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CHARACTERIZATION OF HOLDER AND ZYGMUND CLASSES AS INTERPOLATION SPACES

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ABSTRACT

We give a characterization of certain interpolation spaces between D(A) and $C^0(\overline{\Omega})$, where D(A) is the domain of a second-order strongly elliptic operator having continuous coefficients, with a regular oblique derivative boundary condition; $C^0(\overline{\Omega})$ is the space of continuous functions defined in the closure of a bounded connected open set with smooth boundary.

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0. INTRODUCTION

In this paper we are concerned with the characterization of certain interpolation spaces between the domain of a second-order strongly elliptic operator, with a regular oblique derivative boundary condition, and the space of continuous functions in which the domain is embedded.

The interpolation spaces considered here can be abstractly defined as follows: let E be a Banach space, and let $A:D(A)\subseteq E\to E$ be a closed linear operator which generates a bounded semi-group e^{tA} in E; for each $\theta\in]0,1[$ set

$$D_A(\theta,\infty)=\{x\in E:\sup_{t>0} t^{-\theta}\|e^{tA}x-x\|_E<\infty\},$$

$$D_A(\theta)=\{x\in D_A(\theta,\infty):\lim_{t\to 0^+} t^{-\theta}\|e^{tA}x-x\|_E=0\}.$$

Although this definition seems to depend on the semi-group e^{tA}, it can be shown that in fact the spaces $D_A(\theta,\infty)$ and $D_A(\theta)$ depend only on E and the domain D(A); actually $D_A(\theta,\infty)$ coincides with Lions' interpolation space $(D(A),E)_{1-\theta,\infty}$ (see Lions [10], Lions-Peetre [11]), whereas $D_A(\theta)$ is the so-called "continuous interpolation space" $(D(A),E)_{1-\theta}$ introduced by Da Prato-Grisvard [8].

The spaces $D_A(\theta,\infty)$ and $D_A(\theta)$ have been recently studied by several authors in connection with the theory of abstract evolution equations: see Da Prato-Grisvard [8], Ardito-Ricciardi [5], Da Prato-Sinestrari [9], Sinestrari [16], [17], Lunardi [12], [13], [14], [15], Acquistapace-Terreni [1], [2], [3]. An important feature of these spaces is their "maximal regularity" property. Maximal regularity means the following: if f is continuous with values in a Banach space Y, then the evolution problem

$$\begin{cases} u'-Au=f \\ u(0)=0 \end{cases}$$

has a unique C^1 solution u such that u' and Au are continuous with values in Y. This property is not true for every Banach space Y (see Baillon [6], Travis [22]), but it holds when $Y=D_A(\theta)$, where A is the infinitesimal generator of an analytic semi-group in some other Banach space E; note that we cannot replace here $D_A(\theta)$ by $D_A(\theta,\infty)$ (see Da Prato-Grisvard [8]). However a similar property is true for $D_A(\theta,\infty)$ (with A as before); namely, if f is continuous with values in E and bounded with values in $D_A(\theta,\infty)$, then the same is true for u' and Au. For a proof of these facts see Sinestrari [17].

Our charecterization of the spaces $D_A(\theta,\infty)$ and $D_A(\theta)$ concerns the case in which $E=C^0(\bar{\Omega})$, where $\Omega\subseteq\mathbb{R}^n$ is a bounded connected open set with smooth boundary, and A is a second-order strongly elliptic operator with a regular first-order differential condition at the boundary. In this situation, A is the infinitesimal generator of an analytic semi-group in E (see Stewart [20]). Under Stewart's assumptions we prove that $D_A(\theta,\infty)$ (resp. $D_A(\theta)$) coincides with the space of 2θ-Hölder (resp. 2θ-"little Hölder") continuous functions if $\theta\in]0,1/2[$, and with the space of differentiable functions which satisfy the boundary condition and whose gradient is $(2\theta-1)$-Hölder (resp. $(2\theta-1)$-"little Hölder") continuous if $\theta\in]1/2,1[$. We recall that a function f is α-Hölder (resp. α-"little Hölder") continuous, $\alpha\in]0,1]$, if

$$|f(x)-f(y)|=O(|x-y|^\alpha) \quad (\text{resp. } |f(x)-f(y)|=o(|x-y|^\alpha))$$
$$\text{as } |x-y|\to 0+.$$

When $\theta=1/2$, we obtain as $D_A(\frac{1}{2},\infty)$ (resp. $D_A(\frac{1}{2})$) the so-called Zygmund classes, i.e. the spaces of continuous functions verifying

$$|f(x)+f(y)-2f(\tfrac{x+y}{2})|=O(|x-y|) \quad (\text{resp. } |f(x)+f(y)-2f(\tfrac{x+y}{2})|=$$
$$= o(|x-y|) \quad \text{as } |x-y|\to 0^+,$$

and satisfying an additional property along the boundary, which is in some sense a weak form of the boundary differential condition.

We prove in addition that the interpolation spaces do not change if we replace D(A) by the (smaller) space $C^2_B(\bar{\Omega})$ of twice continuously differentiable functions which satisfy the boundary condition: in other words, we show that

$$\begin{cases} D_A(\theta,\infty)\equiv(D(A),C^0(\bar{\Omega}))_{1-\theta,\infty}=(C^2_B(\bar{\Omega}),C^0(\bar{\Omega}))_{1-\theta,\infty} \\ D_A(\theta)\equiv(D(A),C^0(\bar{\Omega}))_{1-\theta}=(C^2_B(\bar{\Omega}),C^0(\bar{\Omega}))_{1-\theta} \end{cases} \tag{0.1}$$

The results which we prove here are already known (except for the case $\theta=1/2$) in dimension n=1, with $\Omega=]a,b[$ (see Acquistapace-Terreni [3], Da Prato-Grisvard [8]); in this case (0.1) is obvious since $D(A)=C^2_B(\bar{\Omega})$.

The characterization of $D_A(\theta,\infty)$ and $D_A(\theta)$ in the case of boundary conditions of Dirichlet type has been given by Lunardi [13]; she obtains the Hölder and little - Hölder spaces if $\theta\neq 1/2$, and the Zygmund classes if $\theta=1/2$, with the additional requirement to the functions, in either case, to vanish along $\partial\Omega$. Lunardi's result can be found again by our method, which also allows slightly weaker assumptions about the smoothness of $\partial\Omega$ and of the coefficients of the differential operator A.

Let us describe now the subject of the next sections. In Section 1 we list our definitions and assumptions, and state some preliminary results to be used later on; among these ones, we mention some properties of the Zygmund spaces which do not seem to be completely straightforward. Section 2,3,4 are concerned with the proof of the inclusions

$$X^{\theta}(\overline{\Omega}) \hookrightarrow (C^2_B(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta,\infty}\ ,\ Y^{\theta}(\overline{\Omega}) \hookrightarrow (C^2_B(\overline{\Omega}), C^0(\overline{\Omega}))_{1-\theta}\ ,$$

where $X^{\theta}(\overline{\Omega})$, $Y^{\theta}(\overline{\Omega})$ symbolize here the concrete function spaces which we will characterize as $D_A(\theta,\infty)$,$D_A(\theta)$ respectively. More precisely, in Section 2 we consider the case of the half-space $\overline{\Omega}=\mathbb{R}^n_+$, with a boundary condition $Nu=0$ whose principal part is the derivative with respect to the normal to $\partial\Omega$; Section 3 still concerns the case $\overline{\Omega}=\mathbb{R}^n_+$, with a boundary condition $Bu=0$ with general principal part (i.e. a derivative along a non-tangential direction with respect to $\partial\Omega$); finally in Section 4 we treat the general case in a bounded connected open set with C^3 boundary. In Section 5 we consider the reverse inclusions, i.e. the inclusions

$$D_A(\theta,\infty) \hookrightarrow X^{\theta}(\overline{\Omega})\ ,\ D_A(\theta) \hookrightarrow Y^{\theta}(\overline{\Omega}),$$

where $X^{\theta}(\overline{\Omega})$ and $Y^{\theta}(\overline{\Omega})$ have the same meaning as before. Finally, in Section 6 we draw the conclusions, stating our main theorems.

1. PRELIMINARIES

Let us list some notations and definitions. Let Ω be a (possibly unbounded) connected open set of $\mathbb{R}^n$, $n\geq 1$; further assumptions on Ω will be specified when necessary. We denote by $\overline{\Omega}$ the closure of Ω and by $\partial\Omega$ its boundary.

DEFINITION 1.1. For each $k\in\mathbb{N}$ set:

(i) $C^0(\overline{\Omega})=\{f:\overline{\Omega}\to\mathbb{C}$ uniformly continuous and bounded in $\overline{\Omega}\}$,

$$\|f\|_{C^0(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |f(x)|,$$

(ii) $C^k(\overline{\Omega})=\{f\in C^0(\overline{\Omega}):\ D^{\alpha}f\in C^0(\overline{\Omega})\quad \forall\alpha\in\mathbb{N}^n,\ |\alpha|\leq k\}$,

$$\|f\|_{C^k(\overline{\Omega})} = \sum_{|\alpha|\leq k} \|D^{\alpha}f\|_{C^0(\overline{\Omega})}.$$

We will also denote the norm $\|\cdot\|_{C^0(\overline{\Omega})}$ by $\|\cdot\|_{\infty,\overline{\Omega}}$ or simply $\|\cdot\|_{\infty}$ when no confusion can arise.

For each $x_0\in\overline{\Omega}$ and $r>0$ define $\Omega(x_0,r)=\{x\in\Omega:|x-x_0|<r\}$; if $K\subseteq\overline{\Omega}$ is any set, define also for each $\theta\in]0,1]$:

$$[f]_{\theta,\overline{K}} = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^{\theta}} : x,y\in\overline{K},\ x\neq y\right\}. \tag{1.1}$$

When $K=\Omega$ we simply write $[f]_{\theta}$ instead of $[f]_{\theta,\overline{\Omega}}$. □

DEFINITION 1.2. For each $\theta\in]0,1]$ and $k\in\mathbb{N}$ set:

(i) $C^{0,\theta}(\overline{\Omega})=\{f\in C^0(\overline{\Omega}):\ [f]_{\theta}<\infty\}$,

$$\|f\|_{C^{0,\theta}(\overline{\Omega})} = \|f\|_{C^0(\overline{\Omega})} + [f]_{\theta}\ ;$$

(ii) $h^{0,\theta}(\overline{\Omega})=\{f\in C^{0,\theta}(\overline{\Omega}):\ \lim_{r\to 0^+}\sup_{x_0\in\overline{\Omega}} [f]_{\theta,\overline{\Omega(x_0,r)}} = 0\}$;

(iii) $C^{k,\theta}(\overline{\Omega})=\{f\in C^k(\overline{\Omega}):\ D^{\alpha}f\in C^{0,\theta}(\overline{\Omega})\quad \forall\alpha\in\mathbb{N}^n,\ |\alpha|\leq k\}$,

$$\|f\|_{C^{k,\theta}(\overline{\Omega})} = \sum_{|\alpha|\leq k} \|D^{\alpha}f\|_{C^{0,\theta}(\overline{\Omega})}\ ;$$

(iv) $h^{k,\theta}(\overline{\Omega})=\{f\in C^{k,\theta}(\overline{\Omega}):D^{\alpha}f\in h^{0,\theta}(\overline{\Omega})\quad \forall\alpha\in\mathbb{N}^n,\ |\alpha|\leq k\}$.

Again, let $K\subseteq\overline{\Omega}$ be any set; define for each $\theta\in]0,2]$

$$[f]_{*,\theta,\overline{K}} = \sup\left\{\frac{|f(x)+f(y)-2f(\frac{1}{2}(x+y))|}{|x-y|^{\theta}} : x,y,\frac{x+y}{2}\in\overline{K}, x\neq y\right\} \tag{1.2}$$

when $K=\Omega$ we write $[f]_{*,\theta}$ for $[f]_{*,\theta,\overline{\Omega}}$. □

DEFINITION 1.3. For each $\theta\in]0,2]$ set:

(i) $C^{*,\theta}(\overline{\Omega})=\{f\in C^0(\overline{\Omega}):[f]_{*,\theta}<\infty\}$,

$$\|f\|_{C^{*,\theta}(\overline{\Omega})}=\|f\|_{C^0(\overline{\Omega})}+[f]_{*,\theta};$$

(ii) $h^{*,\theta}(\overline{\Omega})=\{f\in C^{*,\theta}(\overline{\Omega}): \lim_{r\to 0^+}\sup_{x_0\in\overline{\Omega}}[f]_{*,\theta,\overline{\Omega(x_0,r)}}=0\}$. □

REMARK 1.4 (i) It is clear that for $\theta\in]0,1]$ and $k=0,1,2,\ldots$ $\ldots,h^{k,\theta}(\overline{\Omega})$ is a closed subspace of $C^{k,\theta}(\overline{\Omega})$. Similarly $h^{*,\theta}(\overline{\Omega})$ is a closed subspace of $C^{*,\theta}(\overline{\Omega})$ for each $\theta\in]0,2]$.

(ii) The spaces $C^{*,\theta}(\overline{\Omega})$, $h^{*,\theta}(\overline{\Omega})$ have been first studied by Zygmund [24], [25], in the one-dimensional case (see also Butzer-Berens [7]). A wide description in the case $\Omega=\mathbb{R}^n$ can be found in Stein [18]; see also Taibleson [21]. It is well-known that

$$C^{*,\theta}(\overline{\Omega})=C^{0,\theta}(\overline{\Omega}),\quad h^{*,\theta}(\overline{\Omega})=h^{0,\theta}(\overline{\Omega})\quad \forall\theta\in]0,1[,$$
$$C^{*,\theta}(\overline{\Omega})=C^{1,\theta-1}(\overline{\Omega}),\quad h^{*,\theta}(\overline{\Omega})=h^{1,\theta-1}(\overline{\Omega})\quad \forall\theta\in]1,2[,$$

with equivalence of norms; on the other hand we have the proper (continuous) inclusions

$$C^{0,1}(\overline{\Omega})\hookrightarrow C^{*,1}(\overline{\Omega})\hookrightarrow\bigcap_{\theta\in]0,1[}h^{0,\theta}(\overline{\Omega}),$$
$$C^{1}(\overline{\Omega})\hookrightarrow h^{*,1}(\overline{\Omega}),$$
$$C^{1,1}(\overline{\Omega})\hookrightarrow C^{*,2}(\overline{\Omega})\hookrightarrow\bigcap_{\theta\in]1,2[}h^{1,\theta-1}(\overline{\Omega}),$$

whereas $h^{0,1}(\overline{\Omega})=\{\text{constants}\}$, $h^{*,2}(\overline{\Omega})=\{\text{affine functions}\}$ (for bounded Ω). For all these results see Zygmund [24], [25] in the case $n=1$, and Triebel ([23], Section 4.5) in the case of a bounded $\Omega\subseteq\mathbb{R}^n$ with C^∞ boundary; a proof for more general Ω (i.e. when $\partial\Omega$ is of class C^1) can also be given by a direct method which makes use of Zygmund's one-dimensional argument ([25], chapter II, Theorem 3.4), but we omit it for brevity. More properties of the spaces $C^{*,1}(\overline{\Omega})$, $h^{*,1}(\overline{\Omega})$ will be proved in Lemmata 1.18 and 1.19 below.

(iii) Obviously in the semi-norms (1.1) and (1.2) it suffices to consider points x,y with $|x-y|\leq\delta$, δ being any fixed positive number. □

REMARK 1.5. Let Y be a Banach space and let X be any of the symbols C^k, $C^{k,\theta}$, $h^{k,\theta}$, $C^{*,\theta}$, $h^{*,\theta}$. Definitions 1.1, 1.2 and 1.3 can be obviously adapted to the case of functions $f:\overline{\Omega}\to Y$; in this case we denote the corresponding space by $X(\overline{\Omega},Y)$. However if $Y=\mathbb{C}^m$ or $Y=\mathbb{R}^m$ we will still write $X(\overline{\Omega})$ instead of $X(\overline{\Omega},Y)$, provided no confusion can arise.

In particular, suppose $f\in C^k(\overline{\Omega})$, $k\geq 1$, and denote by $D^s f(x)$ the s-th order gradient of f at x, i.e. the $\binom{n+s-1}{s}$-vector whose components are $\{D^\alpha f(x)\}_{\alpha\in\mathbb{N}^n,|\alpha|=s}$; then we have $D^s f\in C^{k-s}(\overline{\Omega},\mathbb{C}^{\binom{n+s-1}{s}})$ or simply $D^s f\in C^{k-s}(\overline{\Omega})$. □

DEFINITION 1.6. Let Y be a Banach space. For each $\lambda\in[0,1[$ set

(i) $C_\lambda(]0,1],Y)=\{u:]0,1]\to Y: t\to t^\lambda u(t)$ is continuous and bounded in $]0,1]\}$,

$$\|u\|_{C_\lambda(]0,1],Y)}=\sup_{t\in]0,1]}\|t^\lambda u(t)\|_Y;$$

(ii) $C_\lambda([0,1],Y)=\{u\in C_\lambda(]0,1],Y): t\to t^\lambda u(t)\in C^0([0,1],Y)\}$. □

Clearly $C_\lambda([0,1],Y)$ is a closed subspace of $C_\lambda(]0,1],Y)$; note that $C_0([0,1],Y)=C^0([0,1],Y)$.

Now we recall the definition of Lions' interpolation spaces.

DEFINITION 1.7. Let Y,E be Banach spaces with $Y\hookrightarrow E$ (continuous inclusion), and let $\theta\in]0,1[$.

