GMT 19/20, lecture 1, 12/3/20 [version: 21/4/2020]

Structure the course

Part 1: basic theory

Hausdorff measures and dimension
Lipschitz maps, area formula
Rectifiable sets

Part 2: currents

General theory
compactness of integral currents
and solution of Plateau Problem

Alternative version of Part 2: theory of finite perimeter sets (the choice will be made later)

1 Introduction to Plateau Problem

This problem is often stated as follows:

"Find the surface Σ with smallest area that spans a given curve Γ"

surface → d-dimensional surface in $\mathbb{R}^n$ or in some Riemann. manifold
area → d-dimensional volume
curve → (d-1)-dimensional surface

! This is not a well-defined problem, but rather a family of loosely defined problems.

One should indeed specify the meaning of many terms:

- find
- surface
- area
- "Σ spans Γ"

In all cases I consider, "find" means "prove the existence of". Existence results are difficult enough!

The meaning of "surface", "area", "Σ spans Γ" is clear in the context of smooth surfaces.

However it is difficult to prove existence results directly in the class of smooth surfaces.

Besides, there may be modelling reasons to consider less regular objects.

For example, soap films are usually singular surfaces!

Existence results are (almost) always obtained by

1.1 The direct method of the Calculus of Variations

Let me recall how the direct method works:
to find (prove the existence of) a minimizer of a certain functional F which is naturally defined on a space $\tilde{X}$ of "regular" objects (functions, surfaces, ...)

(a) we construct a suitable class X of "generalized" objects which includes $\tilde{X}$ and is endowed with a topology with good compactness properties;
(We loosely refer to X as a "compactification" of Y)

(b) we extend F to X so that it is lower semicontinuous;

(c) we prove the existence of a minimizer x_0 of F on X by a standard lower-semicontinuity and compactness argument;

(d) if possible we prove a regularity result showing that x_0 actually belongs to the original space $\tilde{X}$.

Remarks

- Steps (a) and (b) go together, and X is most often NOT compact (think of Sobolev spaces). What actually one does is to construct a topological space X such that F admits an extension to X which is:

  (i) (sequentially) lower-semicontinuous;

  (ii) coercive, that is, sublevels $\{x \in X : F(x) \leq m\}$ are contained in (sequentially) compact sets.

  Then Step (c) is based on the following elementary lemma:

  $$\boxed{\begin{array}{l}\text{If } X \text{ is a topological space and } F: X \longrightarrow [-\infty, +\infty] \text{ is} \\ \text{l.s.c. and coercive, then } F \text{ admits a minimizer on } X.\end{array}}$$

- To find the right topology on X one must balance two opposed requirements: indeed, for the purpose of making F l.s.c., the more open sets there are, the better; on the other hand, to make F coercive, the less open sets there are the better.

- The abstract scheme outlined above should be implemented in a less abstract fashion, to avoid the risk of setting up just some abstract nonsense.

  For example, one may endow $\tilde{X}$ with the discrete topology, let $X = \tilde{X} \cup \{\infty\}$ be the one-point compactification of $\tilde{X}$, and extend F to X by setting $F(\infty) := \inf\{F(x) : x \in \tilde{X}\}$. Thus F is l.s.c. and coercive on X, and ∞ is a minimizer of F on X.
  But clearly such minimizer is completely meaningless!
  In particular one cannot even think of performing step (d)

Basic example

To find a solution (of class C^2) of

$$(*)\quad \begin{cases} \Delta u = 0 & \text{on } \Omega \leftarrow \text{bounded open set in } \mathbb{R}^d \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

given function on $\partial\Omega$

(that is, the harmonic extension of u_0 to Ω) one looks for a minimizer of the Dirichlet energy

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx$$

among all $u : \Omega \to \mathbb{R}$ s.t. $u = u_0$ on $\partial\Omega$.

Here one uses that $\Delta u = 0$ is the Euler-Lagrange equation associated to the (convex) functional E.

