GMT 19/20, Exercise Sheet 1

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Reference: lectures 1 & 2

▷ In the next exercises Y is a metric space (with distance d).

Ex. 1

The following are equivalent:

(a) Y is totally bounded (that is, for every $\varepsilon > 0$ Y can be covered by finitely many (closed) balls with radius $\leq \varepsilon$);

(b) Y is the limit (in Hausdorff distance d_H) of a sequence of finite sets.

In particular every compact metric space Y is the limit of a sequence of finite sets.

Ex. 2

Let

$$\mathcal{C}(Y) := \{ C \subset Y \text{ s.t. } C \text{ is closed}, C \neq \emptyset \}$$

be endowed with the Hausdorff distance d_H. Prove that:

(a) if Y is complete so is $\mathcal{C}(Y)$;

(b) if Y is totally bounded so is $\mathcal{C}(Y)$;

(c) if Y is compact so is $\mathcal{C}(Y)$;

(d) $\{ C \in \mathcal{C}(Y) : C \text{ compact} \}$ is closed in $\mathcal{C}(Y)$;

(e) $\{ C \in \mathcal{C}(Y) : C \text{ connected \& compact} \}$ is closed in $\mathcal{C}(Y)$;

(f) $\{ C \in \mathcal{C}(\mathbb{R}^n) : C \text{ connected} \}$ is not closed in $\mathcal{C}(\mathbb{R}^n)$.

▷ Recall that a path in Y is a continuous map $\gamma : I \to Y$ (closed interval) and $L(\gamma)$ denotes the length of γ.

Ex. 3

Prove that:

(a) $\gamma \mapsto L(\gamma)$ is lower semicontinuous w.r.t. pointwise convergence;

(b) in general $\gamma \longmapsto L(\gamma)$ is not continuous on the space $X := \{\gamma \text{ s.t. } \mathrm{Lip}(\gamma) \leq m\}$ endowed with uniform distance that is, $d_X(\gamma_0, \gamma_1) := \sup_{t \in I} d(\gamma_0(t), \gamma_1(t))$.

▷ Let $\gamma : I \to Y$ be a path in Y, and for every closed interval $J \subset I$ let $L(\gamma, J)$ be the length of the restriction of γ to J.
We say that γ has constant speed v if $L(\gamma, J) = v \cdot \mathrm{length}(J)$ for every $J \subset I$.

Ex. 4

Given $\gamma : [0,1] \longrightarrow Y$, prove that $L(\gamma) \leq \mathrm{Lip}(\gamma)$ and if $L(\gamma) < +\infty$ then equality holds iff γ has constant speed.

Ex. 5

Let $\gamma : \underbrace{[0,1]}_{= I} \to Y$ be a path with $L(\gamma) < +\infty$ and assume that γ is not constant on any subinterval of I.
Then there exists a reparametrization $\tilde{\gamma} = \gamma \circ \sigma$ such that:

(a) $\sigma : I \to I$ is an increasing homeomorphism;

(b) $\tilde{\gamma}$ has constant speed $v = L(\gamma)$. In particular $L(\tilde{\gamma}) = \mathrm{Lip}(\tilde{\gamma})$.

Hint: let σ be the inverse of $\ell : I \to I$, $\ell(t) := \dfrac{L(\gamma, [0,t])}{L(\gamma)}$.
(You must prove that ℓ is an increasing homeom.)

Ex. 6

Let $\gamma : \overbrace{[0,1]}^{\substack{I \\ \parallel}} \to Y$ be a non-constant path with $L(\gamma) < +\infty$. Then there exists a reparametrization $\tilde{\gamma} := \gamma \circ \sigma$ such that:

(a) $\sigma : I \to I$ is strictly increasing and $\sigma(0) = 0$, $\sigma(1) = 1$;

(b) $\tilde{\gamma}$ is continuous, $\tilde{\gamma}(I) = \gamma(I)$, $L(\tilde{\gamma}) = L(\gamma)$;

(c) $\tilde{\gamma}$ has constant speed $v = L(\gamma)$.