(i) We say that $x\in(Y,E)_{\theta,\infty}$ if there exists $u\in C^0([0,1],E)$ such that $u(0)=x$ and moreover,

$$u\in C_\theta(]0,1],Y) \quad , \quad u'\in C_\theta(]0,1],E);$$

$(Y,E)_{\theta,\infty}$ is a Banach space with norm $\|x\|_{(Y,E)_{\theta,\infty}} =$
$=\|x\|_E + |x|_{(Y,E)_{\theta,\infty}}$, where

$$|x|_{(Y,E)_{\theta,\infty}} = \inf\{\|u\|_{C_\theta(]0,1],Y)} + \|u'\|_{C_\theta(]0,1],E)} :$$
$$: u\in C^0([0,1],E), u(0)=x\}$$

(ii) We say that $x\in(Y,E)_\theta$ if there exists $u\in C^0([0,1],E)$ such that $u(0)=x$ and moreover

$$u\in C_\theta([0,1],Y) \quad , \quad u'\in C_\theta([0,1],E). \quad \square$$

REMARK 1.8. Clearly $(Y,E)_\theta$ is a closed subspace of $(Y,E)_{\theta,\infty}$ and in fact it coincides with the closure of Y in the norm of $(Y,E)_{\theta,\infty}$; a proof is in Sinestrari [16]. More details about the spaces $(Y,E)_{\theta,\infty}$ can be found in Lions-Peetre [11], Butzer-Berens [7], Triebel [23]; for the spaces $(Y,E)_\theta$ see Da Prato-Grisvard [8]. $\square$

REMARK 1.9. Let A be the infinitesimal generator of an analytic semi-group e^{tA} (possibly not strongly continuous at $t=0$) on the Banach space E. Thus, in particular, the resolvent set of A contains a sector $S=\{z\in\mathbb{C}: z\neq 0, |\arg z|<\eta\}$ with $\eta\in]\pi/2,\pi]$, and there exists $M>0$ such that

$$\|R(z,A)\|_{L(E)} \le \frac{M}{|z|} \quad \forall z\in S, \qquad (1.3)$$

where $R(z,A)=(z-A)^{-1}$ is the resolvent of A and $L(E)$ is the Banach space of bounded linear operators $T:E\to E$, with the usual norm $\|T\|_{L(E)} = \sup\{\|Tx\|_E : \|x\|_E = 1\}$. Choose $Y=D(A)$, endowed with the graph norm: then for each $\theta\in]0,1[$ the spaces $(D(A),E)_{1-\theta,\infty}$ and $(D(A),E)_{1-\theta}$ are denoted by $D_A(\theta,\infty)$ and $D_A(\theta)$ respectively. In addition the following characterizations hold:

$$\begin{aligned} D_A(\theta,\infty) &= \{x\in E: |x|_\theta^{(1)} = \sup_{t>0} t^{-\theta}\|e^{tA}x-x\|_E < \infty\} \\ &= \{x\in E: |x|_\theta^{(2)} = \sup_{t>0} t^{1-\theta}\|Ae^{tA}x\|_E < \infty\} \\ &= \{x\in E: |x|_\theta^{(3)} = \sup_{t>0} t^{\theta}\|AR(t,A)x\|_E < \infty\}, \end{aligned} \qquad (1.4)$$

and the norms

$$\|x\|_\theta^{(i)} = \|x\|_E + |x|_\theta^{(i)} \qquad i=1,2,3$$

are all equivalent to the norm $\|x\|_{(D(A),E)_{1-\theta,\infty}}$ introduced in Definition 1.7. Hence in what follows we will denote simply by $\|x\|_\theta$ any of the norms $\|x\|_\theta^{(1)}$, $\|x\|_\theta^{(2)}$, $\|x\|_\theta^{(3)}$, $\|x\|_{(D(A),E)_{1-\theta,\infty}}$ in $D_A(\theta,\infty)$ (and by $|x|_\theta$ any of the corresponding semi-norms). Similarly we have

$$\begin{aligned} D_A(\theta) &= \{x\in E: \lim_{t\to 0^+} t^{-\theta}\|e^{tA}x-x\|_E = 0\} \\ &= \{x\in E: \lim_{t\to 0^+} t^{1-\theta}\|Ae^{tA}x\|_E = 0\} \\ &= \{x\in E: \lim_{t\to +\infty} t^{\theta}\|AR(t,A)x\|_E = 0\}. \end{aligned} \qquad (1.5)$$

Moreover if $0<\sigma<\theta<1$ the following continuous inclusions are true:

$$D(A) \hookrightarrow D_A(\theta) \hookrightarrow D_A(\theta,\infty) \hookrightarrow D_A(\sigma) \hookrightarrow \overline{D(A)}.$$

All these properties are proved in Sinestrari [17]. □

Now we list a series of auxiliary results which will be needed in the following sections.

LEMMA 1.10. *Suppose Ω is bounded with $\partial\Omega$ of class C^1 (or, alternatively, suppose $\Omega=\mathbb{R}^n$ or $\overline{\Omega}=\overline{\mathbb{R}}^n_+=\{x\in\mathbb{R}^n : x_n\geq 0\}$. There exists $M_0>0$ such that*

$$\|Du\|_{C^0(\overline{\Omega})} \leq M_0\{\|u\|_{C^0(\overline{\Omega})} + \|D^2u\|_{C^0(\overline{\Omega})}\} \quad \forall u\in C^2(\overline{\Omega}).$$

Proof. By considering separately Re u and Im u, and replacing M_0 by $2M_0$, we can assume that u is real-valued. Suppose first $\Omega=\mathbb{R}^n$ or $\overline{\Omega}=\overline{\mathbb{R}}^n_+$, and fix $x_0\in\overline{\Omega}$ such that $Du(x_0)\neq 0$ (if $Du\equiv 0$ in $\overline{\Omega}$ the result is obvious). By Taylor's formula we have, denoting by $(\cdot|\cdot)_n$ the scalar product in $\mathbb{R}^n$ or in $\mathbb{C}^n$:

$$u(x)=u(x_0)+(Du(x_0)|x-x_0)_n+\frac{1}{2}(D^2u(\xi)\cdot(x-x_0)|x-x_0)_n \quad \forall x\in\overline{\Omega}$$

where ξ is a suitable point in the segment joining x and x_0. We can assume (possibly replacing u by $-u$) that $x_0+t\,\frac{Du(x_0)}{|Du(x_0)|}\in\overline{\Omega}$ $\forall t\geq 0$. Hence if we choose $x=x_0+t\frac{Du(x_0)}{|Du(x_0)|}$, we easily get

$$|Du(x_0)| \leq \frac{2}{t}\|u\|_\infty + \frac{t}{2}\|D^2u\|_\infty \qquad \forall t>0.$$

The expression on the right-hand side attains its minimum value when $t=2\|D^2u\|_\infty^{-1/2}\|u\|_\infty^{1/2}$. Hence

$$|Du(x_0)| \leq 4\,\|u\|_\infty^{1/2}\,\|D^2u\|_\infty^{1/2} \qquad \forall x_0\in\overline{\Omega}$$

and the result follows with $M_0=2$.

Suppose now Ω bounded with $\partial\Omega$ of class C^1 and fix $x_0\in\overline{\Omega}$ with $Du(x_0)\neq 0$. We can select a finite number of open balls W_j, $1\leq j\leq k$, centered in points of $\partial\Omega$, with the following properties:

(a) $\partial\Omega \subseteq \bigcup_{j=1}^{k} V_j$, where $V_j = W_j\cap\overline{\Omega}$,

(b) for $j=1\ldots k$ there exists a diffeomorphism $\psi_j:V_j\to B(0,1)=\{x\in\mathbb{R}^n : |x|<1\}$ of class C^1, such that

$$\psi_j(V_j)=B^+(0,1)=\{x\in B(0,1) : x_n\geq 0\},$$

$$\psi_j(V_j\cap\partial\Omega)=\Sigma(0,1)=\{x\in B(0,1) : x_n=0\}.$$

Define now, for $\varepsilon>0$,

$$\Omega_\varepsilon=\{x\in\Omega : \mathrm{dist}(x,\partial\Omega)>\varepsilon\}, \quad A_\varepsilon=\{x-\sigma\nu(x) : x\in\partial\Omega, \sigma\in[0,\varepsilon]\},$$

where $\nu(x)$ is the unit exterior normal vector at $x\in\partial\Omega$. Obviously in general A_ε is not contained in $\overline{\Omega}$, but this is true for small ε: in this case $A_\varepsilon=\overline{\Omega}-\Omega_\varepsilon$. Hence we can choose $\varepsilon>0$ such that

$$A_{2\varepsilon} = \overline{\Omega} - \Omega_{2\varepsilon} \subseteq \bigcup_{j=1}^{k} V_j \tag{1.6}$$

Now, if $x_0\in\Omega_\varepsilon$, set $x=x_0+\varepsilon\,\frac{Du(x_0)}{|Du(x_0)|}$; by Taylor's formula, as before, we get

$$|Du(x_0)| \leq \frac{2}{\varepsilon}\|u\|_\infty + \frac{\varepsilon}{2}\|D^2u\|_\infty \qquad \forall x_0\in\Omega_\varepsilon. \tag{1.7}$$

On the other hand, suppose $x_0\in\overline{\Omega}-\Omega_\varepsilon$. Take a point $z_0\in\partial\Omega$ such that $|x_0-z_0|=\mathrm{dist}\,(x_0,\partial\Omega)$, then $\nu(z_0)=\frac{z_0-x_0}{|z_0-x_0|}$ and we can assume (possibly replacing u by $-u$) that $(Du(x_0)|\nu(z_0))_n\leq 0$.

Define

$$g = \frac{\dfrac{Du(x_0)}{|Du(x_0)|} - \nu(z_0)}{\left|\dfrac{Du(x_0)}{|Du(x_0)|} - \nu(z_0)\right|};$$

we have:

$$-(q|\nu(z_0))_n = (q\,|\,\frac{Du(x_0)}{|Du(x_0)|})_n \geq \frac{1}{\sqrt{2}}. \qquad (1.8)$$

Set $x=x_0+\varepsilon q$; then the segment joining x_0 and x lies in $\overline{\Omega}$ (indeed, for each $\sigma\in]0,\varepsilon]$ the point $x_0+\sigma q$ belongs to the ball with center $x_0-\sigma\nu(z_0)$ and radius σ). Hence by Taylor's formula and (1.8) we easily get

$$\frac{\varepsilon}{\sqrt{2}}\,|Du(x_0)| \leq 2\|u\|_\infty + \frac{\varepsilon^2}{2}\|D^2u\|_\infty \qquad \forall x_0\in\overline{\Omega}-\Omega_\varepsilon,$$

which, together with (1.7), yields the result. □

LEMMA 1.11. Let Y be a Banach space and let $\lambda\in]0,1[$. For each $u\in C^0([0,1],Y)$ such that $u'\in C_\lambda(]0,1],Y)$ we have:

$$\|u\|_{C^0([0,1],Y)} \leq \|u(1)\|_Y + \frac{1}{1-\lambda}\|u'\|_{C_\lambda(]0,1],Y)}.$$

Proof. We have for each $t\in]0,1]$

$$\|u(t)\|_Y \leq \|u(1)\|_Y + \int_t^1 \|u'(s)\|_Y ds \leq \|u(1)\|_Y + \int_t^1 \frac{ds}{s^\lambda}\,\|u'\|_{C_\lambda(]0,1],Y)}. \;□$$

Let us define now some suitable subspaces of the spaces C^k, $C^{k,\alpha}$, $h^{k,\alpha}$, $C^{*,1}$, $h^{*,1}$ which we introduced in Definitions 1.1, 1.2 and 1.3.

DEFINITION 1.12. Denote by X any of the symbols C^k, $C^{k,\alpha}$, $h^{k,\alpha}$, $C^{*,1}$, $h^{*,1}$ with $\alpha\in]0,1]$ and $k=0,1,2$, and set

$$X_0(\overline{\Omega}) = \{f\in X(\overline{\Omega}):\ f(x)=0\quad \forall x\in\partial\Omega\}. \quad □$$

DEFINITION 1.13. Suppose $\partial\Omega$ is of class C^1 and consider the boundary differential operator

$$Bu(x) = \alpha(x)u(x)+(\beta(x)\,|\,Du(x))_n, \quad x\in\partial\Omega,$$

where $\alpha\in C^0(\partial\Omega,\mathbb{C})$, $\beta\in C^0(\partial\Omega,\mathbb{C}^n)$. If X is any of the symbols C^k, $C^{k,\alpha}$, $h^{k,\alpha}$ with $\alpha\in]0,1]$ and $k=1,2$, set

$$X_B(\overline{\Omega})=\{f\in X(\overline{\Omega}):Bf=0\}. \quad □$$

DEFINITION 1.14. Suppose $\partial\Omega$ is of class C^1, and denote by $\nu(x)$ the unit exterior normal vector at $x\in\partial\Omega$; then $\nu\in C^0(\partial\Omega,\mathbb{R}^n)$. Let B be as in Definition 1.13, and suppose in addition that $\alpha\in C^0(\partial\Omega,\mathbb{R})$, $\beta\in C^0(\partial\Omega,\mathbb{R}^n)$ and $(\beta(x)|\nu(x))_n>0$ $\forall x\in\partial\Omega$. Define

$$[f]_{1,\beta} = \sup\{\frac{|f(x-\sigma\beta(x))-f(x)|}{\sigma}: x\in\partial\Omega, \sigma>0, x-\sigma\beta(x)\in\overline{\Omega}\} \qquad (1.9)$$

Set

(i) $C_\beta^{*,1}(\overline{\Omega})=\{f\in C^{*,1}(\overline{\Omega}):[f]_{1,\beta}<\infty\}$,

$$\|f\|_{C_\beta^{*,1}(\overline{\Omega})} = \|f\|_{C^{*,1}(\overline{\Omega})} + [f]_{1,\beta};$$

(ii) $h_B^{*,1}(\overline{\Omega})=\{f\in C_\beta^{*,1}(\overline{\Omega})\cap h^{*,1}(\overline{\Omega}):\ \lim_{\sigma\to 0^+}\frac{f(x-\sigma\beta(x))-f(x)}{\sigma} = \alpha(x)f(x)\quad \forall x\in\partial\Omega\}$. □

Obviously $h_B^{*,1}(\overline{\Omega})$ is a closed subspace of $C_\beta^{*,1}(\overline{\Omega})$; moreover we have

$$C^{0,1}(\overline{\Omega}) \hookrightarrow C_\beta^{*,1}(\overline{\Omega}) \hookrightarrow C^{*,1}(\overline{\Omega})$$

$$C_B^1(\overline{\Omega}) \hookrightarrow h_B^{*,1}(\overline{\Omega}) \hookrightarrow h^{*,1}(\overline{\Omega})\cap C_\beta^{*,1}(\overline{\Omega}).$$

Concerning Definition 1.14, the following lemma is useful:

LEMMA 1.15. Let Ω be bounded, with $\partial\Omega$ of class C^1, let $\beta\in C^0(\partial\Omega,\mathbb{R}^n)$ satisfy $(\beta(x)|\nu(x))_n>0$ $\forall x\in\partial\Omega$. Then there exists $\overline{\sigma}>0$ such that

$$x-\sigma\beta(x)\in\overline{\Omega}\quad \forall\sigma\in[0,\overline{\sigma}],\ \forall x\in\partial\Omega.$$

Consequently in the semi-norm (1.9) we can take $\sigma\in]0,\sigma_0]$, where σ_0 is any sufficiently small fixed positive number (independent of $x\in\partial\Omega$).

Proof. As $\partial\Omega$ is of class C^1 and compact, we have $(\beta(x)|\nu(x))_n \geq \delta_0>0$ $\forall x\in\partial\Omega$; in addition $\partial\Omega$ can be covered by a finite number of open balls W_j, $1\leq j\leq k$, with the following property: there exist functions $g_j\in C^1(\overline{W}_j,\mathbb{R})$ with $|Dg_j(x)|\geq\delta_j>0$ $\forall x\in\overline{W}_j$, such that

$$W_j\cap\Omega=\{x\in W_j: g_j(x)>0\},\quad W_j\cap\partial\Omega=\{x\in W_j:g_j(x)=0\}.$$

We can also suppose that the covering is minimal, i.e. $\bigcup_{j=1}^{k} W_j-W_s$ does not contain $\partial\Omega$ for each $s=1\ldots k$. Choose $\bar\sigma$ so small that:

(a) $x\in\partial\Omega\cap[W_s-\bigcup_{j\neq s}W_j] \Rightarrow \{x-\sigma\beta(x):\sigma\in[0,\bar\sigma]\}\subseteq W_s$,

(b) $x\in\partial\Omega\cap W_s\cap W_j \Rightarrow \{x-\sigma\beta(x):\sigma\in[0,\bar\sigma]\}\subseteq W_s$ or $\subseteq W_j$.

This is clearly possible. Now take $x\in\partial\Omega$: then, by (a) and (b), for some $j=1\ldots k$ we have $x\in W_j$ and also $x-\sigma\beta(x)\in W_j$ $\forall\sigma\in[0,\bar\sigma]$. In addition, by Taylor's formula,

$$g_j(x-\sigma\beta(x)) = g_j(x)-\sigma(Dg_j(x)|\beta(x))_n+o(\sigma)\quad \text{as } \sigma\to0^+,$$

and, since $\nu(x) = -\dfrac{Dg_j(x)}{|Dg_j(x)|}$ and $g_j(x)=0$, we get

$$g_j(x-\sigma\beta(x))\geq\sigma|Dg_j(x)|(\nu(x)|\beta(x))_n+o(\sigma)\geq\sigma\delta_j\delta_0+o(\sigma)\quad \text{as } \sigma\to0^+.$$

Hence, possibly replacing $\bar\sigma$ by a smaller number, we get

$$g_j(x-\sigma\beta(x))>0\quad \forall\sigma\in]0,\bar\sigma],$$

which implies $x-\sigma\beta(x)\in W_j\cap\Omega\subseteq\Omega\ \forall\sigma\in]0,\bar\sigma]$. □

We will need later another geometric property of Ω, which we

express in the following lemma.