More precisely, one looks for a minimizer of E on the Sobolev space

this is X $\longrightarrow$ $W^{1,2}_{u_0}(\Omega) := \{ u \in W^{1,2}(\Omega) : u = u_0 \text{ on } \partial\Omega \}$.

intended in a suitable weak sense

Existence is guaranteed by:

- E is <u>weakly lower semicontinuous</u> on $W^{1,2}(\Omega)$,
- E is <u>coercive</u>, and more precisely every sublevel $\{u \in W^{1,2}_{u_0} : E(u) \leq m\}$ is <u>weakly compact</u> in $W^{1,2}$.

Finally one proves that u is <u>regular</u>.

More precisely, $u \in C^\infty(\Omega) \cap C^k(\bar{\Omega})$ where k depends on the regularity of $\partial\Omega$ and of the datum u_0.

In the previous example we could not prove the existence of a minimizer u directly in a class of regular functions, say $C^2(\bar{\Omega})$, due to lack of good compactness properties.

The Sobolev space $W^{1,2}(\Omega)$, endowed with the weak topology, provides the right "compactification" of $C^2(\bar{\Omega})$.

To apply a similar strategy to Plateau Problem we need:

- a "compactification" of the class of regular surfaces;
- a lower semicontinuos extension of the area functional to such larger class;
- an extension of the notion of boundary (or of the notion "Σ spans Γ")
- some regularity of the minimal surface obtained in this larger class. (More on this issue in next lecture!)

The theory of (integral) currents provides a setup for this program!

Remarks

To be precise, it provides a "compactification" of the class of ORIENTED surfaces.

The theory of finite perimeter sets serves similar purposes, but it is limited to surface of codimension 1: $d = n-1$.

1.2 Review of main approaches to existence results for Plateau Pb.

1.2.1 First approach: "distributional / measure theoretic"

Based on classes of "generalized" surfaces with good compactness properties; the definitions are based on measure theory and remind that of Sobolev functions.
Essentially two variants

- Finite perimeter sets (E. De Giorgi, mid 1950s)
  $\simeq$ generalized oriented surfaces of dimension $d = n-1$.

- Integral currents (H. Federer & W. Fleming, 1960)
  $\simeq$ generalized oriented surfaces of arbitrary dimension and codimension ($1 \leq d \leq n-1$).

Finite per. sets are particular cases of integral currents.

In this course I will describe the construction of these classes of generalized surface, prove the relevant compactness theorems, and use these to prove existence of solutions to Plateau Problem. "minimal surfaces"

Regularity theory is delicate and not yet fully established. It will not be covered in this course.

WARNING In this course "minimal surface" means minimizer of the area functional (in suitable classes).
Outside this course it means "surface with vanishing mean curvature ($H=0$)", i.e., critical points of the area functional.

$\boxed{1.2.2}$ Second approach: "set theoretic" (NOT IN THIS COURSE)

Let me begin with a 1-dimensional version of P.P.

Steiner Problem

"Find the connected compact set Σ with minimal length that contains a given set Γ"

in a given metric space Y

1-dimensional Hausdorff measure (explained later)

If the ambient space Y is compact a solution can be found as follows:

Let

$$X := \{\Sigma \text{ s.t. } \Gamma \subset \Sigma \subset Y,\ \Sigma \text{ compact \& connected}\}$$

$d_H :=$ Hausdorff distance on the class of closed subsets of Y, that is

$$d_H(C, C') := \inf\{r \text{ s.t. } \underbrace{I_r(C)}_{\text{closed } r\text{-neighb. of } C} \supset C';\ I_r(C') \supset C\}$$

$\mathcal{H}^1 :=$ 1-dim. Hausdorff measure (of subsets of Y)

Then

- X endowed with distance d_H is COMPACT; (follows easily by standard results on Hausdorff distance)
- $\mathcal{H}^1$ is lower semicontinuous on X. (Gołąb's Theorem)

Let's assume that X is not empty and contains at least one element Σ with $\mathcal{H}^1(\Sigma) < +\infty$.

Then $\mathcal{H}^1$ admits a minimizer on X.

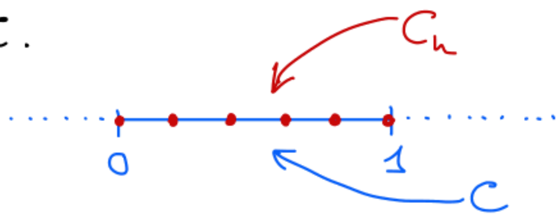
Remarks

(i) In the statement of Steiner Problem the requirement that "Σ is connected and contains Γ" is a substitute for "Σ spans Γ".