Hint: in this case the function ℓ from previous exercise is still increasing but possibly not strictly, and therefore may be not invertible. However some sort of inverse of ℓ works.

Ex. 7

Let $\gamma : \overbrace{[0,1]}^{\substack{I \\ \parallel}} \to Y$ be a path with $L(\gamma) < +\infty$ and fix $\delta > 0$. Then there exists a reparametrization $\tilde{\gamma} := \gamma \circ \sigma$ such that:

(a) $\sigma : I \to I$ is an increasing homeomorphism;

(b) $\mathrm{Lip}(\tilde{\gamma}) \leq L(\gamma) + \delta$.

Hint: let σ be the inverse of $\ell_\delta : I \to I$, $\ell_\delta(t) := \dfrac{L(\gamma, [0,t]) + \delta t}{L(\gamma) + \delta}$.

Ex. 8

Given $x_0, x_1 \in Y$ with $x_0 \neq x_1$, define

$$X := \{ \gamma : [0,1] \to Y \text{ s.t. } \gamma \text{ continuous}, \gamma(0) = x_0, \gamma(1) = x_1 \};$$

$$(*) \qquad L_{min} := \inf \{ L(\gamma) : \gamma \in X \},$$

and assume that $L_{min} < +\infty$. Prove that:

(a) if Y is compact the infimum in $(*)$ is attained;
(complete the sketh of proof given is Lecture 1, § 1.2.3)

(b) if Y is locally compact the infimum in $(*)$ may be not attained;

(c) if the ball $\overline{B(x_0, L_m)}$ is compact the infimum in $(*)$ is attained;

(d) if the infimum in $(*)$ is attained then every minimizer γ has the following property: if $\gamma(t_0) = \gamma(t_1)$ for some $t_0 < t_1$ then γ is constant on $[t_0, t_1]$;

(e) if the infimum in $(*)$ is attained then there exists a minimizer γ which is injective.

Ex. 9

Take X, L_{min} as in Ex. 6, assume that $L_{min} < +\infty$ and let

$$(**) \qquad \tilde{L}_{min} := \inf \left\{ \mathrm{Lip}(\gamma) : \gamma \in X \right\}$$

Prove that:

(a) $\tilde{L}_{min} = L_{min}$;

(b) every minimizer γ of $(**)$ is also a minimizer of $(*)$ and has constant speed;

(c) if Y is compact the infimum in $(**)$ is attained.

A network N in Y is a finite collection of paths $\gamma : \overbrace{[0,1]}^{I} \to Y$ such that the set

$$S(N) := \bigcup_{\gamma \in N} \gamma(I) \quad \longleftarrow \text{support of } N$$

is connected; for such N we define

$$\underbrace{\mathrm{Lip}(N)}_{\text{Lipschitz const. of } N} := \sup_{\gamma \in N} \mathrm{Lip}(\gamma) \ ; \qquad \underbrace{L(N)}_{\text{length of } N} := \sum_{\gamma \in N} L(\gamma) \, .$$

Ex. 10

Given a finite set $\Gamma \subset Y$ define

$$(*) \qquad L_{min} := \inf \{ L(N) : N \text{ s.t. } S(N) \supset \Gamma \} .$$

Assume that Y is compact and $L_{min} < +\infty$.
Prove that this infimum in $(*)$ is attained.

Hint: prove that, given a network N s.t. $S(N) \supset \Gamma$, there exists a network $\tilde{N}$ s.t. $S(N) \supset S(\tilde{N}) \supset \Gamma$, $L(\tilde{N}) \leq L(N)$, and $\# N \leq \# \Gamma - 1$. Then prove that L_{min} is equal to

$$\tilde{L}_{min} := \inf \left\{ L(N) : \begin{array}{l} N \text{ s.t. } S(N) \supset \Gamma, \ \# N \leq \# \Gamma - 1 \\ \mathrm{Lip}(N) \leq L_{min} + 1 \end{array} \right\} ,$$

and that this infimum is attained.

Ex. 11

Prove that there exists a minimizer N_0 of $(*)$ in Ex. 10 such that each $\gamma \in N_0$ is injective.