LEMMA 1.16. Suppose Ω is bounded with $\partial\Omega$ of class C^1. There exist $\sigma_1>0$, $M_1\geq1$ satisfying the following property: if $x,y\in\overline{\Omega}$ and $|x-y|\leq\sigma_1$ there exists a continuously differentiable path $\Gamma:[0,1]\to\overline{\Omega}$ such that

$$\Gamma(0)=x,\quad \Gamma(1)=y,\quad \ell(\Gamma)\leq M_1|x-y|,$$

where $\ell(\Gamma)=\int_0^1|\Gamma'(t)|dt$ is the length of Γ.

Proof. As in the proofs of Lemmata 1.10 and 1.15 we have $\partial\Omega\subseteq\bigcup_{j=1}^{k}V_j$, where $V_j=W_j\cap\overline{\Omega}$ and W_j is an open ball, centered in a point of $\partial\Omega$, having the following property: there exists a diffeomorphism $\psi_j:\overline{W}_j\to\overline{B(0,1)}=\{x\in\mathbb{R}^n:|x|\leq1\}$ of class C^1, such that $|D\psi_j|\neq0$ in $\overline{W}_j$, $\psi_j^{-1}\in C^1(\overline{B(0,1)})$ and $\psi_j(V_j)=B^+(0,1)$, $\psi_j(V_j\cap\partial\Omega)=\Sigma(0,1)$.

We can also suppose that $\bigcup_{j\neq s}V_j$ does not contain $\partial\Omega$ for each $s=1\ldots k$. Define again, for $\varepsilon>0$, $\Omega_\varepsilon=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\varepsilon\}$, $A_\varepsilon=\{x-\sigma\nu(x):\ x\in\partial\Omega,\sigma\in[0,\varepsilon]\}$, and take $\varepsilon>0$ such that (1.6) holds. Next, choose $\sigma_1\in]0,\varepsilon[$ small enough, in order that the following properties are satisfied:

(a) $x\in[V_s-\bigcup_{j\neq s}V_j]\cap[\overline{\Omega}-\Omega_{2\varepsilon}] \Rightarrow \overline{\Omega(x,\sigma_1)}\subseteq V_s$,

(b) $x\in V_s\cap V_j\cap[\overline{\Omega}-\Omega_{2\varepsilon}] \Rightarrow \overline{\Omega(x,\sigma_1)}\subseteq V_s$ or $\subseteq V_j$,

where $\Omega(x,\sigma_1)=\{y\in\Omega:|y-x|<\sigma_1\}$. Clearly this choice is possible. Take now $x,y\in\overline{\Omega}$ with $|x-y|\leq\sigma_1$; three cases can occur:

Case 1: $x,y\in\Omega_\varepsilon$. Then the ball $B(x,\varepsilon)$ is contained in Ω, and $y\in B(x,\sigma_1)\subseteq B(x,\varepsilon)$. Thus we can take as Γ the segment joining x and y.

Case 2: $x,y\in\overline{\Omega}-\Omega_\varepsilon$. By (a) and (b) we deduce that there exists s, $1\le s\le k$, such that $x,y\in V_s$; hence $\psi_s(x)$, $\psi_s(y)\in B^+(0,1)$. Set $\gamma(t)=t\psi_s(x)+(1-t)\psi_s(y)$; then the segment γ lies in $B^+(0,1)$. We take $\psi_s^{-1}\circ\gamma$ as Γ, obtaining

$$\ell(\Gamma) = \int_0^1 |(\psi_s^{-1}\circ\gamma)'(t)|dt= \int_0^1 |(D\psi_s^{-1})\circ\gamma(t)\cdot\gamma'(t)|dt \le$$
$$\le \|D\psi_s^{-1}\|_\infty\cdot\ell(\gamma)=\|D\psi_s^{-1}\|_\infty|\psi_s(x)-\psi_s(y)|\le\|D\psi_s^{-1}\|_\infty\|D\psi_s\|_\infty|x-y|. \tag{1.10}$$

Case 3: $x\in\overline{\Omega}-\Omega_\varepsilon$, $y\in\Omega_\varepsilon$. As $\sigma_1<\varepsilon$, we have $x,y\in\overline{\Omega}-\Omega_{2\varepsilon}$ and, by (1.6), the argument of Case 2 is applicable, so that (1.10) holds. Hence the lemma is proved with

$$M_1 = \max_{1\le s\le k} \|D\psi_s^{-1}\|_\infty\|D\psi_s\|_\infty. \quad \square$$

REMARK 1.17. Lemma 1.16 is obvious when Ω is convex; however in the non-convex case this lemma will be useful is Section 5 in dealing with the spaces $C^{*,1}(\overline{\Omega})$, $h^{*,1}(\overline{\Omega})$. □

Now we need to study some properties of the spaces $C^{*,1}(\overline{\Omega})$, $h^{*,1}(\overline{\Omega})$ in more detail.

LEMMA 1.18. If $f,g\in C^{*,1}(\overline{\Omega})$ (resp. $h^{*,1}(\overline{\Omega})$) then $f\cdot g\in C^{*,1}(\overline{\Omega})$ (resp. $h^{*,1}(\overline{\Omega})$) and

$$[fg]_{*,1} \le [f]_{*,1}\|g\|_\infty + \|f\|_\infty[g]_{*,1} + [f]_{1/2}[g]_{1/2}.$$

Proof. Let $x,y\in\overline{\Omega}$ be such that $\frac{x+y}{2}\in\overline{\Omega}$. Then

$$f(x)g(x)+f(y)g(y)-2\,f(\tfrac{x+y}{2})g(\tfrac{x+y}{2})=$$
$$=[f(x)+f(y)-2f(\tfrac{x+y}{2})]g(x)+f(x)[g(x)+g(y)-2g(\tfrac{y+x}{2})]-$$
$$-2[f(x)-f(\tfrac{x+y}{2})][g(y)-g(\tfrac{x+y}{2})]$$

and taking into account Remark 1.4 (ii), the result follows. □

LEMMA 1.19. Suppose Ω is bounded with $\partial\Omega$ of class C^1 (or, alternatively, suppose Ω is bounded and convex). Let $f\in C^{*,1}(\overline{\Omega})$ (resp. $h^{*,1}(\overline{\Omega})$) and $g\in C^{*,\alpha}(\overline{\Lambda},\mathbb{R}^n)$ (resp. $h^{*,\alpha}(\overline{\Lambda},\mathbb{R}^n)$) where $\overline{\Lambda}=\mathbb{R}^n$ or $\overline{\Lambda}=\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x_n\ge 0\}$, and suppose that $g(\overline{\Lambda})\subseteq\overline{\Omega}$. If $\alpha\in]1,2]$, then $f\circ g\in C^{*,1}(\overline{\Lambda})$ (resp. $h^{*,1}(\overline{\Lambda})$) and there exists $C>0$ such that

$$[f\circ g]_{*,1,\overline{\Lambda}} \le C\{\|f\|_{C^{*,1}(\overline{\Omega})}[g]_{1,\overline{\Lambda}}+[f]_{1/\alpha,\overline{\Omega}}[g]^{1/\alpha}_{*,\alpha,\overline{\Lambda}}\}.$$

Proof. Suppose that Ω is convex. If $x,y\in\overline{\Lambda}$ then $\frac{1}{2}[g(x)+g(y)]\in\overline{\Omega}$; hence we can write

$$[f\circ g(x)+f\circ g(y)-2f\circ g(\tfrac{x+y}{2})]=[f(g(x))+f(g(y))-2f(\tfrac{g(x)+g(y)}{2})]+$$
$$+2[f(\tfrac{g(x)+g(y)}{2})-f(g(\tfrac{x+y}{2}))]$$

and by Remark 1.4 (ii)

$$|f\circ g(x)+f\circ g(y)-2f\circ g(\tfrac{x+y}{2})|\le[f]_{*,1}|g(x)-g(y)|+$$
$$+2[f]_{1/\alpha}|\tfrac{1}{2}[g(x)+g(y)-2g(\tfrac{x+y}{2})]|^{1/\alpha}\le \tag{1.11}$$
$$\le\{[f]_{*,1}[g]_1 + 2^{1-1/\alpha}[f]_{1/\alpha}[g]^{1/\alpha}_{*,\alpha}\}|x-y|,$$

which implies the result.

Suppose now that Ω is not necessarily convex, but $\partial\Omega$ is of class C^1. Then we have, as in the proofs of Lemmata 1.10 and 1.16, $\partial\Omega\subseteq\bigcup_{j=1}^k V_j$, where $V_j=W_j\cap\overline{\Omega}$ and W_j is an open ball centered in a point of $\partial\Omega$ and such that there exists a diffeomorphism $\psi_j:\overline{W}_j\to\overline{B(0,1)}$ which satisfies $\psi_j(V_j)=B^+(0,1)$, $\psi_j(V_j\cap\partial\Omega)=\Sigma(0,1)$. Let $\varepsilon>0$ be such that (1.6) holds. By Remark 1.4 (iii) we can consider only points $x,y\in\overline{\Lambda}$ such that $|x-y|\le\frac{1}{2}\varepsilon[g]_1^{-1}$; this obviously implies $|g(x)-g(y)|\le\varepsilon/2$.

Thus, let $x,y\in\overline{\Lambda}$ with $|x-y|\le\frac{1}{2}\varepsilon[g]_1^{-1}$. If $\frac{1}{2}[g(x)+g(y)]\in\overline{\Omega}$, we can apply the above argument, and (1.11) still holds: this is

certainly the case if either g(x) or g(y) belongs to $\Omega_{\varepsilon/2}$.
Hence we can reduce ourselves to the following situation:

$$|g(x)-g(y)|\le\varepsilon/2;\qquad g(x),g(y),g(\tfrac{x+y}{2})\in\bar\Omega;\qquad \tfrac12[g(x)+g(y)]\notin\bar\Omega. \tag{1.12}$$

Set $\sigma=\max\{|g(x)-g(y)|,\ |g(x)-g(\frac{x+y}{2})|,\ |g(y)-g(\frac{x+y}{2})|\}$;

obviously

$$\sigma\le[g]_1\cdot|x-y|\le\varepsilon/2 \tag{1.13}$$

We want now to find a point $w\in\Omega$ satisfying

$$\tfrac12[w+g(x)],\ \tfrac12[w+g(y)],\ \tfrac12[w+g(\tfrac{x+y}{2})],\ \tfrac12\{w+\tfrac12[g(x)+g(y)]\}\in\Omega \tag{1.14}$$

and

$$|g(x)-w|\le3\sigma,\quad |g(y)-w|\le3\sigma,\quad |g(\tfrac{x+y}{2})-w|\le\tfrac72\sigma \tag{1.15}$$

Suppose that this has been done: then by (1.14) we can write

$$\begin{aligned}&[f\circ g(x)+f\circ g(y)-2f\circ g(\tfrac{x+y}{2})]=[f(g(x))+f(w)-2f(\tfrac12(w+g(x)))]+\\&+[f(g(y))+f(w)-2f(\tfrac12(w+g(y)))]+2[f(\tfrac12(w+g(x)))+f(\tfrac12(w+g(y)))-\\&-2f(\tfrac12[w+\tfrac12(g(x)+g(y))])]-2[f(w)+f(g(\tfrac{x+y}{2}))-\\&-2f(\tfrac12(w+g(\tfrac{x+y}{2})))]+4[f(\tfrac12[w+\tfrac12(g(x)+g(y))])-\\&-f(\tfrac12(w+g(\tfrac{x+y}{2})))]\end{aligned}$$

and hence, by (1.15) and (1.13)

$$\begin{aligned}&|f\circ g(x)+f\circ g(y)-2f\circ g(\tfrac{x+y}{2})|\le[f]_{*,1}\{|g(x)-w|+|g(y)-w|+|g(x)-g(y)|+\\&+|g(\tfrac{x+y}{2})-w|\}+4^{1-1/\alpha}\,[f]_{1/\alpha}|g(x)+g(y)-2g(\tfrac{x+y}{2})|^{1/\alpha}\le\\&\le\tfrac{21}{2}[f]_{*,1}[g]_1|x-y|+4^{1-1/\alpha}[f]_{1/\alpha}[g]_{*,\alpha}^{1/\alpha}|x-y|.\end{aligned}$$

This implies the result, provided (1.14) and (1.15) hold.
In order to find the point $w\in\Omega$ satisfying (1.14) and (1.15), we

start with observing that, by the definition of σ, we have $\mathrm{dist}(\frac12[g(x)+g(y)],\partial\Omega)\le\sigma/2$. Choose a point $z\in\partial\Omega$ realizing such distance: then

$$0<|\tfrac12[g(x)+g(y)]-z|\le\sigma/2 \tag{1.16}$$

and the unit exterior normal vector at z is

$$\nu(z)=\frac{\frac12[g(x)+g(y)]-z}{|\frac12[g(x)+g(y)]-z|}.$$

Define now $w=z-2\sigma\nu(z)$: by (1.6) and (1.13) $w\in\Omega$ and

$$\mathrm{dist}(w,\partial\Omega)=2\sigma; \tag{1.17}$$

in particular, $\frac12[w+\frac12(g(x)+g(y))]\in\Omega$ because it lies in the segment joining w and z.
By (1.17), we see that (1.14) will follow if we show that the points $\frac12[w+g(x)]$, $\frac12[w+g(y)]$, $\frac12[w+g(\frac{x+y}{2})]$ lie in a ball centered in w with radius less than 2σ; as we will see, this will also imply (1.15). Indeed, we have by (1.16)

$$\begin{aligned}&|\tfrac12[w+g(x)]-w|=\tfrac12|g(x)-w|\le\tfrac12|g(x)-z|+\tfrac12|2\sigma\nu(z)|\le\\&\le\tfrac12|g(x)-\tfrac12[g(x)+g(y)]|+\tfrac12|\tfrac12[g(x)+g(y)]-z|+\\&+\sigma\le\tfrac32\sigma\end{aligned}$$

and in particular $|g(x)-w|\le3\sigma$; similarly

$$|\tfrac12[w+g(y)]-w|\le\tfrac32\sigma,\qquad |g(y)-w|\le3\sigma,$$

and finally

$$\begin{aligned}&|\tfrac12[w+g(\tfrac{x+y}{2})]-w|=\tfrac12|g(\tfrac{x+y}{2})-w|\le\tfrac12|g(\tfrac{x+y}{2})-\tfrac12[g(x)+g(y)]|+\\&+\tfrac12|\tfrac12[g(x)+g(y)]-z|+\sigma\le\tfrac14|g(\tfrac{x+y}{2})-g(x)|+\\&+\tfrac14|g(\tfrac{x+y}{2})-g(y)|+\tfrac54\sigma\le\tfrac74\sigma,\end{aligned}$$

which also implies $|g(\frac{x+y}{2})-w|\le \frac{7}{2}\sigma$. Thus, (1.14) and (1.15) are proved and this concludes the proof. □

We finish this section with a version of the well-known Sobolev's Theorem. Set

$L^p(\Omega)=\{f:\Omega\to\mathbb{C}:$ f is Lebesgue measurable and

$\|f\|_{L^p(\Omega)}=[\int_\Omega|f(x)|^p dx]^{1/p}<\infty\}$,

$H^{1,p}(\Omega)=\{f\in L^p(\Omega): \frac{\partial f}{\partial x_i}\in L^p(\Omega),\ i=1,\dots,n\}$,

where $p\in[1,\infty[$. Similarly one defines the spaces $L^p(\Omega,\mathbb{R}^m)$, $H^{1,p}(\Omega,\mathbb{R}^m)$ and $L^p(\Omega,\mathbb{C}^m)$, $H^{1,p}(\Omega,\mathbb{C}^m)$. The result is the following:

LEMMA 1.20. *Suppose $\partial\Omega$ is of class C^1 and set $\Omega(x_0,r)=\{x\in\Omega: |x-x_0|<r\}$, where $x_0\in\overline{\Omega}$ and $r>0$. Suppose $q\ge p>n$ and $\alpha=1-n/p$: then $H^{1,p}(\Omega)\hookrightarrow h^{0,\alpha}(\overline{\Omega})$; moreover there exist $C_1,C_2>0$ such that for each $x_0\in\overline{\Omega}$, $r>0$, and $u\in H^{1,p}(\Omega)$*

$$[u]_{\alpha,\overline{\Omega(x_0,r)}}\le C_1\|Du\|_{L^p(\Omega(x_0,r))}\le C_2\, r^{n/p-n/q}\|Du\|_{L^q(\Omega(x_0,r))}.$$

Proof. The first inequality is Sobolev-Morrey's inequality (see e.g. Adams [4], Lemma 5.17), the second one follows by Hölder's inequality. □

Finally we remark that in the forthcoming sections any number C appearing in the estimates will denote a constant which is independent of the estimated quantities.

