat{\mathcal{G}} := \{ \widehat{B} \mid B \in \mathcal{G} \}$ satisfies

$$\bigcup_{B \in \mathcal{G}} \widehat{B} \supset \bigcup_{B \in \mathcal{F}} B$$

$$\widehat{\overline{B(x,r)}} := \overline{B(x, 5r)}$$

(so, if $\mathcal{F}$ covers E then $\widehat{\mathcal{G}}$ covers E)



This $\mathcal{F}$ shows that the assumpt. $r_i \le R < +\infty$ is necessary

Proof Case 1 $\mathcal{F}$ is countable and can be ordered so that radii are <u>decreasing</u>

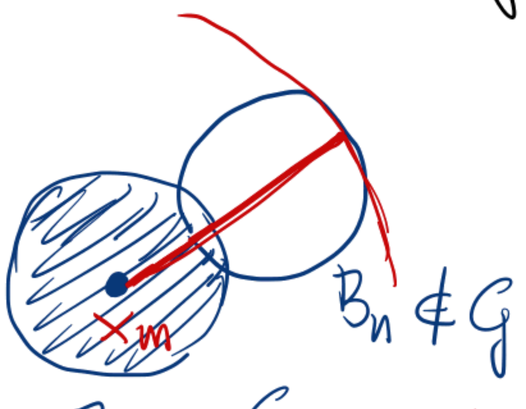
$r_1 \geq r_2 \geq r_3 \dots$

Construction of $\mathcal{G}$:

I add B_n to $\mathcal{G}$ if it <u>does not</u> intersect any of the balls already in $\mathcal{G}$. Otherwise I throw it away!

<u>Then</u> • $\mathcal{G}$ is disjoint by construction!

• if $B_n \notin \mathcal{G}$ then $\exists\, m \leq n$ s.t. $B_m \in \mathcal{G}$ and $B_m \cap B_n \neq \emptyset$



$B_n \subset B(x_m, 3r_m)$
$\cap$
$\hat{B}_m$

Thus $\bigcup_{B \in \mathcal{G}} \hat{B} \supset \bigcup_{B \in \mathcal{F}} B$

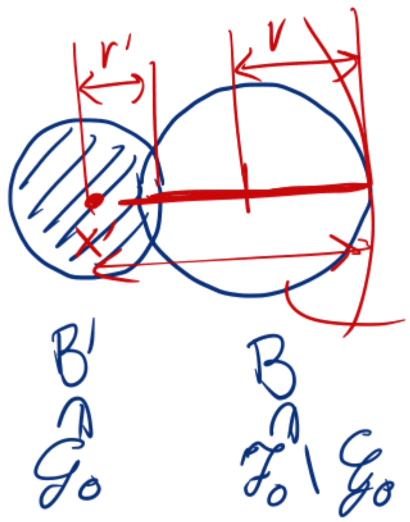
Case 2 | General case.

- Divide $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots$

  where $\mathcal{F}_n := \left\{ B_i = B(x_i, r_i) \in \mathcal{F} \;\middle|\; \frac{R}{2^{n+1}} \le r_i \le \frac{R}{2^n} \right\}$

- Take $\mathcal{G}_0$ maximal disj. subfamily of $\mathcal{F}_0$. Then $\bigcup_{B \in \mathcal{G}_0} \hat{B} \supset \bigcup_{B \in \mathcal{F}_0} B$

  Indeed if $B \in \mathcal{F}_0 \setminus \mathcal{G}_0$ then $\exists\, B'' \in \mathcal{G}_0$ s.t. $B \cap B' \neq \emptyset$ then



$B \subset \widehat{B'}$

$\le r' + 2r \le 5r'$

- $\mathcal{G}_1$ is a maximal subset of $\mathcal{F}_1$ which is disjoint and "disjoint from $\mathcal{G}_0$"

Then $\mathcal{G}_0 \cup \mathcal{G}_1$ is disjoint

and $\bigcup_{B \in \mathcal{G}_0 \cup \mathcal{G}_1} \hat{B} \supset \bigcup_{B \in \mathcal{F}_1} B$

as before!

- and so on!

Finally I take $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots$

$\square$