Moreover, given $\gamma_0, \gamma_1 \in N_0$, $\gamma_0 \neq \gamma_1$, $\gamma_0(I) \cap \gamma_1(I)$ contains at most 1 point.

Ex. 10 and 11 show that the minimization of $L(N)$ among all networks N such that $S(N)$ contains Γ leads to a satisfactory variant of Steiner problem.
One can show that this variant is completely equivalent to the one described in Lecture 1, § 1.2.2.
Note that for existence it is essential that Γ is finite.

In the next exercises Y is an n-dimensional surface of class C^1 in $\mathbb{R}^m$, possibly with boundary.

Some exercises require some knowledge of the theory of Sobolev Spaces and Calculus of Variations.

Ex. 12

Assume that $\gamma : [0,1] \to Y$ is a continuous path in the Sobolev class $W^{1,1}$ (as a function with values in $\mathbb{R}^m$)
Prove that

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)| \, dt .$$

The inequality $\leqslant$ is easy. The proof of $\leqslant$ is quite delicate: the one I know uses the points of L^1-approximate continuity of $\dot{\gamma}$ and a covering argument.

Ex. 13

Fix $x_0, x_1 \in Y$ with $x_0 \neq x_1$, and define

$$X_p := \left\{ \gamma : [0,1] \to Y \text{ s.t. } \begin{array}{l} \gamma \text{ is contin. and } W^{1,p} \\ \gamma(0) = x_0 ,\ \gamma(1) = x_1 \end{array} \right\} .$$

Prove that if Y is compact then $\|\dot{\gamma}\|_p$ admits a minimizer over all $\gamma \in X_p$.

Ex. 14

Following the scheme outlined in Lecture 1, § 1.2.3, prove that a minimizer γ_0 of $\|\dot{\gamma}\|_p$ on X_p is also a minimizer of $\|\dot{\gamma}\|_1$ (i.e., the length of γ) on X_1, and that γ_0 has constant speed and more precisely $|\dot{\gamma}(t)| = \|\dot{\gamma}\|_1$ for a.e. $t \in [0,1]$.

Recall that a linear map $T: \mathbb{R}^d \to \mathbb{R}^n$ is conformal if $T = \Lambda R$ where Λ is a dilation of $\mathbb{R}^n$ (i.e., $\Lambda : x \mapsto \lambda x$ with $\lambda \geqslant 0$) and $R : \mathbb{R}^d \to \mathbb{R}^n$ is a linear isometry (i.e., $|Rx| = |x| \ \forall x \in \mathbb{R}^d$).
In other words, a matrix $M \in \mathbb{R}^{n\times d}$ is conformal if $M = \lambda R$ with $\lambda \geqslant 0$ and $R \in \mathbb{R}^{n\times d}$ s.t. $R^t R = I_d \leftarrow$ dxd identity matrix.

For $d=n=2$, conformal matrices are

$$\mathcal{M} := \underbrace{\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a,b \in \mathbb{R} \right\}}_{\mathcal{M}^+} \cup \underbrace{\left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} : a,b \in \mathbb{R} \right\}}_{\mathcal{M}^-}$$

Ex. 15

Let Ω be an open set in $\mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}^2$ be a map of class C^1. Then

We identify $\mathbb{R}^2$ and $\mathbb{C}$

(a) if f is holomorphic or anti-holomorphic then f is conformal;

(b) if f is conformal, Ω is connected and $\nabla f \neq 0$ on Ω, then f is either holomorphic or anti-holomorphic.

(c) Does the conclusion in (b) hold without assuming $\nabla f \neq 0$?

The answer to (c) is positive at least for f of class $C^1 \cap W^{2,1}_{loc}$.
Hint:
Set $A^0 := \{ x \in \Omega : \nabla f(x) = 0 \}$ and $A^{\pm} := \{ x \in \Omega \setminus A^0 : \nabla f(x) \in \mathcal{M}^{\pm} \}$, and prove that the function $g : \Omega \to \mathbb{R}^2$ given by

$$g(x) := \begin{cases} \overline{\frac{\partial f}{\partial x_1}(x)} & \text{if } x \in A^- \\ 0 & \text{if } x \in A^0 \\ \frac{\partial f}{\partial x_1}(x) & \text{if } x \in A^+ \end{cases}$$

is holomorphic. [Here I need $g \in W^{1,1}_{loc}$, which I prove using the assumption $f \in W^{2,1}_{loc}$.]