2. THE CASE OF A HALF-SPACE WITH A NORMAL BOUNDARY CONDITION

Let us consider the closed half-space

$\mathbb{R}^n_+=\mathbb{R}^{n-1}\times[0,\infty[=\{(x,y):x\in\mathbb{R}^{n-1},\ y\ge0\}$;

obviously the unit exterior normal vector at each point of the boundary $\Sigma=\mathbb{R}^{n-1}\times\{0\}$ is $\nu=-e^n$. Let $\alpha\in C^2(\mathbb{R}^{n-1},\mathbb{R})$ be a real non-negative function; define the boundary operator

$$Nu(x)=\alpha(x)u(x,0)+(\nu|Du(x,0))_n=\alpha(x)u(x,0)-\frac{\partial u}{\partial y}(x,0),\quad x\in\mathbb{R}^{n-1}\qquad(2.1)$$

and consider the spaces (see Definitions 1.13 and 1.14)

(a) $C^k_N(\mathbb{R}^n_+)=\{f\in C^k(\mathbb{R}^n_+):Nf=0\}$, $k=1,2$;

(b) $C^{1,\alpha}_N(\mathbb{R}^n_+)=\{f\in C^{1,\alpha}(\mathbb{R}^n_+):Nf=0\}$,

$h^{1,\alpha}_N(\mathbb{R}^n_+)=\{f\in h^{1,\alpha}(\mathbb{R}^n_+):Nf=0\}$, $\alpha\in]0,1[$;

(c) $C^{*,1}_\nu(\mathbb{R}^n_+)=\{f\in C^{*,1}(\mathbb{R}^n_+):[f]_{1,\nu}=\sup\{\frac{|f(x,y)-f(x,0)|}{y}:y>0,x\in\mathbb{R}^{n-1}\}<\infty\}$,

$h^{*,1}_N(\mathbb{R}^n_+)=\{f\in C^{*,1}_\nu(\mathbb{R}^n_+)\cap h^{*,1}(\mathbb{R}^n_+):\lim_{y\to0^+}\frac{f(x,y)-f(x,0)}{y}=\alpha(x)f(x,0)$

$\forall x\in\mathbb{R}^{n-1}\}$.

These spaces are complete with respect to the norms

(a) $\|f\|_{C^k(\mathbb{R}^n_+)}$; (b) $\|f\|_{C^{1,\alpha}(\mathbb{R}^n_+)}$; (c) $\|f\|_{C^{*,1}(\mathbb{R}^n_+)}+[f]_{1,\nu}$.

Our goal is the following theorem:

THEOREM 2.1. *Let $\alpha\in C^2(\mathbb{R}^{n-1},\mathbb{R})$ with $\alpha\ge0$ and let N be the operator defined in (2.1). The following continuous inclusions hold:*

$$(C^2_N(\mathbb{R}^n_+),\ C^0(\mathbb{R}^n_+))_{1-\theta,\infty}\hookrightarrow\begin{cases}C^{0,2\theta}(\mathbb{R}^n_+) & \text{if } \theta\in]0,1/2[,\\ C^{*,1}_\nu(\mathbb{R}^n_+) & \text{if } \theta=1/2,\\ C^{1,2\theta-1}_N(\mathbb{R}^n_+) & \text{if } \theta\in]1/2,1[;\end{cases}$$

which also implies $|g(\frac{x+y}{2})-w|\leq \frac{7}{2}\sigma$. Thus, (1.14) and (1.15) are proved and this concludes the proof. □

We finish this section with a version of the well-known Sobolev's Theorem. Set

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where $p\in[1,\infty[$. Similarly one defines the spaces $L^p(\Omega,\mathbb{R}^m)$, $H^{1,p}(\Omega,\mathbb{R}^m)$ and $L^p(\Omega,\mathbb{C}^m)$, $H^{1,p}(\Omega,\mathbb{C}^m)$. The result is the following:

<u>LEMMA</u> 1.20. <u>Suppose $\partial\Omega$ is of class C^1 and set</u> $\Omega(x_0,r)=\{x\in\Omega:|x-x_0|<r\}$, <u>where</u> $x_0\in\bar\Omega$ <u>and</u> $r>0$. <u>Suppose</u> $q\geq p>n$ <u>and</u> $\alpha=1-n/p$: <u>then</u> $H^{1,p}(\Omega)\hookrightarrow h^{0,\alpha}(\bar\Omega)$; <u>moreover there exist</u> $C_1,C_2>0$ <u>such that</u> for each $x_0\in\bar\Omega$, $r>0$, <u>and</u> $u\in H^{1,p}(\Omega)$

$$[u]_{\alpha,\overline{\Omega(x_0,r)}}\leq C_1\|Du\|_{L^p(\Omega(x_0,r))}\leq C_2\, r^{n/p-n/q}\|Du\|_{L^q(\Omega(x_0,r))}.$$

<u>Proof</u>. The first inequality is Sobolev-Morrey's inequality (see e.g. Adams [4], Lemma 5.17), the second one follows by Hölder's inequality. □

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$$Nu(x)=\alpha(x)u(x,0)+(\nu|Du(x,0))_n=\alpha(x)u(x,0)-\frac{\partial u}{\partial y}(x,0),\quad x\in\mathbb{R}^{n-1} \tag{2.1}$$

and consider the spaces (see Definitions 1.13 and 1.14)

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$h^{1,\alpha}_N(\mathbb{R}^n_+)=\{f\in h^{1,\alpha}(\mathbb{R}^n_+):Nf=0\}$, $\quad \alpha\in]0,1]$;

(c) $C^{*,1}_\nu(\mathbb{R}^n_+)=\{f\in C^{*,1}(\mathbb{R}^n_+):[f]_{1,\nu}=\sup\{\frac{|f(x,y)-f(x,0)|}{y}:y>0,x\in\mathbb{R}^{n-1}\}<\infty\}$,

$h^{*,1}_N(\mathbb{R}^n_+)=\{f\in C^{*,1}_\nu(\mathbb{R}^n_+)\cap h^{*,1}(\mathbb{R}^n_+):\lim_{y\to0^+}\frac{f(x,y)-f(x,0)}{y}=\alpha(x)f(x,0)$ $\forall x\in\mathbb{R}^{n-1}\}$.

These spaces are complete with respect to the norms

(a) $\|f\|_{C^k(\mathbb{R}^n_+)}$; (b) $\|f\|_{C^{1,\alpha}(\mathbb{R}^n_+)}$; (c) $\|f\|_{C^{*,1}(\mathbb{R}^n_+)}+[f]_{1,\nu}$.

Our goal is the following theorem:

<u>THEOREM</u> 2.1. <u>Let</u> $\alpha\in C^2(\mathbb{R}^{n-1},\mathbb{R})$ <u>with</u> $\alpha\geq0$ <u>and let</u> N <u>be the operator defined in (2.1). The following continuous inclusions hold:</u>

$$(C^2_N(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta,\infty}\hookrightarrow\begin{cases}C^{0,2\theta}(\mathbb{R}^n_+) & \text{if } \theta\in]0,1/2[,\\ C^{*,1}_\nu(\mathbb{R}^n_+) & \text{if } \theta=1/2,\\ C^{1,2\theta-1}_N(\mathbb{R}^n_+) & \text{if } \theta\in]1/2,1[;\end{cases}$$

$$(C_N^2(\mathbb{R}^n_+),\ C^0(\mathbb{R}^n_+))_{1-\theta} \hookrightarrow \begin{cases} h^{0,2\theta}(\mathbb{R}^n_+) & \underline{\text{if}}\ \theta\in]0,1/2[\\ h_N^{*,1}(\mathbb{R}^n_+) & \underline{\text{if}}\ \theta=1/2 \\ h_N^{1,2\theta-1}(\mathbb{R}^n_+) & \underline{\text{if}}\ \theta\in]1/2,1[\ . \end{cases}$$

Proof. By Definition 1.7, we have to show that if

$$f \in \begin{cases} C^{0,2\theta}(\mathbb{R}^n_+)\ (\text{resp. } h^{0,2\theta}(\mathbb{R}^n_+)) & \text{if } \theta\in]0,1/2[\\ C_\nu^{*,1}(\mathbb{R}^n_+)\ \ (\text{resp. } h_N^{*,1}(\mathbb{R}^n_+)) & \text{if } \theta=1/2 \\ C_N^{1,2\theta-1}(\mathbb{R}^n_+)\ (\text{resp. } h_N^{1,2\theta-1}(\mathbb{R}^n_+)) & \text{if } \theta\in]1/2;1[\end{cases}$$

then there exists a function $u(t,x,y)$, defined in $[0,1]\times\mathbb{R}^n_+$, with the following properties:

$$\begin{cases} u\in C^0([0,1]\times\mathbb{R}^n_+)),\ u(0,\cdot,\cdot)=f; \\ u\in C_{1-\theta}(]0,1],C^2(\mathbb{R}^n_+))\ (\text{resp. } u\in C_{1-\theta}([0,1],C^2(\mathbb{R}^n_+))) \\ u_t\in C_{1-\theta}(]0,1],C^0(\mathbb{R}^n_+))\ (\text{resp. } u_t\in C_{1-\theta}([0,1],C^0(\mathbb{R}^n_+))) \\ [Nu(t,\cdot,\cdot)](x)=0\quad \forall x\in\mathbb{R}^{n-1},\ \forall t\in]0,1]. \end{cases}$$

Let us start with the following remark: by Lemma 1.10, we have

$$\|t^{1-\theta}Du(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n_+)} \le C\{\|t^{1-\theta}u(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n_+)} + \|t^{1-\theta}D^2u(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n_+)}\},$$

and by Lemma 1.11

$$\sup_{t\in]0,1]} \|t^{1-\theta}u(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n_+)} \le \|u(1,\cdot,\cdot)\|_{C^0(\mathbb{R}^n_+)} + \frac{1}{\theta}\sup_{t\in]0,1]}\|t^{1-\theta}u_t(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n_+)}\ ;$$

hence it will be sufficient to prove that

$$\begin{cases} u\in C^0([0,1]\times\mathbb{R}^n_+),\quad u(0,x,y)=f(x,y)\quad \forall(x,y)\in\mathbb{R}^n_+, \\ D^2u,\ u_t\in C_{1-\theta}(]0,1],C^0(\mathbb{R}^n_+))\ (\text{resp. } C_{1-\theta}([0,1],C^0(\mathbb{R}^n_+))) \\ [Nu(t,\cdot,\cdot)](x)=0\quad \forall x\in\mathbb{R}^{n-1},\ \forall t\in]0,1]. \end{cases} \tag{2.2}$$

Here and in the following the symbol D denotes the gradient with respect to the only coordinates $(x,y)=(x_1,x_2,\dots,x_{n-1},y)$.
We will proceed as follows. First we will construct an extension F of f to the whole $\mathbb{R}^n$, in such a way that F is as smooth as f and satisfies the additional condition along Σ whenever f does. Next, we will construct a function $w(t,x,y)$, defined in $[0,1]\times\mathbb{R}^n$, satisfying (2.2) in $[0,1]\times\mathbb{R}^n$; this is done by taking the convolution $[\phi^t * F](x,y)$, where ϕ^t is a mollifier of parameter $t^{1/2}$, and adding to it a suitable term in order to satisfy the condition $Nw(t,\cdot,\cdot)=0$. Finally, the restriction of w to $[0,1]\times\mathbb{R}^n_+$ will be the desired function u.

Step 1. The extension of f is the following:

$$F(x,y)=\begin{cases} f(x,y) & x\in\mathbb{R}^{n-1},\ y\ge 0 \\ f(x,-y)-2\alpha(x)\displaystyle\int_y^0 \exp[\alpha(x)(y-s)]f(x,-s)\,ds, & x\in\mathbb{R}^{n-1},\ y<0. \end{cases} \tag{2.3}$$

REMARK 2.2 - (i) If α vanishes somewhere, this definition does not assure the uniform continuity of F: in this case we replace $F(x,y)$ by $F(x,y)\cdot a(y)$, where $a\in C^\infty(\mathbb{R})$ is such that $a\equiv1$ in $[-1,\infty[$ and $a\equiv0$ in $]-\infty,-2]$.

(ii) If the boundary operator (2.1) is of Neumann type, i.e. $\alpha(x)\equiv0$, then the extension F defined in (2.3) reduces to the even extension of f: hence it has the property that, for smooth f, the function $y\to\frac{\partial F}{\partial y}(x,y)$ (whose evaluation at $y=0$ yields the boundary operator) is odd.
The same property holds in the general case, namely the extension F given by (2.3) is constructed in such a way that, for smooth f,

the function $y \to \alpha(x)F(x,y) - \frac{\partial F}{\partial y}(x,y)$ turns out to be odd around y=0. This obviously guarantees that $NF=0$ whenever $Nf=0$; it is a remarkable fact that, in addition, F has exactly the same degree of smoothness of f, even in the case of non-differentiable f, i.e. when no boundary condition is required for f. This will be shown in the next proposition. □

PROPOSITION 2.3 - Let f belong to any of the following spaces:

(i) $C^0(\mathbb{R}^n_+)$; (ii) $C^{0,2\theta}(\mathbb{R}^n_+)$ (resp. $h^{0,2\theta}(\mathbb{R}^n_+)$) with $\theta \in]0,1/2[$;

(iii) $C^{*,1}_{\nu}(\mathbb{R}^n_+)$ (resp. $h^{*,1}_N(\mathbb{R}^n_+)$) ; (iv) $C^1_N(\mathbb{R}^n_+)$;

(v) $C^{1,2\theta-1}_N(\mathbb{R}^n_+)$ (resp. $h^{1,2\theta-1}_N(\mathbb{R}^n_+)$) with $\theta \in]1/2,1[$.

Then the function F defined in (2.3) satisfies accordingly:

(i) $F \in C^0(\mathbb{R}^n)$; (ii) $F \in C^{0,2\theta}(\mathbb{R}^n)$ (resp. $F \in h^{0,2\theta}(\mathbb{R}^n)$) ;

(iii) $F \in C^{*,1}(\mathbb{R}^n)$ and $\sup\{\frac{|F(x,y)-F(x,0)|}{|y|} : x \in \mathbb{R}^{n-1}, y \neq 0\} < \infty$

(resp. $F \in h^{*,1}(\mathbb{R}^n)$ and $\lim_{y \to 0} \frac{F(x,y)-F(x,0)}{y} =$

$= \alpha(x) f(x,0)\ \forall x \in \mathbb{R}^{n-1}$);

(iv) $F \in C^1(\mathbb{R}^n)$ and $NF=0$; (v) $F \in C^{1,2\theta-1}(\mathbb{R}^n)$ (resp $F \in h^{1,2\theta-1}(\mathbb{R}^n)$) and $NF=0$.

Moreover we have in any case

$$\|F\| \leq c\,\|f\|$$

in the corresponding norm; in particular, in case (iii) we set

$$\|F\| = \|F\|_{C^{*,1}(\mathbb{R}^n)} + \sup\{\frac{|F(x,y)-F(x,0)|}{|y|} : x \in \mathbb{R}^{n-1},\ y \neq 0\}.$$

Proof. (i)-(ii) The results follow by straightforward computations.
(iii) This proof is more delicate. Suppose $f \in C^{*,1}_{\nu}(\mathbb{R}^n_+)$ and let

$(x,y),\ (x',y') \in \mathbb{R}^n$; we have to estimate the quantity $F(x,y)+F(x',y')-2F(\frac{x+x'}{2}, \frac{y+y'}{2})$. This is easy if y,y' have the same sign, so we can suppose $y>0>y'$ and two cases can occur:

(a) $y > \frac{y+y'}{2} \geq 0 > y'$; (b) $y > 0 > \frac{y+y'}{2} > y'$.

In case (a) we can write

$$[F(x,y)+F(x',y')-2F(\frac{x+x'}{2},\frac{y+y'}{2})] = f(x,y)+f(x',-y') - 2\alpha(x')\int_{y'}^{0} \exp[\alpha(x')(y-s)]\,f(x',-s)\,ds - 2f(\frac{x+x'}{2},\frac{y+y'}{2}) =$$
$$= [f(x,y)+f(x',-y')-2f(\frac{x+x'}{2},\frac{y-y'}{2})] + 2[f(\frac{x+x'}{2},\frac{y-y'}{2}) - f(\frac{x+x'}{2},0)] - 2[f(\frac{x+x'}{2},\frac{y+y'}{2}) - f(\frac{x+x'}{2},0)] - 2\alpha(x')\int_{y'}^{0} \exp[\alpha(x')(y'-s)]\,f(x',-s)\,ds \tag{2.4}$$

and consequently, since $|y+y'| \vee |y'| \leq |y-y'|$, we get

$$|F(x,y)+F(x',y')-2F(\frac{x+x'}{2},\frac{y+y'}{2})| \leq [f]_{*,1}[|x-x'|+|y+y'|] + 2[f]_{1,\nu}[|\frac{y-y'}{2}|+|\frac{y+y'}{2}|] + 2\|\alpha\|_\infty \|f\|_\infty |y'| \leq c\|f\|_{C^{*,1}_{\nu}}[|x-x'|+|y-y'|]. \tag{2.5}$$

In case (b) we have

$$[F(x,y)+F(x',y')-2F(\frac{x+x'}{2},\frac{y+y'}{2})] = f(x,y)+f(x'-y') - 2\alpha(x')\int_{y'}^{0} \exp[\alpha(x')(y'-s)]\,f(x',-s)\,ds - 2f(\frac{x+x'}{2},-\frac{y+y'}{2}) + 4\alpha(\frac{x+x'}{2})\int_{\frac{y+y'}{2}}^{0} \exp[\alpha(\frac{x+x'}{2})(\frac{y+y'}{2}-s)]\,f(\frac{x+x'}{2},-s)\,ds = \tag{2.6}$$
$$= [f(x,y)+f(x',-y')-2f(\frac{x+x'}{2},\frac{y-y'}{2})] + 2[f(\frac{x+x'}{2},\frac{y-y'}{2}) - f(\frac{x+x'}{2},0)] - 2[f(\frac{x+x'}{2},-\frac{y+y'}{2}) - f(\frac{x+x'}{2},0)] - 2\alpha(x')\int_{y'}^{0} \exp[\alpha(x')(y'-s)]\,f(x',-s)\,ds + 4\alpha(\frac{x+x'}{2})\int_{\frac{y+y'}{2}}^{0} \exp[\alpha(\frac{x+x'}{2})(\frac{y+y'}{2}-s)]\,f(\frac{x+x'}{2},-s)\,ds$$

which implies, since $|y+y'|\le|y'|\le|y-y'|$,

$$|F(x,y)+F(x',y')-2F(\frac{x+x'}{2},\ \frac{y+y'}{2})|\le [f]_{*,1}[\,|x-x'|+|y+y'|\,]+$$

$$+2\,[f]_{1,\nu}[\,|\frac{y-y'}{2}|+|\frac{y+y'}{2}|\,]+2\|\alpha\|_\infty\|f\|_\infty[\,|y'|+2\,|\frac{y+y'}{2}|\,]\le \tag{2.7}$$

$$\le c\,\|f\|_{C_\nu^{*,1}(\mathbb{R}^n_+)}\,[\,|x-x'|+|y-y'|\,].$$

By (2.5) and (2.7) we deduce that $F\in C^{*,1}(\mathbb{R}^n)$ and $\|F\|_{C^{*,1}(\mathbb{R}^n)}\le$

