GMT 19/20

Lecture 1   12/3/20

Intro

Structure of course

part 1
- Hausdorff measures and dim.
- Lipschitz maps and area formula
- Rectifiable sets

part 2
- Currents (or Finite Per. Sets) and solution of Plateau Pb.

Plateau Problem

" Find the surface Σ with minimal area that spans a given curve Γ "

d-dim. surface in $\mathbb{R}^n$ (or M^n)

d-dim. volume

(d-1)-dim. surf.

- "Find"? means "prove the existence of"
- general dimension and codimension.

The problem is actually <u>not</u> well defined.

- What is a surface Σ?
- What is area?
- What is the meaning of "Σ spans Γ"?

Clear answer if we restrict to regular objects.

But in the class of regular objects it is difficult to prove existence, and sometimes there is NO existence

(There are also modelling reasons to consider more general objects!)

Concerning existence: direct method of Calc. Var. that is:

" semicontinuity and compactness "

Basic example:

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \qquad u_0 \text{ given}$$

$$F(u) := \int_\Omega \frac{1}{2} |\nabla u|^2 dx$$

minimize $F(u)$ on $\boxed{W^{1,2}_{u_0}(\Omega)}$

then you prove regularity of u, and recover a classical sol.

For Plateau Pb. we need to find

- A compactification of surfaces
- a l.s.c. extension of the area functional
- an extension of the notion of bdry which is <u>stable</u>
- prove regularity (if possible).

That what the theory of currents is about.

This is not the only possible approach!

Quick review of other approaches

- Purely set-theoretic approach

<u>An example</u>

$d = 1$ (Steiner problem)

" Find the connected compact set Σ with minimal length that contains a given (finite?) set Γ "

1-dim. $\mathcal{H}^1$ Hausdorff meas.

Solution by direct method.

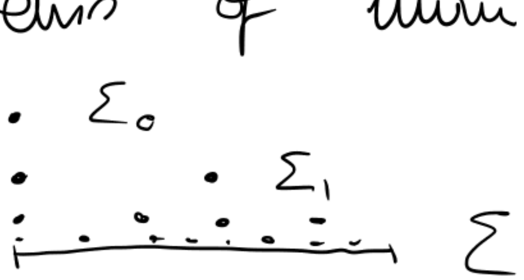
$\mathcal{K} = \{ \Sigma$ compact, connected, $\Sigma \supset \Gamma \}$

$d_H =$ Hausdorff distance

$\mathcal{H}^1$ is l.s.c. on $\mathcal{K}$! (Golab's Th.)

Existence of minimizers is easy!

<u>Rem</u>



$\Sigma_n \xrightarrow[d_H]{} \Sigma$

semicontinuity of length wrt d_H fails!!

What about higher d?

(Golab's theorem has no equivalent!)

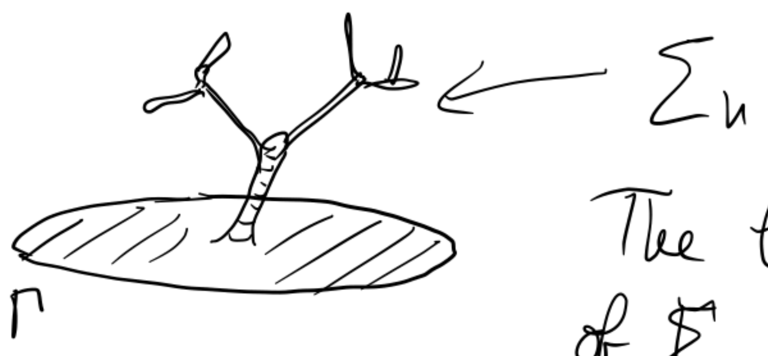
Attempt of extension

Γ given curve

Σ compact set such that $\Gamma \subset \Sigma$ & Γ can be retracted to a point in Σ

This is a good class of sets (compact wrt d_H if some boundedness is assumed)

But area is not l.s.c.



The limit in d_H of Σ_n can be any closed set containing the disk Σ

This approach (purely set theoretic) was followed by Reifenberg in 1960

Proof is highly non trivial.

Variants of this approach proposed by others

Harrison (+ De Lellis, Ghiraldin, Maggi)

after 2000

2 | Parametric Approach

Fix D model/reference surface (disk in the plane, or some other d-dimensional domain)

Fix $\Gamma := \gamma(\partial D)$, $\gamma : \partial D \to \mathbb{R}^n$ given

Let $\Sigma := \phi(D)$, $\phi : D \to \mathbb{R}^n$

Then "Σ spans Γ" translate into the bdry condition $\phi = \gamma$ on ∂D.

Moreover

$\sim$ correct if ϕ is injective

$$\mathrm{vol}_d(\Sigma) = \int_D J\phi \, ds \leftarrow F(\phi)$$

if $d = 2$, $n = 3$

$$= \int_D \left| \frac{\partial \phi}{\partial s_1} \times \frac{\partial \phi}{\partial s_2} \right| ds_1 ds_2$$

minimize $F(\phi)$ over all $\phi : D \to \mathbb{R}^n$ s.t. $\phi = \gamma$ on ∂D.