Now there are now two possibilities:

1) $A^0 = g^{-1}(0)$ has an accumulation point in Ω, then $g = 0$ on Ω (because g is holom. and Ω is connected) and f is constant (in part. f is hol.).

2) A^0 has no accumulation points in Ω, then A^+ or A^- is empty $(*)$, and then f is antiholomorphic or holomorphic;

Given A^-, A^+ open disjoint, nonempty subsets of a connected open set $\Omega \subset \mathbb{R}^2$, then $\Omega \setminus (A^+ \cup A^-)$ cannot be discrete.

Ex. 16

Let $M \in \mathbb{R}^{n \times d}$ with $1 \leq d \leq n$. Then

$$\det(M^t M) \leqslant \frac{|M|^{2d}}{d^d}$$

$|M|$ = Frobenius norm of M $:= \left(\sum_{ij} M_{ij}^2\right)^{1/2} = \mathrm{tr}(M^t M)$.

and equality holds iff M is conformal.

Hint: use the singular value decomposition $M = R \Lambda \tilde{R}$ where $\tilde{R} \in SO(d)$, $\Lambda \in \mathbb{R}^{d \times d}$ is diagonal with positive entries, $R \in \mathbb{R}^{n \times d}$ satisfies $R^t R = I_d$.

Ex. 17

For $1 \leqslant d \leqslant n$, let Ω be an open set in $\mathbb{R}^d$ and $\phi : \Omega \to \mathbb{R}^n$ a map of class C^1. Prove that

$$\underbrace{\int_\Omega J\phi \, dx}_{\text{area functional}} \leqslant \frac{1}{d^{d/2}} \int_\Omega |\nabla \phi|^d \, dx \, ,$$

and equality holds iff ϕ is conformal.

This statement generalizes the inequality $F(\phi) \leqslant E(\phi)$ proved (only for $n=3$) in Leture 1, § 1.2.3.

Ex. 18

Let $M \in \mathbb{R}^{2m\times 2}$ and for every $j=1,\dots,m$ let M_j be the 2×2 minor formed by the rows $2j-1$, $2j$. Then

(a) $\det(M^t M) \geqslant \left(\sum_{j=1}^m \det M_j\right)^2$;

(b) equality holds in (a) if $M_j \in \mathcal{M}^+ := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a,b\in\mathbb{R} \right\}$;

(c) equality holds in (a) iff the colums of M span a complex subspace of $\mathbb{R}^{2m}$, which is identified with $\mathbb{C}^m$ by

$$(x_1,\dots,x_{2m}) \simeq (x_1+ix_2, x_3+ix_4, \dots, x_{2m-1}+ix_{2m}).$$

Hint for (a): set $\phi(M) := \det(M^t M)$; $\Phi(M) := (\sum \det M_j)^2$ and notice that $\phi(M) = \phi(RM)$ for every $R\in O(2m)$, $\Phi(M) = \Phi(RM)$ for every R of the form

$$R = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_m \end{pmatrix} \qquad \text{with } R_1,\dots,R_m \in SO(2).$$

Then show that it suffices to prove (a) when all minors M_j are upper triangular...

▷ Inequality (a) is a simple case of Wirtinger inequality.

Ex. 19

Let D be a closed domain in $\mathbb{R}^2$ with boundary of class C^1, oriented counter-clockwise, and let $f: D \to \mathbb{R}^2$ be a map of class C^1.
Then

$$\int_D \det(\nabla f)\,dx = \int_{\partial D} f_1 \, \frac{\partial f_2}{\partial \tau} \, d\mathcal{H}^1$$

length measure on ∂D

tangential derivative of f_2

Thus $\int_D \det(\nabla f)\,dx$ depends only on the restriction of f to ∂D.