$\le c\,\|f\|_{C_\nu^{*,1}(\mathbb{R}^n)}$; on the other hand it is easily seen that for

each $x\in\mathbb{R}^{n-1}$ and $y\in\mathbb{R}$

$$|F(x,y)-F(x,0)|\le\begin{cases}[f]_{1,\nu}|y| & \text{if } y\ge 0\\ [f]_{1,\nu}|y|+\|\alpha\|_\infty\|f\|_\infty|y| & \text{if } y<0\end{cases},$$

which implies

$$\|F\|_{C^{*,1}(\mathbb{R}^n)}+\sup\{\frac{|F(x,y)-F(x,0)|}{|y|}\ :x\in\mathbb{R}^{n-1},y\ne 0\}\le c\|f\|_{C_\nu^{*,1}(\mathbb{R}^n_+)}.$$

Suppose now, in addition, that $f\in h_N^{*,1}(\mathbb{R}^n_+)$. We have to show that

$$|F(x,y)+F(x',y')-2F(\frac{x+x'}{2},\frac{y+y'}{2})|=o(|x-x'|+|y-y'|) \tag{2.8}$$

as $|x-x'|+|y-y'|\to 0^+$ and

$$\lim_{y\to 0}\ \frac{F(x,y)-F(x,0)}{y}=\alpha(x)f(x,0)\quad \forall x\in\mathbb{R}^{n-1}. \tag{2.9}$$

The proof of (2.8) is easy if y,y' have the same sign. Otherwise, we again reduce to case (a) or (b). In case (a) by (2.4) we deduce as $|x-x'|+|y-y'|\to 0^+$

$$[F(x,y)-F(x',y')-2F(\frac{x+x'}{2},\frac{y+y'}{2})]=o(|x-x'|+|y+y'|)+$$

$$+(y-y')[\alpha(\frac{x+x'}{2})f(\frac{x+x'}{2},0)+o(1)]-(y+y')[\alpha(\frac{x+x'}{2}\quad)+o(1)]+$$

$$+2y'[\alpha(x')f(x',0)+o(1)]=o(|x-x'|+|y+y'|)-2y'[\alpha(\frac{x+x'}{2})f(\frac{x+x'}{2},0)-$$

$$\alpha(x')f(x',0)]=o(|x-x'|+|y-y'|).$$

In case (b), by (2.6) similarly we get as $|x-x'|+|y-y'|\to 0^+$

$$[F(x,y)+F(x',y')-2F(\frac{x+x'}{2},\frac{y+y'}{2})]=o(|x-x'|+|y+y'|)+$$

$$+(y-y')[\alpha(\frac{x+x'}{2})f(\frac{x+x'}{2},0)+o(1)]+(y+y')[\alpha(\frac{x+x'}{2})f(\frac{x+x'}{2},0)+o(1)]+$$

$$+2y'[\alpha(x')f(x',0)+o(1)]-2(y+y')[\alpha(\frac{x+x'}{2})f(\frac{x+x'}{2},0)+o(1)]=$$

$$=o(|x-x'|+|y+y'|)+2y'[\alpha(x')f(x',0)-\alpha(\frac{x+x'}{2})f(\frac{x+x'}{2},0)]=$$

$$=o(|x-x'|+|y-y'|).$$

This proves (2.8). Finally it is easy to see that as $y\to 0$

$$F(x,y)-F(x,0)=\begin{cases}y[\alpha(x)f(x,0)+o(1)] & \text{if } y\ge 0\\ -y[\alpha(x)f(x,0)+o(1)]+2y[\alpha(x)f(x,0)+o(1)] & \text{if } y<0,\end{cases}$$

which implies (2.9). This concludes the proof of (iii).

(iv) Let $f\in C_N^1(\mathbb{R}^n_+)$. It is easy to see that $\frac{\partial F}{\partial x_i}\in C^0(\mathbb{R}^n)$,$i=1,\dots,n-1$;

about $\frac{\partial F}{\partial y}$, we have clearly

$$\frac{\partial F}{\partial y}(x,y)=\begin{cases}\frac{\partial f}{\partial y}\ (x,y) & x\in\mathbb{R}^{n-1},y>0\\ -\frac{\partial f}{\partial y}\ (x,-y)+2\alpha(x)f(x,-y)-2[\alpha(x)]^2\int_y^0\exp[\alpha(x)(y-s)]f(x,-s)ds, & \\ x\in\mathbb{R}^{n-1},y<0;\end{cases}$$

when $y=0$ we note that

$$\lim_{y\to 0^+}\frac{\partial F}{\partial y}(x,y)=\frac{\partial f}{\partial y}(x,0)\quad,\ \lim_{y\to 0^-}\frac{\partial F}{\partial y}(x,y)=-\frac{\partial f}{\partial y}(x,0)+2\alpha(x)f(x,0),$$

and since $Nf=0$, the two limits are equal. This shows that there exists $\frac{\partial F}{\partial y}(x,y)\big|_{y=0}$ and $\frac{\partial F}{\partial y}(x,0)=\frac{\partial f}{\partial y}(x,0)$, so that in particular

$NF(x)=0$. Clearly $\frac{\partial F}{\partial y}\in C^0(\mathbb{R}^n)$ and it is easy to see that the required estimate holds.

(v) By (iv) we know that $F \in C^1(\mathbb{R}^n)$ and $NF=0$; the result follows easily. □

Step 2. We want to construct a function $w(t,x,y)$, defined in $[0,1] \times \mathbb{R}^n$, with the following properties:

$$\begin{aligned}
&(i) \quad w \in C^0([0,1] \times \mathbb{R}^n), \ w(0,\cdot,\cdot)=F\\
&(ii) \quad D^2 w \in C_{1-\theta}(]0,1], C^0(\mathbb{R}^n)) \ (\text{resp. } D^2 w \in C_{1-\theta}([0,1], C^0(\mathbb{R}^n)) \text{ and}\\
&\qquad \lim_{t\to 0^+} t^{1-\theta} \| D^2 w(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)} = 0) \qquad (2.10)\\
&(iii) \quad [Nw(t,\cdot,\cdot)](x)=0 \quad \forall x \in \mathbb{R}^{n-1}, \ \forall t \in]0,1]\\
&(iv) \quad w_t \in C_{1-\theta}(]0,1], C^0(\mathbb{R}^n)) \ (\text{resp. } w_t \in C_{1-\theta}([0,1], C^0(\mathbb{R}^n)) \text{ and}\\
&\qquad \lim_{t\to 0^+} t^{1-\theta} \| w_t(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)} = 0).
\end{aligned}$$

First of all set for each $t \in]0,1]$ and $(x,y) \in \mathbb{R}^n$

$$\phi^t(x,y) = t^{-n/2} \phi(t^{-1/2}x, t^{-1/2}y)$$

where $\phi \in C^\infty(\mathbb{R}^n)$ is an even (in each variable) non-negative function with support contained in the ball $\overline{B(0,1)} = \{(x,y) \in \mathbb{R}^n : |x|^2+y^2 \le 1\}$ and such that $\int_{\mathbb{R}^n} \phi(x,y)\,dx\,dy = 1$. Next, define

$$v(t,x,y) = \begin{cases} F(x,y) & \text{if } t=0, (x,y) \in \mathbb{R}^n \\ [\phi^t * F](x,y) = t^{-n/2} \int_{\mathbb{R}^n} \phi(t^{-1/2}(x-\xi), t^{-1/2}(y-\eta)) \cdot \\ \quad \cdot F(\xi,\eta)\,d\xi\,d\eta & \text{if } t \in]0,1], (x,y) \in \mathbb{R}^n \end{cases} \qquad (2.11)$$

and

$$g(t,x) = \begin{cases} 0 & \text{if } t=0, \ x \in \mathbb{R}^{n-1} \\ \alpha(x) v(t,x,0) - \frac{\partial v}{\partial y}(t,x,0) = \alpha(x)[\phi^t * F](x,0) - \\ \quad -[\frac{\partial \phi^t}{\partial y} * F](x,0) & \text{if } t \in]0,1], x \in \mathbb{R}^{n-1}; \end{cases} \qquad (2.12)$$

finally set

$$w(t,x,y) = v(t,x,y) - \frac{(1-y)\ \theta(y)}{1+\alpha(x)} g(t,x), \quad t \in [0,1], (x,y) \in \mathbb{R}^n, \qquad (2.13)$$

where $\theta \in C^\infty(\mathbb{R})$, $\theta \equiv 1$ in $[-1,1]$, $\theta \equiv 0$ outside $[-2,2]$ and $0 \le \theta \le 1$.

REMARK 2.4 - It can be verified that the function v defined by (2.11) satisfies conditions (i), (ii) and (iv) of (2.10), whereas in general condition (2.10) (iii) needs not to be true. For this reason we have to introduce the function g defined in (2.12) which is suitably constructed in order that (2.10) (iii) holds automatically. We will see that, consequently, the function w given by (2.13) satisfies (2.10).

The auxiliary function g is unnecessary if in the boundary operator (2.1) we have $\alpha(x) \equiv \alpha = $ constant, because in this case it can be shown that (2.10) (iii) holds, i.e.

$$(N[\phi^t * F])(x) = \alpha[\phi^t * F](x,0) - \frac{\partial}{\partial y}[\phi^t * F](x,0) = 0 \quad \forall x \in \mathbb{R}^{n-1}.$$

This is clear if F is smooth, i.e. if f is smooth: indeed, as observed in Remark 2.2, for a differentiable F the function (with fixed $x \in \mathbb{R}^{n-1}$)

$$G(y) = \alpha F(x,y) - \frac{\partial F}{\partial y}(x,y)$$

is odd; as the kernel ϕ^t is an even function, the convolution

$$[\phi^t * G](y) = \alpha[\phi^t * F](x,y) - \frac{\partial}{\partial y}[\phi^t * F](x,y)$$

is also odd, and therefore it vanishes at $y=0$. However, even if f is not smooth enough to give sense to Nf, a direct computation can show that $N[\phi^t * F]=0$. We will prove this fact indirectly by formula (2.14) below, since that equality reduces to $g(t,x) \equiv 0$ if $\alpha(x) \equiv$ constant. □

We have to verify (2.10) for the function w defined in (2.13).

We start with the following result:

PROPOSITION 2.5 (i) w is twice differentiable in $]0,1]\times\mathbb{R}^n$.

(ii) $[Nw(t,\cdot,\cdot)](x)=\alpha(x)w(t,x,0)-\frac{\partial w}{\partial y}(t,x,0)=0 \quad \forall t\in]0,1], \forall x\in\mathbb{R}^{n-1}$.

(iii) $w\in C^0([0,1]\times\mathbb{R}^n)$ and $w(0,x,y)=F(x,y) \quad \forall (x,y)\in\mathbb{R}^n$.

Proof. (i) It is a straightforward consequence of the mollifying properties of the convolution and of the regularity of α.

(ii) For each $y\in\mathbb{R}$ we have

$$\alpha(x)w(t,x,y)-\frac{\partial w}{\partial y}(t,x,y)=\alpha(x)v(t,x,y)-\frac{\partial v}{\partial y}(t,x,y)-$$
$$-\frac{(1-y)\theta(y)}{1+\alpha(x)}\alpha(x)g(t,x)+\frac{-\theta(y)+(1-y)\theta'(y)}{1+\alpha(x)}g(t,x)$$

and choosing $y=0$ we get

$$[Nw(t,\cdot,\cdot)](x)=\alpha(x)v(t,x,0)-\frac{\partial v}{\partial y}(t,x,0)-g(t,x);$$

by (2.11) and (2.12) the result follows.

(iii) We need an alternative expression for g, namely

$$g(t,x)=\alpha(x)[\phi^t * F](x,0)-[\phi^t*(F\cdot\alpha)](x,0). \tag{2.14}$$

To prove (2.14) it suffices to show that

$$\frac{\partial v}{\partial y}(t,x,0)=[\phi^t*(F\cdot\alpha)](x,0). \tag{2.15}$$

In fact, we have

$$\frac{\partial v}{\partial y}(t,x,0)=t^{-\frac{n+1}{2}}\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial y}(t^{-1/2}(x-\xi),-t^{-1/2}\eta)F(\xi,\eta)\,d\eta\, d\xi$$
$$=t^{-\frac{n+1}{2}}\int_{\mathbb{R}^{n-1}}\int_0^\infty\frac{\partial\phi}{\partial y}(t^{-1/2}(x-\xi),-t^{-1/2}\eta)f(\xi,\eta)\,d\eta\, d\xi+$$
$$+t^{-\frac{n+1}{2}}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0\frac{\partial\phi}{\partial y}(t^{-1/2}(x-\xi),-t^{-1/2}\eta)f(\xi,-\eta)\,d\eta\, d\xi-$$
$$-t^{-\frac{n+1}{2}}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0\frac{\partial\phi}{\partial y}(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\cdot$$
$$\cdot 2\alpha(\xi)\int_\eta^0\exp[\alpha(\xi)(\eta-s)]f(\xi,-s)\,ds\,d\eta\,d\xi.$$

As $y\to\frac{\partial\phi}{\partial y}(x,y)$ is an odd function, the first two terms cancel each other. Thus by an integration by parts in the variable η we obtain

$$\frac{\partial v}{\partial y}(t,x,0)=-t^{-n/2}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\cdot$$
$$\cdot 2\alpha(\xi)[-f(\xi,-\eta)+\alpha(\xi)\int_\eta^0\exp[\alpha(\xi)(\eta-s)]f(\xi,-s)ds]\,d\eta\, d\xi.$$

On the other hand, as $y\to\phi(x,y)$ is an even function, we have

$$2t^{-n/2}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\alpha(\xi)f(\xi,-\eta)\,d\eta\, d\xi=$$
$$=t^{-n/2}\int_{\mathbb{R}^{n-1}}[\int_{-\infty}^0\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\alpha(\xi)f(\xi,-\eta)\,d\eta+$$
$$+\int_0^\infty\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\alpha(\xi)f(\xi,\eta)\,d\eta]\,d\xi,$$

and consequently

$$\frac{\partial v}{\partial y}(t,x,0)=t^{-n/2}\int_{\mathbb{R}^{n-1}}\int_0^\infty\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\alpha(\xi)f(\xi,\eta)\,d\eta\, d\xi$$
$$+t^{-n/2}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^0\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\alpha(\xi)[f(\xi,-\eta)-$$
$$-2\alpha(\xi)\int_\eta^0\exp[\alpha(\xi)(x-s)]f(\xi,-s)ds]\,d\eta\, d\xi=$$
$$=t^{-n/2}\int_{\mathbb{R}^n}\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\alpha(\xi)F(\xi,\eta)\,d\eta\, d\xi=[\phi^t*(F\cdot\alpha)](x,0).$$

This proves (2.15) and hence (2.14).

Now, to prove (iii) it is enough to verify that

$$\lim_{t\to 0^+}\|w(t,\cdot,\cdot)-F\|_{C^0(\mathbb{R}^n)}=0 \tag{2.16}$$

By (2.14) we get as $t\to 0^+$

$$g(t,x)\to\alpha(x)F(x,0)-\alpha(x)F(x,0)=0 \quad \text{uniformly in } x\in\mathbb{R}^{n-1};$$

on the other hand we have $v\in C^0([0,1]\times\mathbb{R}^n)$, $v(0,\cdot,\cdot)=F$ and

$$\lim_{t\to 0^+}\|v(t,\cdot,\cdot)-F\|_{C^0(\mathbb{R}^n)} = 0.$$

(These facts follow since F is uniformly continuous). Thus by (2.13) we get

$$w(t,x,y)\to F(x,y)-\frac{(1-y)\theta(y)}{1+\alpha(x)}g(0,x) = F(x,y) \text{ uniformly in } (x,y)\in\mathbb{R}^n,$$

i.e. (2.16) holds. The proof is complete. □

Conditions (i) and (iii) of (2.10) are proved; we have now to verify (ii) and (iv) of (2.10).

LEMMA 2.6. We have for each $t\in]0,1]$

$$\|D^2w(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)}\le c\{\|\phi^t*F\|_{C^2(\mathbb{R}^n)}+\|\phi^t*(F\cdot\alpha)\|_{C^2(\mathbb{R}^n)}\}.$$

Proof. It is a straightforward consequence of (2.13) and (2.14). □

Thus to prove (ii) of (2.10) we have to estimate the C^2-norms of the convolutions ϕ^t*F, $\phi^t*(F\cdot\alpha)$, and, in view of Lemma 1.10, it will be sufficient to estimate their C^0-norms and the C^0-norms of their second-order gradients. This is the goal of the next two lemmata.