Try to find a minimizer on some Sobolev Space, e.g. $W^{1,2}_\gamma(D)$

What is the problem?

- F is policonvex and therefore weakly l.s.c. on $W^{1,2}_\gamma(D)$
- but F lacks coercivity.

What does this mean?

F is invariant by reparametriz.

$$F(\phi \circ \sigma) = F(\phi) \quad \forall\ \sigma : D \to D \ \text{diffeo.}$$

Using this fact I can construct minimizing sequences ϕ_n of F that converge to "very bad" ϕ.

For example, let ϕ be a minimizer of F.

Then any sequence $\phi_n = \phi \circ \sigma_n$ with $\sigma_n : D \to D$ diffeo is a minimizing seq.

I can choose σ_n so that the limit of $\phi \circ \sigma_n$ is constant!

$$\phi \circ \sigma_n \longrightarrow \text{constant map.}$$

NO COMPACTNESS of minimizing seq.

The parametric approach works in dimension $d=1$ and $d=2$

<u>Why?</u>

<u>$d=1$</u> $\Gamma = \{x_0, x_1\}$, find the geodesic connecting x_0 and x_1

$\boxed{D = [0,1]}$ (on some ambient space M^n which is a Riem. man.)

You minimize

$$\cancel{F(\phi) = \int_0^1 |\phi'(s)|\, ds}$$

$$E(\phi) = \int_0^1 |\phi'(s)|^2 ds \quad \text{on } W^{1,2}_{\dots}(D)$$

the minimizer ϕ of $E(\phi)$ is also a minimizer of $F(\phi)$ + <u>$|\phi'|$ is constant</u>

Explanation

1) $$E(\phi) = \int_0^1 |\phi'|^2 \, ds$$

Jensen $\longrightarrow$ $$\geq \left(\int_0^1 |\phi'| \, ds\right)^2 = (L(\phi))^2$$

$=$ holds if $|\phi'|$ is constant!

2) Every $\phi : D = [0,1] \longrightarrow \mathbb{R}^d$ admits a reparametrization $\tilde{\phi} := \phi \circ \sigma$ with constant speed ($|\tilde{\phi}'| = \text{const.}$)

3) Let ϕ_0 be a minimizer of E.
Let ϕ be a competitor for F
then

$$L^2(\phi) = L^2(\tilde{\phi}) = E(\tilde{\phi}) \geq E(\phi_0)$$
$$\geq L^2(\phi_0)$$

$\Longrightarrow$ ϕ_0 minimizes L

$\Longrightarrow$ if $\phi = \phi_0$ then $E(\phi_0) = L^2(\phi_0) \Rightarrow |\phi_0'| = \text{cons}$

A similar trick works for $d=2$

- D = disk = $\overline{B(0,1)}$ in $\mathbb{R}^2$
- $\phi : D \to \mathbb{R}^{\cancel{n}\,3}$ $F(\phi) = \int_D \left|\frac{\partial\phi}{\partial s_1} \times \frac{\partial\phi}{\partial s_2}\right| ds$
- $E(\phi) = \int_D \frac{1}{2} |\nabla\phi|^2 ds = \int_D \frac{1}{2}\left(\left|\frac{\partial\phi}{\partial s_1}\right|^2 + \left|\frac{\partial\phi}{\partial s_2}\right|^2\right) ds$

Then a minimizer of $E(\phi)$ on the class $W^{1,2}(D)$ is a also a minimizer of F.

$\phi = \gamma$ on ∂D up to reparam. of $\partial D = S^1$

Proof

1) $E(\phi) = \int_D \frac{1}{2}\left(\left|\frac{\partial\phi}{\partial s_1}\right|^2 + \left|\frac{\partial\phi}{\partial s_2}\right|^2\right) ds$

"=" iff $\left|\frac{\partial\phi}{\partial s_1}\right| = \left|\frac{\partial\phi}{\partial s_2}\right|$ $\rightarrow$ $\geq \int_D \left|\frac{\partial\phi}{\partial s_1}\right| \left|\frac{\partial\phi}{\partial s_2}\right| ds$

"=" iff $\frac{\partial\phi}{\partial s_1} \perp \frac{\partial\phi}{\partial s_2}$ $\rightarrow$ $\geq \int_D \left|\frac{\partial\phi}{\partial s_1} \times \frac{\partial\phi}{\partial s_2}\right| ds = F(\phi)$

$E(\phi) = F(\phi)$ iff $d\phi(s)$ is conformal

2] For every Σ param. by ϕ there exists a reparam.

$\tilde{\phi} := \phi \circ \sigma$, $\sigma : D \to D$

σ is not necess. the identity on ∂D

s.t. $\tilde{\phi}$ is conformal (that is, $E(\tilde{\phi}) = F(\tilde{\phi})$)

(Lichtenstein theorem.)

3] Same proof as before!

This parametric approach was developed by Douglas & Radó in 1930s.

It only work for $d=2$

Because of lack of conformal parametrizations in dimension $d>2$.