Ex. 20

Let D be as in Ex. 19, and let $f : D \to \mathbb{R}^{2m}$ be a map of class C^1. Then

$$\int_D Jf \, dx \geq \left| \int_{\partial D} \sum_{j=1}^{m} f_{2j-1} \frac{\partial f_{2j}}{\partial \tau} \, d\mathcal{H}^1 \right| ,$$

and equality holds if f is holomorphic in the interior of D.
(We identify $\mathbb{R}^2$ with $\mathbb{C}$ and $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ as in Ex. 18.)

Hint: use Ex. 18 and Ex. 19.

Ex. 21

Let D be as in Ex. 19, let $f : D \to \mathbb{R}^{2m} \simeq \mathbb{C}^m$ be an injective map of class C^1 which is holomorphic in the interior of D, and set

$$\Gamma := f(\partial D), \qquad \Sigma := f(D) .$$

Then

$$\operatorname{area}(\Sigma) \leq \operatorname{area}(\tilde{\Sigma})$$

for every surface $\tilde{\Sigma}$ of class C^1 in $\mathbb{R}^{2m}$ parametrized by D and with boundary Γ.

This exercise gives a partial proof (for $d=2$) of the claim in Example 3, § 1.3, Lecture 2.

Ex. 22

For every $1 \leq j \leq m$, let $p_j : \mathbb{R}^{2m} \to \mathbb{R}^2$ be the projection given by

$$p_j(x_1, \dots, x_{2m}) := (x_{2j-1}, x_{2j}) .$$

Let V be a 2-dimensional subspace of $\mathbb{R}^{2m}$, and let Σ be a bounded Borel set in V. Then

$$\operatorname{area}(\Sigma) \geqslant \sum_{j=1}^{m} \operatorname{area}(p_j(\Sigma)) ,$$

and equality holds iff V is a complex subspace of $\mathbb{R}^{2m} \simeq \mathbb{C}^m$. (We identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ as in Ex. 18.)

Hint: parametrize V by a linear map $x \mapsto Mx$ and use Ex. 18.

Ex. 23

Let Σ be a 2-dimensional compact surface of class C^1 in $\mathbb{R}^{2m}$ (possibly with boundary, and possibly not orientable), and for $j = 1, \dots, m$ take p_j as in Ex. 22. Prove that

(a) $\operatorname{area}(\Sigma) \geqslant \sum_{j=1}^{m} \operatorname{area}(p_j(\Sigma))$;

(b) equality holds in (a) iff the tangent planes to Σ are complex subspaces of $\mathbb{R}^{2m} \simeq \mathbb{C}^m$;

(c) $\operatorname{area}(\Sigma) \geqslant \sum_{j=1}^{m} \operatorname{area}(\Sigma_j)$ where

$$\Sigma_j := \{ y \in \mathbb{R}^2 \text{ s.t. } y \notin \pi_j(\partial\Sigma) \text{ and } \deg_2(\Sigma, p_j, y) = 1 \} .$$

degree modulo 2 at y of (the restriction of) p_j to Σ

Hint: use Ex. 22.

Ex. 24

In $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$ define

$$\Gamma := (S^1 \times \{0\}) \cup (\{0\} \times \overbrace{S^1}^{\{x\in\mathbb{R}^2 \,:\, |x|=1\}}) , \qquad \Sigma_0 := (\overbrace{D^2}^{\{x\in\mathbb{R}^2 \,:\, |x|\leq 1\}} \times \{0\}) \cup (\{0\} \times D^2),$$

and let Σ be a 2-dimensional compact (possibly not orientable) surface of class C^1 with $\partial\Sigma = \Gamma$. Then

$$\mathrm{area}(\Sigma) \geq 2\pi = \mathrm{area}(\Sigma_0) .$$

(Compare with the claim in Example 1 in § 1.3, Lecture 2.)

<u>Hint</u>: use statement (c) in Ex. 23 and degree theory to prove that $\Sigma_1 = \Sigma_2 = D^2$.