LEMMA 2.7. For each $f\in C^0(\mathbb{R}^n_+)$, let F be defined by (2.3). Then

$$\|\phi^t*F\|_{C^0(\mathbb{R}^n)}+\|\phi^t*(F\cdot\alpha)\|_{C^0(\mathbb{R}^n)}\le c\,\|f\|_{C^0(\mathbb{R}^n)} \qquad \forall t\in]0,1].$$

Proof. Obviously

$$\|\phi^t*F\|_{C^0(\mathbb{R}^n)}+\|\phi^t*(F\alpha)\|_{C^0(\mathbb{R}^n)}\le c\{\|F\|_{C^0(\mathbb{R}^n)}+\|F\alpha\|_{C^0(\mathbb{R}^n)}\}\le c\|F\|_{C^0(\mathbb{R}^n)}$$

and Proposition 2.3(i) yields the result. □

LEMMA 2.8 (i) If $\theta\in]0,1/2[$ and $f\in C^{0,2\theta}(\mathbb{R}^n_+)$, then

$$\|D^2[\phi^t*F]\|_{C^0(\mathbb{R}^n)}+\|D^2[\phi^t*(F\cdot\alpha)]\|_{C^0(\mathbb{R}^n)}\le c\,t^{\theta-1}\|f\|_{C^{0,2\theta}(\mathbb{R}^n_+)}$$

$\forall t\in]0,1]$.

(ii) If $\theta=1/2$ and $f\in C^{*,1}_\nu(\mathbb{R}^n_+)$, then

$$\|D^2[\phi^t*F]\|_{C^0(\mathbb{R}^n)}+\|D^2[\phi^t*(F\cdot\alpha)]\|_{C^0(\mathbb{R}^n)}\le c\,t^{-1/2}\|f\|_{C^{*,1}_\nu(\mathbb{R}^n_+)}$$

$\forall t\in]0,1]$.

(iii) If $\theta\in]1/2,1[$ and $f\in C_N^{1,2\theta-1}(\mathbb{R}^n_+)$, then

$$\|D^2[\phi^t*F]\|_{C^0(\mathbb{R}^n)}+\|D^2[\phi^t*(F\cdot\alpha)]\|_{C^0(\mathbb{R}^n)}\le c\,t^{\theta-1}\|f\|_{C^{1,2\theta-1}(\mathbb{R}^n)}$$

$\forall t\in]0,1]$.

If, moreover, in cases (i), (ii), (iii) we assume $f\in h^{0,2\theta}(\mathbb{R}^n_+)$, $f\in h_N^{*,1}(\mathbb{R}^n_+)$, $f\in h_N^{1,2\theta-1}(\mathbb{R}^n_+)$ respectively, then we get

$$t^{1-\theta}\{\|D^2[\phi^t*F]\|_{C^0(\mathbb{R}^n)}+\|D^2[\phi^t*(F\cdot\alpha)]\|_{C^0(\mathbb{R}^n)}\}=o(1) \text{ as } t\to 0^+. \tag{2.17}$$

Proof. (i) We have for each $(x,y)\in\mathbb{R}^n$ and $t\in]0,1]$

$$D^2[\phi^t*F](x,y)=t^{-\frac{n+2}{2}}\int_{\mathbb{R}^n}(D^2\phi)(t^{-1/2}(x-\xi),t^{-1/2}(y-\eta))F(\xi,\eta)\,d\eta\,d\xi = \tag{2.18}$$

$$=t^{-1}\int_{\mathbb{R}^n}(D^2\phi)(z,w)F(x-t^{1/2}z,y-t^{1/2}w)\,dw\,dz.$$

As the integral over $\mathbb{R}^n$ of $D^2\phi$ vanishes, in the last integral we can replace $F(x-t^{1/2}z,y-t^{1/2}w)$ by $[F(x-t^{1/2}z,y-t^{1/2}w)-F(x,y)]$, obtaining

$$|D^2[\phi^t*F](x,y)|\le c\,t^{\theta-1}[F]_{2\theta,\overline{B((x,y),t^{1/2})}} \qquad \forall(x,y)\in\mathbb{R}^n,\ \forall t\in]0,1] \tag{2.19}$$

where $B((x,y),t^{1/2})=\{(z,w)\in\mathbb{R}^n : |z-x|^2+(w-y)^2<t\}$. A similar procedure applied to $D^2[\phi^t*(F\cdot\alpha)]$ leads to

$$|D^2[\phi^t*(F\cdot\alpha)](x,y)|\le ct^{\theta-1}[F\cdot\alpha]_{2\theta,\overline{B((x,y),t^{1/2})}}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1]. \tag{2.20}$$

As $[F\alpha]_{2\theta}\le[F]_{2\theta}\|\alpha\|_\infty+\|F\|_\infty[\alpha]_{2\theta}$, by (2.20) we deduce that

$$|D^2[\phi^t*(F\cdot\alpha)](x,y)|\le ct^{\theta-1}\|F\|_{C^{0,2\theta}(\mathbb{R}^n)}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1].$$

and (i) follows by Proposition 2.3(ii). If, in addition, $f\in h^{0,2\theta}(\mathbb{R}^n)$, then clearly (2.17) follows by (2.18) and (2.20).

(ii) Again we have (2.18). As $D^2\phi$ is an even function, whose integral over $\mathbb{R}^n$ vanishes, in the last integral of (2.18) we can replace $F(x-t^{1/2}z,y-t^{1/2}w)$ by $\frac{1}{2}[F(x-t^{1/2}z,y-t^{1/2}w)+F(x+t^{1/2}z,y+t^{1/2}w)-2F(x,y)]$; thus we check

$$|D^2[\phi^t*F](x,y)|\le ct^{-1/2}[F]_{*,1,\overline{B((x,y),t^{1/2})}}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1]. \tag{2.21}$$

and similarly

$$|D^2[\phi^t*(F\alpha)](x,y)|\le ct^{-1/2}[F\alpha]_{*,1,\overline{B((x,y),t^{1/2})}}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1]. \tag{2.22}$$

hence by Lemma 1.18 we easily obtain

$$|D^2[\phi^t*(F\alpha)](x,y)|\le ct^{-1/2}\|F\|_{C^{*,1}(\mathbb{R}^n)}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1],$$

and by Proposition 2.3(iii) we get (ii). If moreover $f\in h_N^{*,1}(\mathbb{R}^n_+)$ then (2.17) follows easily by (2.21), (2.22) and Lemma 1.18.

(iii) As now $F\in C^1(\mathbb{R}^n)$, instead of (2.18) we can write for each $(x,y)\in\mathbb{R}^n$ and $t\in]0,1]$ (with an obvious meaning of the notations):

$$D^2[\phi^t*F](x,y)=D[\phi^t*(DF)](x,y)=t^{-\frac{n+1}{2}}\int_{\mathbb{R}^n}D\phi(t^{-1/2}(x-\xi),t^{1/2}(y-\eta))\cdot$$

$$\cdot DF(\xi,\eta)\,d\eta\, d\xi=t^{-1/2}\int_{\mathbb{R}^n}D\phi(z,w)DF(x-t^{1/2}z,y-t^{1/2}w)\,dw\,dz.$$

As the integral over $\mathbb{R}^n$ of $D\phi$ vanishes, we replace $DF(x-t^{1/2}z,y-t^{1/2}w)$ by $[DF(x-t^{1/2}z,y-t^{1/2}w)-DF(x,y)]$, obtaining

$$|D^2[\phi^t*F](x,y)|\le ct^{\theta-1}[DF]_{2\theta-1,\overline{B((x,y),t^{1/2})}}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1]. \tag{2.23}$$

Now we have

$$[D(F\alpha)]_{2\theta-1}\le[DF]_{2\theta-1}\|\alpha\|_\infty+\|DF\|_\infty[\alpha]_{2\theta-1}+[F]_{2\theta-1}\|D\alpha\|_\infty+\|F\|_\infty[D\alpha]_{2\theta-1},$$

which implies

$$|D^2[\phi^t*(F\cdot\alpha)](x,y)|\le ct^{\theta-1}\|F\|_{C^{1,2\theta-1}(\mathbb{R}^n)}\quad \forall (x,y)\in\mathbb{R}^n,\ \forall t\in]0,1];$$

hence the result follows by Proposition 2.3 (v). If, in particular, $f\in h_N^{1,2\theta-1}(\mathbb{R}^n_+)$, by (2.23) and (2.24) we also obtain (2.17). □

By Lemmata 2.6, 2.7 and 2.8 it follows that condition (2.10)(ii) holds.

The proof of condition (2.10)(iv) is a little more delicate. As in the case of D^2w (Lemma 2.6) it is easily seen that w_t can be estimated in terms of $\frac{\partial}{\partial t}[\phi^t*F]$ and $\frac{\partial}{\partial t}[\phi^t*(F\alpha)]$; but the C^0-norms of these functions are not controlled by the appropriate power of t. The point is that, however, such derivatives appear in the expression of w_t in a suitable combination, which can in fact be estimated by the required power of t.

<u>LEMMA</u> 2.9 (i) <u>If</u> $\theta\in]0,1/2[$ <u>and</u> $f\in C^{0,2\theta}(\mathbb{R}^n_+)$, <u>then</u>

$$\|w_t(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)}\le ct^{\theta-1}\|f\|_{C^{0,2\theta}(\mathbb{R}^n_+)}\quad \forall t\in]0,1].$$

(ii) <u>If</u> $\theta=1/2$ <u>and</u> $f\in C_\nu^{*,1}(\mathbb{R}^n_+)$, <u>then</u>

$$\|w_t(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)}\le ct^{-1/2}\|f\|_{C_\nu^{*,1}(\mathbb{R}^n_+)}\quad \forall t\in]0,1].$$

(iii) <u>If</u> $\theta\in]1/2,1[$ <u>and</u> $f\in C_N^{1,2\theta-1}(\mathbb{R}_+^n)$, <u>then</u>

$$\|w_t(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)} \le ct^{\theta-1}\|f\|_{C^{1,2\theta-1}(\mathbb{R}_+^n)} \qquad \forall t\in]0,1] .$$

<u>If, moreover, in cases (i), (ii), (iii) we assume</u> $f\in h^{0,2\theta}(\mathbb{R}_+^n)$, $f\in h_N^{*,1}(\mathbb{R}_+^n)$, $f\in h_N^{1,2\theta-1}(\mathbb{R}_+^n)$ <u>respectively, then</u>

$$t^{1-\theta}\|w_t(t,\cdot,\cdot)\|_{C^0(\mathbb{R}^n)}=o(1) \text{ as } t\to 0^+ \tag{2.25}$$

<u>Proof</u>. (i) By (2.13), (2.11) and (2.12), we have clearly

$$w_t(t,x,y)=\frac{\partial}{\partial t}[\phi^t*F](x,y)-\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)\frac{\partial}{\partial t}[\phi^t*F](x,0)-$$
$$-\frac{\partial}{\partial t}[\phi^t*(F\cdot\alpha)](x,0)] . \tag{2.26}$$

Hence for each $(x,y)\in\mathbb{R}^n$ and $t\in]0,1]$ we have

$$w_t(t,x,y)=-\frac{n}{2}t^{-\frac{n+2}{2}}\int_{\mathbb{R}^n}\{\phi(t^{-1/2}(x-\xi),t^{1/2}(y-\eta))-$$
$$-\phi(t^{-1/2}(x-\xi),-t^{-1/2}\eta)\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(\xi)]\}F(\xi,\eta)d\eta d\xi-$$
$$-\frac{1}{2}t^{-\frac{n+3}{2}}\int_{\mathbb{R}^n}\{\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_i}(t^{-1/2}(x-\xi),t^{-1/2}(y-\eta))(x_i-\xi_i)-$$
$$-\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_i}(t^{-1/2}(x-\xi),-t^{1/2}\eta)(x_i-\xi_i)\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(\xi)]\}F(\xi,\eta)d\eta d\xi$$
$$-\frac{1}{2}t^{-\frac{n+3}{2}}\int_{\mathbb{R}^n}\{\frac{\partial\phi}{\partial y}(t^{-1/2}(x-\xi),t^{-1/2}(y-\eta))(y-\eta)-$$
$$-\frac{\partial\phi}{\partial y}(t^{-1/2}(x-\xi),-t^{1/2}\eta)\eta\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(\xi)]\}F(\xi,\eta)d\eta d\xi.$$

Each term on the right-hand side is the sum of two integrals: in the first integral of each term set $z=t^{-1/2}(x-\xi)$, $w=t^{-1/2}(y-\eta)$, in the second integral of each term set $z=t^{-1/2}(x-\xi)$, $w=-t^{-1/2}\eta$.

After the change of variables it is easily seen that

$$w_t(t,x,y)=-\frac{1}{2}t^{-1}\int_{\mathbb{R}^n}[n\phi(z,w)+\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_i}(z,w)z_i+\frac{\partial\phi}{\partial y}(z,w)w]\cdot$$
$$\cdot\{F(x-t^{1/2}z,y-t^{1/2}w)-\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(x-t^{1/2}z)]\cdot \tag{2.27}$$
$$\cdot F(x-t^{1/2}z,-t^{1/2}w)\}dwdz \qquad \forall(x,y)\in\mathbb{R}^n,\ \forall t\in]0,1] .$$

As

$$\int_{\mathbb{R}^n}[n\phi(z,w)+\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_i}(z,w)z_i+\frac{\partial\phi}{\partial y}(z,w)w]dwdz=0 \tag{2.28}$$

we can replace in (2.27) $F(x-t^{1/2}z,y-t^{1/2}w)$ by $[F(x-t^{1/2}z,y-t^{1/2}w)-F(x,y)]$; consequently

$$|w_t(t,x,y)|\le ct^{\theta-1}\{[F]_{2\theta,\overline{B((x,y),t^{1/2})}}+[\alpha]_{2\theta,\overline{B(x,t^{1/2})}}\cdot\|F\|_\infty\}. \tag{2.29}$$

Hence

$$|w_t(t,x,y)|\le ct^{\theta-1}\|F\|_{C^{0,2\theta}(\mathbb{R}^n)} \qquad \forall(x,y)\in\mathbb{R}^n,\forall t\in]0,1] ,$$

and (i) follows by Proposition 2.3(ii). If moreover $f\in h^{0,2\theta}(\mathbb{R}_+^n)$, then (2.29) implies (2.25).

(ii) As in (i), we arrive to (2.27). Now we observe that the function $(z,w)\to n\phi(z,w)+\sum_{i=1}^{n-1}\frac{\partial\phi}{\partial x_i}(z,w)z_i+\frac{\partial\phi}{\partial y}(z,w)w$ is even, and in addition (2.28) holds; this allows us to replace in (2.27) $F(x-t^{1/2}z,y-t^{1/2}w)$ by $\frac{1}{2}[F(x-t^{1/2}z,y-t^{1/2}w)+F(x+t^{1/2}z,y+t^{1/2}w)-2F(x,y)]$. Hence we get

$$|w_t(t,x,y)|\le ct^{-1/2}\{[F]_{*,1}+\|D\alpha\|_\infty\|F\|_\infty\}\le ct^{-1/2}\|F\|_{C^{*,1}(\mathbb{R}^n)}$$
$$\forall(x,y)\in\mathbb{R}^n,\ \forall t\in]0,1] ,$$

and (ii) follows by Proposition 2.3 (iii). Suppose in particular that $f\in h_N^{*,1}(\mathbb{R}^n_+)$; then to obtain (2.25) we have to do another replacement in (2.27), namely the term $-\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(x-t^{1/2}z)]\cdot F(x-t^{1/2}z,-t^{1/2}w)$ has to be replaced by

$$-\frac{(1-y)\theta(y)}{1+\alpha(x)}\cdot\frac{1}{2}\{[\alpha(x)-\alpha(x-t^{1/2}z)]\;F(x-t^{1/2}z,-t^{1/2}w)+[\alpha(x)-\alpha(x+t^{1/2}z)]\;F(x+t^{1/2}z,t^{1/2}w)\};$$

then we get

$$|w_t(t,x,y)|\le c\;t^{-1/2}\{[F]_{*,1,\overline{B((x,y),t^{1/2})}}+[F\cdot\alpha]_{*,1,\overline{B((x,y),t^{1/2})}}\},$$

and recalling Lemma 1.18 and Proposition 2.3(iii) we check (2.25).