(ii) In the proof above is essential that the sets in X are connected, in the sense that $\mathcal{H}^1$ is not lower semic. on the space of all compact subsets of Y (endowed with the Hausdorff distance d_H).

Let indeed $Y := \mathbb{R}$, $C := [0,1]$, $C_n := \left\{ \frac{k}{n} : k = 0, \dots, n \right\}$ for $n = 1, 2, \dots$.

Then $C_n \xrightarrow[d_H]{} C$.



(More precisely, $d_H(C; C_n) = 1/2n \quad \forall n$).

On the other hand $\mathcal{H}^1(C_n) = \text{length}(C_n) = 0 \quad \forall n$, while $\mathcal{H}^1(C) = \text{length}(C) = 1$.

Extension to higher dimension ($d > 1$)?

Even for $d = 2$ extension presents serious problems.

First issue: given a curve Γ, what could be the class X of admissible "surfaces" Σ?

A possible candidate is

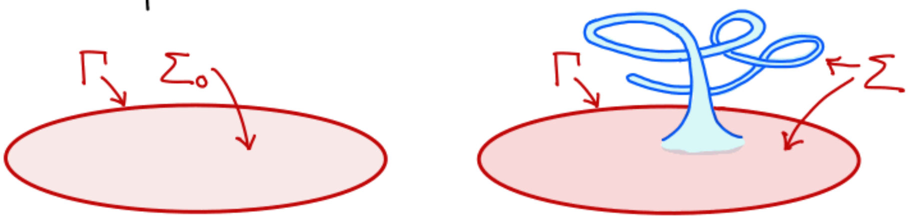
$$X := \left\{ \Sigma \text{ s.t. } \Sigma \text{ compact}, \ \Sigma \supset \Gamma, \ \Gamma \text{ is homotopic in } \Sigma \text{ to a point} \right\}$$

Then it is natural to define area(Σ) for $\Sigma \in X$ as $\mathcal{H}^2(\Sigma)$ where $\mathcal{H}^2$ is the 2-dim. Hausdorff measure (on Y).

! The main difficulty is that $\mathcal{H}^2$ is not lower semicontinuous on X w.r.t. Hausdorff distance.

(There is no version of Gołąb Theorem for $\mathcal{H}^2$)

Let indeed Γ be a circle in $\mathbb{R}^3$, Σ_0 be the disk spanned by Γ, and consider a smooth surface Σ diffeomorphic to Σ_0 as follows:



Note that the blue part of Σ can have area as small as desired, and at the same time it can include any given finite set F (with $F \cap \Sigma_0 = \emptyset$).

In particular we can take a sequence of finite sets F_n which converge in d_H to a closed ball B that contains the disk Σ_0 and construct a sequence of surfaces $\Sigma_n \subset B$ as above such that

$$\mathcal{H}^2(\Sigma_n) = \text{area}(\Sigma_n) \leq \underbrace{A}_{\text{area of } \Sigma_0} + \frac{1}{n} .$$

Then $\Sigma_n \xrightarrow[d_H]{} B$.

But $\mathcal{H}^2(\Sigma_n) \longrightarrow A$ while $\mathcal{H}^2(B) = +\infty$.

However a "set theoretic approach" to Plateau Problem in general dimension and codimension was successfully carried out by E.R. Reifenberg in mid 1960s.

Important recent contribution by: J. Harrison and by C. De Lellis + F. Maggi + al.

$\boxed{1.2.3}$ Third approach: "parametric" (NOT IN THIS COURSE)

The idea is to consider surfaces and curves as sets parametrized by a disc and an interval, respectively.
Let me begin with a basic 1-dimensional problem

Existence of geodesics. I

Let Y be a compact metric space.