(iii) As now $F\in C^1(\mathbb{R}^n)$, we proceed in a different way. We write

$$[\phi^t*F](x,y)=\int_{\mathbb{R}^n}\phi(z,w)F(x-t^{1/2}z,y-t^{1/2}w)\,dwdz,$$

$$[\phi^t*(F\cdot\alpha)](x,y)=\int_{\mathbb{R}^n}\phi(z,w)F(x-t^{1/2}z,y-t^{1/2}w)\;\alpha(x-t^{1/2}z)\,dwdz;$$

thus, starting from (2.26) we easily get for each $(x,y)\in\mathbb{R}^n$ and $t\in]0,1]$:

$$\begin{aligned}w_t(t,x,y)=&-\frac{1}{2}t^{-1/2}\int_{\mathbb{R}^n}\phi(z,w)\Big[\sum_{i=1}^{n-1}z_i\;\Big\{\frac{\partial F}{\partial x_i}(x-t^{1/2}z,y-t^{1/2}w)-\\&-\frac{\partial F}{\partial x_i}(x-t^{1/2}z,-t^{1/2}w)\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(x-t^{1/2}z)]\Big\}+\\&+w\Big\{\frac{\partial F}{\partial y}(x-t^{1/2}z,y-t^{1/2}w)-\frac{\partial F}{\partial y}(x-t^{1/2}z,-t^{1/2}w)\frac{(1-y)\theta(y)}{1+\alpha(x)}[\alpha(x)-\alpha(x-t^{1/2}z)]\Big\}\\&+\sum_{i=1}^{n-1}z_i\;\frac{(1-y)\theta(y)}{1+\alpha(x)}\;F(x-t^{1/2}z,-t^{1/2}w)\;\frac{\partial\alpha}{\partial x_i}\;(x-t^{1/2}z)\Big]\,dwdz.\end{aligned}$$

As $(z,w)\to\phi(z,w)z_i$ and $(z,w)\to\phi(z,w)w$ are odd functions, we can replace $\frac{\partial F}{\partial x_i}(x-t^{1/2}z,y-t^{1/2}w)$, $\frac{\partial F}{\partial y}(x-t^{1/2}z,y-t^{1/2}w)$ and $F(x-t^{1/2}z,y-t^{1/2}w)\;\frac{\partial\alpha}{\partial x_i}(x-t^{1/2}z)$ respectively by $[\frac{\partial F}{\partial x_i}(x-t^{1/2}z,y-t^{1/2}w)-\frac{\partial F}{\partial x_i}(x,y)]$, $[\frac{\partial F}{\partial y}(x-t^{1/2}z,y-t^{1/2}w)-\frac{\partial F}{\partial y}(x,y)]$

and $[F(x-t^{1/2}z,y-t^{1/2}w)\frac{\partial\alpha}{\partial x_i}(x-t^{1/2}z)-F(x,y)\frac{\partial\alpha}{\partial x_i}(x)]$; hence we obtain

$$|w_t(t,x,y)|\le ct^{\theta-1}\{[DF]_{2\theta-1,\overline{B((x,y),t^{1/2})}}+\|DF\|_\infty[\alpha]_{2\theta-1,\overline{B(x,t^{1/2})}}+[F\cdot D\alpha]_{2\theta-1,\overline{B((x,y),t^{1/2})}}\},\tag{2.30}$$

and consequently

$$|w_t(t,x,y)|\le ct^{\theta-1}\|F\|_{C^{1,2\theta-1}(\mathbb{R}^n)}\qquad\forall(x,y)\in\mathbb{R}^n,\forall t\in]0,1].$$

By Proposition 2.3(v) we get (iii). If in addition $f\in h_N^{1,2\theta-1}(\mathbb{R}^n_+)$, by (2.30) we also get (2.25). □

By Lemma 2.9 condition (iv) of (2.10) is proved. This concludes Step 2.

To complete the proof of Theorem 2.1 we have just to set

$$u=w_{|[0,1]\times\mathbb{R}^n_+}.$$

As w satisfies (2.10), it is clear that u satisfies (2.2). By Definition 1.7 and Lemmata 1.10 and 1.11, this means $f\in(C^2_N(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta,\infty}$ (resp. $f\in(C^2_N(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta}$).

Theorem 2.1 is proved. □

3. THE CASE OF A HALF-SPACE WITH A NON-TANGENTIAL BOUNDARY CONDITION

In this section we consider again the half-space $\mathbb{R}^n_+=\mathbb{R}^{n-1}\times[0,\infty[$, with general (non-tangential) boundary conditions.

Let $\alpha\in C^2(\mathbb{R}^{n-1},\mathbb{R})$ and $\beta\in C^2(\mathbb{R}^{n-1},\mathbb{R}^n)$ be such that

$$\alpha(x)\geq 0\ ,\quad \beta_n(x)=(\beta(x)|e^n)_n\leq -\delta_0<0.$$

It is not restrictive to assume that $\beta_n(x)=-1$ $\forall x\in\mathbb{R}^{n-1}$. Again, we denote by (x,y) the points of $\mathbb{R}^n_+$ $(x\in\mathbb{R}^{n-1},y\geq 0)$. Setting $\beta^0(x)=(\beta_1(x),\dots,\beta_{n-1}(x))$, we have $\beta^0\in C^2(\mathbb{R}^{n-1},\mathbb{R}^{n-1})$ and we can write $\beta(x)=(\beta^0(x),-1)$.

Define the boundary operator

$$(Bu)(x)=\alpha(x)u(x,0)+(\beta(x)|Du(x,0))_n\quad ,\ x\in\mathbb{R}^{n-1}. \tag{3.1}$$

We recall the definition of the spaces (see Definitions 1.13 and 1.14)

(a) $C^k_B(\mathbb{R}^n_+)=\{f\in C^k(\mathbb{R}^n_+):Bf=0\}$, $\quad k=1,2;$

(b) $C^{1,\alpha}_B(\mathbb{R}^n_+)=\{f\in C^{1,\alpha}(\mathbb{R}^n_+):Bf=0\}$, $h^{1,\alpha}_B(\mathbb{R}^n_+)=\{f\in h^{1,\alpha}(\mathbb{R}^n_+):Bf=0\}$, $\alpha\in]0,1]$;

(c) $C^{*,1}_\beta(\mathbb{R}^n_+)=\{f\in C^{*,1}(\mathbb{R}^n_+):[f]_{1,\beta}=\sup\{\frac{|f(x-\sigma\beta^0(x),\sigma)-f(x,0)|}{\sigma}$: $x\in\mathbb{R}^{n-1}$, $\sigma>0\}<\infty\}$,

$h^{*,1}_B(\mathbb{R}^n_+)=\{f\in C^{*,1}_\beta(\mathbb{R}^n_+)\cap h^{*,1}(\mathbb{R}^n_+)$: $\lim_{\sigma\to 0}\frac{f(x-\sigma\beta^0(x),\sigma)-f(x,0)}{\sigma}=$ $=\alpha(x)f(x,0)\quad\forall x\in\mathbb{R}^{n-1}\}$.

Obviously when $\beta^0=0$, i.e. $\beta=-e^n$, these spaces reduce to the spaces C^k_N, $C^{1,\alpha}_N$, $h^{1,\alpha}_N$, $C^{*,1}_\nu$, $h^{*,1}_N$ of Section 2. In particular, they are all Banach spaces with the norms

(a) $\|f\|_{C^k(\mathbb{R}^n_+)}$, (b) $\|f\|_{C^{1,\alpha}(\mathbb{R}^n_+)}$, (c) $\|f\|_{C^{*,1}(\mathbb{R}^n_+)}+[f]_{1,\beta}$.

We want to prove the following result:

<u>THEOREM</u> 3.1 - <u>Let</u> $\alpha\in C^2(\mathbb{R}^{n-1},\mathbb{R})$, $\beta^0\in C^2(\mathbb{R}^{n-1},\mathbb{R}^{n-1})$ <u>with</u> $\alpha\geq 0$, <u>and set</u> $\beta=(\beta^0,-1)$. <u>If</u> B <u>is the operator defined in</u> (3.1), <u>the following continuous inclusions hold</u>:

$$(C^2_B(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta,\infty}\hookrightarrow\begin{cases}C^{0,2\theta}(\mathbb{R}^n_+) \text{ if } \theta\in]0,1/2[\\ C^{*,1}_\beta(\mathbb{R}^n_+) \text{ if } \theta=1/2\\ C^{1,2\theta-1}_B(\mathbb{R}^n_+) \text{ if } \theta\in]1/2,1[\end{cases};$$

$$(C^2_B(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta}\hookrightarrow\begin{cases}h^{0,2\theta}(\mathbb{R}^n_+) \text{ if } \theta\in]0,1/2[\\ h^{*,1}_B(\mathbb{R}^n_+) \text{ if } \theta=1/2\\ h^{1,2\theta-1}_B(\mathbb{R}^n_+) \text{ if } \theta\in]\tfrac12,1[.\end{cases}$$

<u>Proof</u>. We want to reduce ourselves to the situation of the preceding section, i.e. to the case $\beta=-e^n$. Let $\phi\in C^\infty([0,\infty[)$ be a function with support contained in $[0,1]$ and such that

$$\phi(0)=1,\quad 0\leq\phi(s)\leq 1\quad\forall s\geq 0,\quad \int_0^1\phi(s)ds=\varepsilon$$

where $\varepsilon<\|D\beta^0\|_\infty^{-1}$. Consider the function ω: $\mathbb{R}^n_+\to\mathbb{R}^n_+$ defined by

$$\omega(\xi,s)=(\xi-\int_0^s\phi(\sigma)d\sigma\cdot\beta^0(\xi),s)\ ,\qquad \xi\in\mathbb{R}^{n-1},\ s\geq 0 \tag{3.2}$$

It is easy to see that ω is twice differentiable in $\mathbb{R}^n_+$ and is one-to-one. In addition ω^{-1} is also twice differentiable since the jacobian of ω is non-singular: indeed, we have

$$(D\omega)(\xi,s)=\left(\begin{array}{c|c} I_{n-1}-\int_0^s\phi(\sigma)d\sigma\cdot D\beta^0(\xi) & -\phi(s)\beta^0(\xi)\\ \hline 0 & 1\end{array}\right)$$

and consequently $|\det[(D\omega)(\xi,s)]| = |\det[I_{n-1} - \int_0^s \phi(\sigma)d\sigma \cdot D\beta^0(\xi)]|$;

on the other hand we have

$$\sup_{(\xi,s)\in\mathbb{R}^n_+} |\int_0^s \phi(\sigma)d\sigma \cdot D\beta^0(\xi)| = \varepsilon \|D\beta^0\|_\infty < 1,$$

which implies $|\det(D\omega)(\xi,s)| \geq \delta > 0 \quad \forall (\xi,s)\in\mathbb{R}^n_+$.

Let us define now a mapping $T: u \to T(u)$ by

$$v(\xi,s) = T(u)(\xi,s) = (u\circ\omega)(\xi,s), \qquad u\in C^0(\mathbb{R}^n_+).$$

<u>PROPOSITION</u> 3.2 - <u>The transformation</u> T <u>maps isomorphically</u>:

(i) $C^k_B(\mathbb{R}^n_+)$ <u>into</u> $C^k_N(\mathbb{R}^n_+)$, $k=1,2$;

(ii) $C^{1,\alpha}_B(\mathbb{R}^n_+)$ <u>and</u> $h^{1,\alpha}_B(\mathbb{R}^n_+)$ <u>into</u> $C^{1,\alpha}_N(\mathbb{R}^n_+)$ <u>and</u> $h^{1,\alpha}_N(\mathbb{R}^n_+)$, $\alpha\in]0,1]$;

(iii) $C^{*,1}_\beta(\mathbb{R}^n_+)$ <u>and</u> $h^{*,1}_B(\mathbb{R}^n_+)$ <u>into</u> $C^{*,1}_\nu(\mathbb{R}^n_+)$ <u>and</u> $h^{*,1}_N(\mathbb{R}^n_+)$;

(iv) $C^{0,\alpha}(\mathbb{R}^n_+)$ (<u>resp</u>. $h^{0,\alpha}(\mathbb{R}^n_+)$) <u>into itself</u>, $\alpha\in]0,1]$;

(v) $C^0(\mathbb{R}^n_+)$ <u>into itself</u>.

<u>Proof</u>. It is clear that T preserves the regularity of type C^0, $C^{0,\alpha}$, $h^{0,\alpha}$, C^1, $C^{1,\alpha}$, $h^{1,\alpha}$ and C^2. Suppose next that u is differentiable and Bu=0. Then v also is differentiable and

$$[Nv](\xi) = \alpha(\xi)v(\xi,0) - \frac{\partial v}{\partial s}(\xi,0) = [\alpha(\xi)v(\xi,s) - \frac{\partial v}{\partial s}(\xi,s)]_{s=0} =$$
$$= [\alpha(\xi)u(\omega(\xi,s)) - \sum_{j=1}^{n-1} \frac{\partial u}{\partial x_j}(\omega(\xi,s)) \frac{\partial \omega^j}{\partial s}(\xi,s) - \frac{\partial u}{\partial y}(\omega(\xi,s)) \frac{\partial \omega^n}{\partial s}(\xi,s)]_{s=0} =$$
$$= \alpha(\xi)u(\xi,0) + \sum_{j=1}^{n-1} \frac{\partial u}{\partial x_j}(\xi,0)\beta^0_j(\xi) - \frac{\partial u}{\partial y}(\xi,0) = [Bu](\xi) = 0.$$

Thus we have proved all statements but (iii). Now let $u\in C^{*,1}_\beta(\mathbb{R}^n_+)$: by Lemma 1.19, we have $v = u\circ\omega \in C^{*,1}(\mathbb{R}^n_+)$ and

$$[v]_{*,1} \leq \{\|u\|_{C^{*,1}(\mathbb{R}^n_+)} [\omega]_1 + 2^{1/2}[u]_{1/2}[\omega]^{1/2}_{*,2}\} \leq c\|u\|_{C^{*,1}(\mathbb{R}^n_+)}. \tag{3.3}$$

In addition we have

$$v(\xi,s) - v(\xi,0) = u(\omega(\xi,s)) - u(\xi,0) = [u(\omega(\xi,s)) - u(\xi - s\beta^0(\xi),s)] + + [u(\xi - s\beta^0(\xi),s) - u(\xi,0)] \tag{3.4}$$

and hence

$$|v(\xi,s) - v(\xi,0)| \leq [u]_{1/2} |\omega(\xi,s) - (\xi - s\beta^0(\xi),s)|^{1/2} + [u]_{1,\beta}\cdot s;$$

on the other hand by (3.2) we check

$$|\omega(\xi,s) - (\xi - s\beta^0(\xi),s)| = |\beta^0(\xi)|\cdot|-\int_0^s \phi(\sigma)d\sigma + s| \leq \|\beta^0\|_\infty |\int_0^s\int_0^\sigma \phi'(r)drd\sigma| \leq \frac{1}{2}\|\beta^0\|_\infty \|\phi'\|_\infty s^2 = ks^2 \tag{3.5}$$

and therefore

$$|v(\xi,s) - v(\xi,0)| \leq \{2^{-1/2}[u]_{1/2}\|\beta^0\|^{1/2}_\infty \|\phi'\|^{1/2}_\infty + [u]_{1,\beta}\}s.$$

This shows that $v\in C^{*,1}_\nu(\mathbb{R}^n_+)$ and that $\|v\|_{C^{*,1}_\nu(\mathbb{R}^n_+)} \leq c\|u\|_{C^{*,1}_\beta(\mathbb{R}^n_+)}$.

Suppose now that $u\in h^{*,1}_B(\mathbb{R}^n_+)$; if $|\xi-\xi'|^2 + (s-s')^2 \leq r^2$, similarly to (3.3) we easily get

$$[v]_{*,1,\overline{B(\xi,s),r)}\cap\mathbb{R}^n_+} \leq \{[u]_{*,1,\overline{B(\omega(\xi,s),[\omega]_1\cdot r)}\cap\mathbb{R}^n_+}\cdot[\omega]_1 +$$
$$+ 2^{1/2}[u]_{1/2,\overline{B(\omega(\frac{\xi+\xi'}{2},\frac{s+s'}{2}),[\omega]_{*,2}\cdot r^2)}\cap\mathbb{R}^n_+}\cdot[\omega]^{1/2}_{*,2}\} = o(1) \text{ as } r\to 0^+$$

hence $v\in h^{*,1}(\mathbb{R}^n_+)$. Next, by (3.4) and (3.5) we deduce that

$$\left|\frac{v(\xi,s)-v(\xi,0)}{s} - \alpha(\xi)v(\xi,0)\right| \leq [u]_{1/2,\overline{B(\omega(\xi,s),ks^2)}\cap\mathbb{R}^n_+}\cdot k^{1/2}$$
$$+ \left|\frac{u(\xi-s\beta^0(\xi),s)-u(\xi,0)}{s} - \alpha(\xi)u(\xi,0)\right|$$
$$= o(1) \quad \text{as } s\to 0^+$$

which implies $v\in h_N^{*,1}(\mathbb{R}^n_+)$. The proof is complete. □

Theorem 3.1 follows now easily. Fix $\theta\in]0,1[$ and take accordingly f in one of the spaces $C^{0,2\theta}(\mathbb{R}^n_+)$, $C_\beta^{*,1}(\mathbb{R}^n_+)$, $C_B^{1,2\theta-1}(\mathbb{R}^n_+)$ (resp. $h^{0,2\theta}(\mathbb{R}^n_+)$, $h_B^{*,1}(\mathbb{R}^n_+)$, $h_B^{1,2\theta-1}(\mathbb{R}^n_+)$). Then $Tf=f\circ\omega$ is in the corresponding space, specified in Proposition 3.2, and the mapping is continuous. We can now apply Theorem 2.1, obtaining $Tf\in(C_N^2(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta,\infty}$ (resp. $Tf\in(C_N^2(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta}$), with continuous inclusion. By definition, this means there exists a function $v=v(t,\xi,s)\in C^0([0,1]\times\mathbb{R}^n_+)$ satisfying

$$\begin{cases} v(0,\xi,s)=f\circ\omega(\xi,s) & \forall(\xi,s)\in\mathbb{R}^n_+, \\ v\in C_{1-\theta}(]0,1],C^2(\mathbb{R}^n_+)) & (\text{resp. } v\in C_{1-\theta}([0,1],C^2(\mathbb{R}^n_+)) \\ v_t\in C_{1-\theta}(]0,1],C^0(\mathbb{R}^n_+)) & (\text{resp. } v_t\in C_{1-\theta}([0,1],C^0(\mathbb{R}^n_+)), \\ [Nv(t,\cdot,\cdot)](\xi)=0 & \forall\xi\in\mathbb{R}^{n-1},\ \forall t\in]0,1]. \end{cases}$$

Hence if we set $u(t,x,y)=v(t,\omega^{-1}(x,y))$, we easily deduce

$$\begin{cases} u(0,x,y)=f(x,y) & \forall(x,y)\in\mathbb{R}^n_+, \\ u\in C_{1-\theta}(]0,1],C^2(\mathbb{R}^n_+)) & (\text{resp. } u\in C_{1-\theta}([0,1],C^2(\mathbb{R}^n_+))) \\ u_t\in C_{1-\theta}(]0,1],C^0(\mathbb{R}^n_+)) & (\text{resp. } u\in C_{1-\theta}([0,1],C^0(\mathbb{R}^n_+))) \\ [Bu(t,\cdot,\cdot)](x)=0 & \forall x\in\mathbb{R}^{n-1},\ \forall t\in]0,1]; \end{cases}$$

hence $f\in(C_B^2(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta,\infty}$ (resp. $f\in(C_B^2(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta}$) and the continuity of the inclusion follows. Theorem 3.1 is proved. □