"Given $x_0, x_1 \in Y$
find the continuous path $\gamma : \underbrace{[0,1]}_{I} \to Y$
of minimal length
that connects x_0 and x_1"

I keep the distinction between a path $\gamma : I \to Y$ and the curve parametrized by γ namely the set $\gamma(I)$

that is, $\gamma(0) = x_0, \gamma(1) = x_1$

the length of a path γ is

$$L(\gamma) := \inf \sum_{i=1}^{m} d(\gamma(t_{i-1}), \gamma(t_i))$$

where the infimum is taken over all $m = 1, 2, \ldots$ and all $0 \le t_0 \le t_1 \le \cdots \le t_n \le 1$.

We prove the existence of a solution γ (called a "geodesic" connecting x_0 and x_1) under the assumption that

$$L_{min} := \inf_{\gamma \in X} L(\gamma) \text{ is finite,}$$

where

$$X := \{\gamma : [0,1] \to Y \text{ s.t. } \gamma \text{ cont.}, \gamma(0) = x_0, \gamma(1) = x_1\}.$$

To this end we set

$$X^* := \{\gamma \in X \text{ s.t. } \underbrace{\mathrm{Lip}(\gamma)}_{\text{Lipschitz constant of } \gamma} \le L_{min} + 1, \gamma(0) = x_0, \gamma(1) = x_1\}$$

and note that:

- X^* endowed with the supremum distance is compact.
  (Arzelà - Ascoli Theorem)

$d(\gamma_0, \gamma_1) := \sup_{t \in [0,1]} d(\gamma_0(t), \gamma_1(t))$

- The length functional L is lower semicontinuous on X (w.r.t. the supremum distance).
- Then L admits a minimizer $\bar{\gamma}$ on X^*.
- $$L_{min} := \inf_{\gamma \in X} L(\gamma) \overset{(*)}{=} \inf_{\gamma \in X^*} L(\gamma) = L(\bar{\gamma})$$
  and then $\bar{\gamma}$ is a geodesic connecting x_0 and x.

To prove $(*)$ we use the following lemma: every γ with $L(\gamma) < +\infty$ admits a reparametrization $\tilde{\gamma}$ in X^*.

$\tilde{\gamma}$ is a reparam. of γ if $\tilde{\gamma} = \gamma \circ \sigma$ with $\sigma : [0,1] \to [0,1]$ is an increasing homeomorphism. Note that $L(\tilde{\gamma}) = L(\gamma)$.

Existence of geodesics. II

Assume now that Y is an n-dimensional surface of class C^1 in $\mathbb{R}^m$ (or an n-dimensional Riemannian manifold) and $\gamma \in X$ is also of class C^1.
Then
$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\, dt .$$

!

Now we would to prove the existence of a minimizer of L (over $X \cap C^1$) using the direct method. But there is a difficulty, due to the lack of compactness in the "natural" Sobolev space $W^{1,1}$.

There is however an interesting "trick". Let
$$E(\gamma) := \int_0^1 |\dot{\gamma}(t)|^2 dt ,$$

and let γ_0 be a minimizers of E over $X \cap C^1$.

The existence of such minimizer is a standard application of the direct method: one first proves the existence of a minimizer γ_0 over $X \cap W^{1,2}$ and then proves that γ_0 is regular.

Then

> **Proposition** The path γ_0 is also a minimizer of L over $X \cap W^{1,1}$. Moreover $|\dot{\gamma}_0|$ is (a.e.) constant.

Proof

(1) For every $\gamma \in X \cap C^1$

Jensen Inequality

$$E(\gamma) = \int_0^1 |\dot{\gamma}|^2 dt \geqslant \left(\int_0^1 |\dot{\gamma}|\, dt\right)^2 = (L(\gamma))^2$$

(2) Equality in (i) holds iff $|\dot{\gamma}|$ is constant.

(3) Every $\gamma \in X \cap C^1$ admits a reparametrization $\tilde{\gamma} = \gamma \circ \sigma$ with "constant speed", that is, $|\dot{\gamma}| = $ constant.

Then, for every $\gamma \in X \cap C^1$,

(4) $$(L(\gamma))^2 \underset{\substack{\uparrow \\ \tilde{\gamma} \text{ as in (3)}}}{=} L(\tilde{\gamma})^2 \underset{\substack{\uparrow \\ (2)}}{=} E(\gamma) \underset{\substack{\uparrow \\ \text{choice} \\ \text{of } \gamma_0}}{\geqslant} E(\gamma_0) \underset{\substack{\uparrow \\ (1)}}{\geqslant} (L(\gamma_0))^2 .$$

Thus γ_0 minimizer $L(\gamma)$.

Taking $\gamma := \gamma_0$ in (4) we get $(L(\gamma_0))^2 = E(\gamma_0)$ which implies $|\dot{\gamma}_0| = $ constant by (2). $\square$

Remark The key ingredient of the proof is the fact that L is invariant under reparametrizations.

Solution of Plateau Problem for $d=2$

Let D be the closed disk $\overline{B(0,1)}$ in $\mathbb{R}^2$.
Let Σ be a surface in $\mathbb{R}^n$ parametrized by $\phi : D \to \mathbb{R}^n$ that is, $\Sigma := \phi(D)$.
Then if ϕ is sufficiently regular and injective

$$\text{area}(\Sigma) = F(\phi) := \int_D J\phi \, ds$$

where

$$J\phi(s) := \left(\det\left(\nabla^t\phi(s) \cdot \nabla\phi(s)\right)\right)^{1/2} = \Big(\underbrace{\textstyle\sum}_{\substack{\text{sum taken over}\\ \text{all } 2\times 2 \text{ minors}\\ M \text{ of } \nabla\phi(s)}} (\det M)^2\Big)^{1/2}.$$

If $n=3$ $\quad J\phi(s) = \left|\dfrac{\partial\phi}{\partial s_1} \times \dfrac{\partial\phi}{\partial s_2}\right|$ vector product in $\mathbb{R}^3$

Let Γ be a given regular curve in $\mathbb{R}^n$ parametrized by a path $\gamma : S^1 = \partial D \to \mathbb{R}^n$.
Then "Σ spans Γ" means $\Gamma = \partial\Sigma$, and is implied by the boundary condition

$$\phi = \gamma \text{ on } \partial D.$$

We would like to prove the existence of a minimizer of F using the direct method on a suitable Sobolev space, for example $W^{1,2}_\gamma := \{\phi \in W^{1,2}(D, \mathbb{R}^n) \text{ s.t. } \phi = \gamma \text{ on } \partial D\}$

To this end, note that F can be written as $F(\phi) := \int_D f(\nabla u)\, ds$ where f is policonvex and therefore F is weakly lower-semicontinuous on $W^{1,2}$.

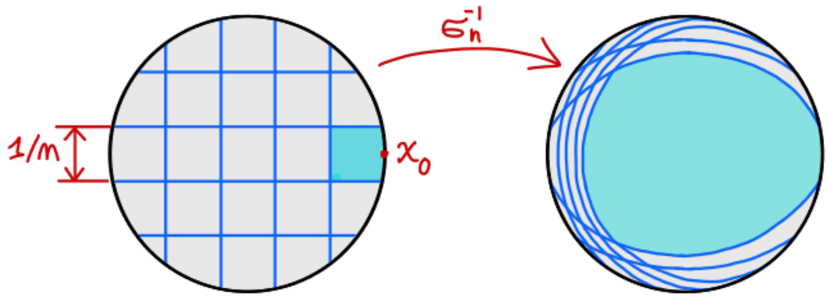
$$\boxed{\text{The problem is that } F \text{ is not coercive, that is, the sublevels } \{\phi \text{ s.t. } F(\phi) \le m\} \text{ are not weakly compact.}}$$

The reason for this lack of compactness lies in the fact that F is invariant under reparametrization, that is,

$$F(\phi) = F(\phi \circ \sigma)$$

for every $\sigma : D \to D$ diffeomorphism of class C^1.

Indeed we can find diffeomorphisms $\sigma_n : D \to D$ for $n = 1, 2, \dots$ that agree with the identity on ∂D and converge pointwise to a constant x_0 in the interior of D:



Thus $\phi \circ \sigma_n$ converge (in the interior of D) to the constant map $\phi(x_0)$, which does not agree with the identity on ∂D, in particular convergence cannot be in $W^{1,2}$-weak.

There is however a trick similar to the 1-dimensional case. Let

$$E(\phi) := \frac{1}{2} \int_D |\nabla \phi|^2 \, ds \qquad \text{(Dirichlet energy)}$$

Then, for $n = 3$ (but similar computations works for all n) and ϕ regular enough:

$$\text{(1)} \quad E(\phi) = \int_D \frac{1}{2}\left(\left|\frac{\partial \phi}{\partial s_1}\right|^2 + \left|\frac{\partial \phi}{\partial s_2}\right|^2 \right) ds$$

$$\underset{(a)}{\geq} \int_D \left|\frac{\partial \phi}{\partial s_1}\right| \left|\frac{\partial \phi}{\partial s_2}\right| ds \underset{(b)}{\geq} \int_D \left| \frac{\partial \phi}{\partial s_1} \times \frac{\partial \phi}{\partial s_2} \right| ds = F(\phi)$$

(2) Inequality (a) in (1) is an equality iff $\left|\frac{\partial\phi}{\partial s_1}\right| = \left|\frac{\partial\phi}{\partial s_2}\right|$.

Inequality (b) is an equality iff $\frac{\partial\phi}{\partial s_1} \perp \frac{\partial\phi}{\partial s_2}$.

Thus the inequality in (1) is an equality $(E(\phi)=F(\phi))$ iff

$$\left|\frac{\partial\phi}{\partial s_1}\right| = \left|\frac{\partial\phi}{\partial s_2}\right| \quad \& \quad \frac{\partial\phi}{\partial s_1} \perp \frac{\partial\phi}{\partial s_2} \quad \text{at every } s \in D$$

This is equivalent to say that the differential $d_s\phi$ of ϕ at s intended as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$, is a composition of an isometry and a dilation of a factor $\lambda = \left|\frac{\partial\phi}{\partial s_1}\right| = \left|\frac{\partial\phi}{\partial s_2}\right|$, that is $d_s\phi$ is a conformal linear map (it preserves angles)

(3) Every ϕ admits a reparametrization $\tilde{\phi} = \phi \circ \sigma$ which is conformal (Lichtenstein theorem) that is, $E(\tilde{\phi}) = F(\tilde{\phi})$.

Then, if ϕ_0 minimizes E, it also minimizes F.
Moreover the map ϕ_0 is harmonic & conformal $\rightarrow$ that is, $d_s\phi$ is conformal for every $s \in D$.

harmonic: because it minimizes the Dirichlet energy

Indeed, for every other ϕ,

$$(4) \qquad F(\phi) \underset{\substack{\uparrow \\ \text{take } \tilde{\phi} \\ \text{as in (3)}}}{=} F(\tilde{\phi}) \underset{\substack{\uparrow \\ (2)}}{=} E(\tilde{\phi}) \underset{\substack{\uparrow \\ \text{choice of } \phi_0}}{\geqslant} E(\phi_0) \underset{\substack{\uparrow \\ (1)}}{\geqslant} F(\phi_0).$$

Moreover taking $\phi := \phi_0$ in (4) we get $F(\phi_0) = E(\phi_0)$ which implies that ϕ_0 is conformal by (2).

This computation suggests that one can solve Plateau Pb. by minimizing the Dirichlet energy E instead of the area functional F.

Final remarks

This approach was carried out by J. Douglas and T. Radó in the mid 1930s.

It presents many difficulties (but it also gives some of the best results in term of regularity.

One of the difficulty is in Step (2): the conformal reparametrization $\tilde{\phi} = \phi \circ \sigma$ is given by an homeomorphism $\sigma : D \to D$ which in general CANNOT be the identity on ∂D. Thus $\tilde{\phi}$ does not satisfies the boundary condition $\tilde{\phi} = \gamma$ on ∂D.

This means that the solution of Plateau Problem is obtained by minimizing $E(\phi)$ with the (nonstandard) boundary condition

$$\phi = \gamma \circ \sigma \text{ on } \partial D \text{ for some}$$
$$\text{homeomorphism } \sigma : \partial D \to \partial D \quad (\sigma \text{ is } \underline{\text{not}} \text{ fixed})$$

Finally, this approach does not work for $d > 2$ because Step (2) fails, it is not true that every map admits a conformal reparametrizations, due to the fact that conformal maps are quite rigid when the domain has dimension $\geqslant 3$.

Here ends the review of main approaches to Plateau Problem