4. THE CASE OF A BOUNDED CONNECTED OPEN SET

Let Ω be a bounded connected open set of $\mathbb{R}^n$, $n\geq1$, with boundary $\partial\Omega$ of class C^3. For each $x\in\partial\Omega$ denote by $\nu(x)$ the unit exterior normal at x; then $\nu\in C^2(\partial\Omega,\mathbb{R}^n)$.
Let $\alpha\in C^2(\partial\Omega,\mathbb{R})$ and $\beta\in C^2(\partial\Omega,\mathbb{R}^n)$ be such that

$$\alpha(x)\geq0, \qquad (\beta(x)|\nu(x))_n\geq\delta_0>0 \qquad \forall x\in\partial\Omega,$$

and define the boundary operator

$$[Bu](x)=\alpha(x)u(x)+(\beta(x)|Du(x))_n, \quad x\in\partial\Omega. \tag{4.1}$$

We will use the spaces of functions (see Definitions 1.13 and 1.14) $C_B^2(\bar\Omega)$, $C_B^{1,\alpha}(\bar\Omega)$, $h_B^{1,\alpha}(\bar\Omega)$, $C_\beta^{*,1}(\bar\Omega)$, $h_B^{*,1}(\bar\Omega)$, as well as the Hölder spaces $C^{0,\alpha}(\bar\Omega)$, $h^{0,\alpha}(\bar\Omega)$.
Our goal is to prove the following result:

<u>THEOREM</u> 4.1 - <u>Let $\Omega\subseteq\mathbb{R}^n$ be a bounded connected open set, with $\partial\Omega$ of class C^3; let $\alpha\in C^2(\partial\Omega,\mathbb{R})$, $\beta\in C^2(\partial\Omega,\mathbb{R}^n)$ with $\alpha\geq0$ and $(\beta|\nu)_n\geq\delta_0>0$. If B is the operator defined in (4.1), then the following continuous inclusions hold:</u>

$$(C_B^2(\bar\Omega),C^0(\bar\Omega))_{1-\theta,\infty}\hookrightarrow\begin{cases} C^{0,2\theta}(\bar\Omega) & \underline{\text{if}}\ \theta\in]0,1/2[\\ C_B^{*,1}(\bar\Omega) & \underline{\text{if}}\ \theta=1/2 \\ C_B^{1,2\theta-1}(\bar\Omega) & \underline{\text{if}}\ \theta\in]1/2,1[\end{cases};$$

$$(C_B^2(\bar\Omega),C^0(\bar\Omega))_{1-\theta}\hookrightarrow\begin{cases} h^{0,2\theta}(\bar\Omega) & \underline{\text{if}}\ \theta\in]0,1/2[\\ h_B^{*,1}(\bar\Omega) & \underline{\text{if}}\ \theta=1/2 \\ h_B^{1,2\theta-1}(\bar\Omega) & \underline{\text{if}}\ \theta\in]1/2,1[. \end{cases}$$

<u>Proof</u>. Fix $\theta\in]0,1[$ and, accordingly, take f in $C^{0,2\theta}(\bar\Omega)$, $C_\beta^{*,1}(\bar\Omega)$ or $C_B^{1,2\theta-1}(\bar\Omega)$ (resp. $h^{0,2\theta}(\bar\Omega)$, $h_B^{*,1}(\bar\Omega)$ or $h_B^{1,2\theta-1}(\bar\Omega)$).

According with Definition 1.7 and Lemmata 1.10 and 1.11, we look for a function $w=w(t,x)\in C^0([0,1]\times\overline{\Omega})$ such that:

$$\begin{cases} w(0,x)=f(x) \quad \forall x\in\overline{\Omega}, \\ \sup_{t\in]0,1]} t^{1-\theta}\|w_t(t,\cdot)\|_{C^0(\overline{\Omega})}<\infty \quad (\text{resp. } \lim_{t\to 0^+} t^{1-\theta}\|w_t(t,\cdot)\|_{C^0(\overline{\Omega})}=0), \\ \sup_{t\in]0,1]} t^{1-\theta}\|D^2w(t,\cdot)\|_{C^0(\overline{\Omega})}<\infty \quad (\text{resp. } \lim_{t\to 0^+} t^{1-\theta}\|D^2w(t,\cdot)\|_{C^0(\overline{\Omega})}=0), \\ [Bw(t,\cdot)](x)=0 \quad \forall x\in\partial\Omega,\ \forall t\in]0,1]. \end{cases} \tag{4.2}$$

Our method consists in transforming the given function f, by an usual localization argument, into a finite set of functions $\{F_i,G_j\}$ of two different kinds:

(a) a function F_i, as smooth as f, defined in a ball, with zero boundary conditions;

(b) a function G_j, as smooth as f, defined in the half space $\mathbb{R}^n_+$ and satisfying $\Lambda_jG_j=0$ on the boundary whenever Bf=0 on $\partial\Omega$; here Λ_j is a suitable first-order differential operator of type (4.1).

The localization argument is not completely standard, for it is carried on by the construction of a finite partition of unity in $\overline{\Omega}$ with special properties along $\partial\Omega$: namely, we need that the localizing functions near $\partial\Omega$ transform the boundary operator B into an operator Λ_j of the same kind. This is done by choosing the functions μ_j, localizing near $\partial\Omega$, in such a way that at each $x\in\partial\Omega$ their gradients $D\mu_j(x)$ are orthogonal to the vector $\beta(x)$ appearing in (4.1). This construction is performed in the first step of the proof.

The second step is the verification that the localized functions F_i or G_j in fact satisfy the conditions stated either in case (a) or in case (b). In case (a), F_i has compact support in a ball U_i, so we apply a result of Lunardi [13] to prove that $F_i\in(C^2_0(U_i),C^0_0(U_i))_{1-\theta,\infty}$ (resp. $F_i\in(C^2_0(U_i),C^0_0(U_i))_{1-\theta}$); in case (b) we can apply the results of Section 3, obtaining $G_j\in(C^2_{\Lambda_j}(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta,\infty}$ (resp. $G_j\in(C^2_{\Lambda_j}(\mathbb{R}^n_+),C^0(\mathbb{R}^n_+))_{1-\theta}$).

Thus, according with Definition 1.7, we have $F_i=u_i(0),G_j=v_j(0)$ for some suitable functions $u_i(t),v_j(t)$.

In the final step we show that the function w(t), which is obtained by gluing together the functions $u_i(t)$, $v_j(t)$, in fact satisfies (4.2).

Step 1 - Here we will construct the required partition of unity. We start from the localizing functions near $\partial\Omega$: for each $x_0\in\partial\Omega$ the construction gives a suitable function μ, defined in a certain neighbourhood V of x_0 in $\overline{\Omega}$; a compactness argument then yields a finite number of localizing functions μ_j with the required properties. Next, we complete, in a standard way, the set of localizing functions by a finite number of suitable functions η_i, defined in balls contained in Ω. Finally we normalize the functions η_i,μ_j, obtaining the desired partition of unity.

To begin with, fix $x_0\in\partial\Omega$. Our first goal is to construct two neighbourhoods V", V' of x_0 in $\overline{\Omega}$ with V" $\subseteq$ V', and a function $\mu:\overline{\Omega}\to\mathbb{R}$ with the following properties:

$$\begin{cases} \mu\in C^2(\overline{\Omega}),\ 0\le\mu\le 1 \text{ in } \overline{\Omega}, \\ \mu=0 \text{ in } \overline{\Omega}-V',\quad \mu\equiv 1 \text{ in } V'' \end{cases} \tag{4.3}$$

and

$$(\beta(x)\,|\,D\mu(x))_n = 0 \qquad \forall x\in\partial\Omega. \tag{4.4}$$

To do this, first of all note that, since $\partial\Omega$ is of class C^3, there exists a neighbourhood W of x_0 in $\mathbb{R}^n$ and a diffeomorphism $\psi : W \to \mathbb{R}^n$ of class C^3, such that, denoting $W\cap\overline{\Omega}$ by V:

$$\begin{cases} \psi(V)=\mathbb{R}^n_+ \\ \psi(V\cap\partial\Omega)=\Sigma = \{\xi\in\mathbb{R}^n_+ : \xi_n=0\}, \\ \psi^{-1}\in C^3(\mathbb{R}^n) . \end{cases} \tag{4.5}$$

Set

$$\gamma_h(\xi)= \sum_{r=1}^{n} \frac{\partial\psi^h}{\partial x_r} (\psi^{-1}(\xi))\beta_r(\psi^{-1}(\xi)) \ , \ \xi\in\Sigma , \quad h=1,\dots,n; \tag{4.6}$$

then $\gamma\in C^2(\Sigma,\mathbb{R}^n)$ and it is easily seen that

$$((Du)o\psi^{-1}(\xi)\,|\,\beta o\psi^{-1}(\xi))_n=(D(uo\psi^{-1})(\xi)\,|\,\gamma(\xi))_n \quad \forall\xi\in\Sigma,\ \forall u\in C^1(V). \tag{4.7}$$

Moreover it is not difficult to verify that

$$\gamma_n(\xi)=(\gamma(\xi)\,|\,e^n)_n=-(\beta o\psi^{-1}(\xi)\,|\,\nu o\psi^{-1}(\xi))_n\cdot \frac{|(Dg)o\psi^{-1}(\xi)|}{|D(go\psi^{-1})(\xi)|} \quad \forall\xi\in\Sigma ,$$

which implies

$$\gamma_n(\xi) \le -\delta_1<0 \qquad \forall\xi\in\Sigma . \tag{4.8}$$

Thus, the non-tangential vector $\beta(x)$ is transformed by ψ into the non-tangential vector $\gamma(\xi)$ given by (4.6). Hence we can define

$$\lambda_h(\xi) = -\frac{\gamma_h(\xi)}{\gamma_n(\xi)} \ , \ \xi\in\Sigma \ , \ h=1,\dots,n; \tag{4.9}$$

thus λ is twice continuously differentiable, $\lambda_n(\xi)=-1$ $\forall\xi\in\Sigma$, and by (4.7)

$$((Du)o\psi^{-1}(\xi)\,|\,\beta o\psi^{-1}(\xi))_n=-\gamma_n(\xi)(D(uo\psi^{-1})(\xi)\,|\,\lambda(\xi))_n \tag{4.10}$$

$\forall\xi\in\Sigma$, $\forall u\in C^1(V)$.

Next, we want to transform the vector λ, defined in (4.9), into $-e^n$; hence we need to construct a function $\omega:\mathbb{R}^n_+\to\mathbb{R}^n_+$ just as in

(3.2) of Section 3, with β replaced here by λ. To be more precise, write $\xi^0=(\xi_1,\dots,\xi_{n-1})$ and $\lambda^0(\xi)=(\lambda_1(\xi),\dots,\lambda_{n-1}(\xi))$, so that $\xi=(\xi^0,\xi_n)$ $\forall\xi\in\mathbb{R}^n_+$ and $\lambda(\xi)=(\lambda^0(\xi),-1)$ $\forall\xi\in\Sigma$; then set

$$\omega(\xi)=(\xi^0-\int_0^{\xi_n}\phi(\sigma)d\sigma\cdot\lambda^0(\xi^0,0),\xi_n), \qquad \xi=(\xi^0,\xi_n)\in\mathbb{R}^n_+ ,$$

where, as in Section 3, $\phi\in C^\infty([0,\infty[)$ is a function with support contained in $[0,1]$ and satisfying

$$\phi(0)=1, \quad 0\le\phi(s)<1 \quad \forall s>0, \quad \int_0^1\phi(s)ds=\varepsilon<\|D\lambda^0(\cdot,0)\|_\infty^{-1} .$$

Then $\omega:\mathbb{R}^n_+\to\mathbb{R}^n_+$ is twice differentiable and one-to-one, with ω^{-1} also twice differentiable; note that

$$\omega(\xi)=\xi, \qquad \frac{\partial\omega}{\partial\xi_n}(\xi)=-\lambda(\xi) \qquad \forall\xi=(\xi^0,0)\in\Sigma \tag{4.11}$$

Now let $\theta\in C^\infty([0,\infty[)$ be such that $0\le\theta\le1$, $\theta\equiv0$ outside $[0,1]$, $\theta\equiv1$ in $[0,1/2]$, and set

$$\zeta(\xi)=\theta(|\xi|) \qquad \xi\in\mathbb{R}^n_+ ;$$

then, clearly, $\zeta\in C^\infty(\mathbb{R}^n_+)$, $0\le\zeta\le1$, $\zeta\equiv0$ outside $B^+(0,1)=\{\xi\in\mathbb{R}^n_+ : |\xi|<1\}$, $\xi\equiv1$ in $B^+(0,1/2)$, and moreover

$$\frac{\partial\zeta}{\partial\xi_n}(\xi)=0 \qquad \forall\xi=(\xi^0,0)\in\Sigma \tag{4.12}$$

i.e. the gradient of ζ is orthogonal to $-e^n$. By (4.12) and (4.11) we easily get

$$(D(\zeta o\omega^{-1})(\xi)\,|\,\lambda(\xi))_n=0 \qquad \forall\xi\in\Sigma . \tag{4.13}$$

We are finally ready to define the desired function μ satisfying (4.3) and (4.4): set

$$\mu(x) = \begin{cases} \zeta\circ\omega^{-1}\circ\psi(x) & \text{if } x\in V, \\ 0 & \text{if } x\in\bar\Omega - V; \end{cases} \tag{4.14}$$

then it is easily seen that (4.3) holds with

$$V'=\psi^{-1}(\omega(B^+(0,1))),\quad V''=\psi^{-1}(\omega(B^+(0,1/2))),$$

while by (4.13), (4.9) and (4.7) we get (4.4).

Up to now, for each fixed $x_0\in\partial\Omega$ we have constructed the function μ given by (4.14), which satisfies (4.3) and (4.4), as well as the sets V, V', V'' verifying $V''\subseteq V'\subseteq V$ and the functions ψ,γ,λ given respectively by (4.5), (4.6), (4.9), for which (4.8) and (4.10) hold. By the compactness of $\partial\Omega$, we can select a finite number of points $x_0^j\in\partial\Omega$, $1\le j\le k$, such that the corresponding neighbourhoods V_j'', $1\le j\le k$, cover $\partial\Omega$. Accordingly, we have also selected the corresponding functions $\psi_j,\gamma_j,\lambda_j,\mu_j$ and the sets V_j', V_j''; in particular, for $j=1,\dots,k$ we have $V_j''\subseteq V_j'\subseteq V_j$ and:

$$\psi_j(V_j)=\mathbb{R}^n_+,\quad \psi_j(V_j\cap\partial\Omega)=\Sigma,\quad \psi_j^{-1}\in C^3(\mathbb{R}^n) \tag{4.15}$$

$$\gamma_j(\xi)=(D\psi_j)(\psi_j^{-1}(\xi))\cdot\beta(\psi_j^{-1}(\xi)) \qquad \forall\xi\in\Sigma \tag{4.16}$$

$$(\gamma_j)_n(\xi)=(\gamma_j(\xi)|e^n)_n \le -\delta_1<0 \qquad \forall\xi\in\Sigma \tag{4.17}$$

$$\lambda_j(\xi) = -\frac{1}{(\gamma_j)_n(\xi)}\gamma_j(\xi) \qquad \forall\xi\in\Sigma \tag{4.18}$$

$$(\beta\circ\psi_j^{-1}(\xi)\,|\,(Du)\circ\psi_j^{-1}(\xi))_n=-(\gamma_j)_n(\xi)(\lambda_j(\xi)\,|\,D(u\circ\psi_j^{-1})(\xi))_n \tag{4.19}$$

$\forall\xi\in\Sigma$, $\forall u\in C^1(V_j)$

$$\begin{cases} \mu_j\in C^2(\bar\Omega),\quad 0\le\mu_j\le 1 \\ \mu_j\equiv 0 \text{ in } \bar\Omega-V_j',\quad \mu_j\equiv 1 \text{ in } V_j'' \end{cases} \tag{4.20}$$

$$(D\mu_j(x)\,|\,\beta(x))_n=0 \qquad \forall x\in\partial\Omega . \tag{4.21}$$

The functions μ_j are the required localizing functions near $\partial\Omega$. To complete the set of localizing functions, we cover the compact set $\bar\Omega - \bigcup_{j=1}^{k} V_j''$ by a finite set of open balls $\{U_i, 1\le i\le m\}$ such that $U_i\subset\subset\Omega$; next, we choose a refinement $\{U_i'', 1\le i\le m\}$ of $\{U_i\}$, such that the family $\{U_i'', V_j'', 1\le i\le m, 1\le j\le k\}$ still covers $\bar\Omega$, and select another family of open sets $\{U_i', 1\le i\le m\}$ satisfying $U_i''\subset\subset U_i'\subset\subset U_i$. Finally, we take for $i=1,\dots,m$ a function $\eta_i\in C^2(\mathbb{R}^n)$ such that $0\le\eta_i\le 1$, $\eta_i\equiv 0$ outside U_i', $\eta_i\equiv 1$ in U_i''.
The set $\{\mu_j,\eta_i, 1\le j\le k, 1\le i\le m\}$ is the required set of localizing functions. To conclude Step 1, we have only to normalize the functions μ_j, η_i in order to get a partition of unity. Thus, we set

$$\begin{aligned} \sigma_i(x)&=\Big[\sum_{r=1}^{m}\eta_r(x)+\sum_{s=1}^{k}\mu_s(x)\Big]^{-1}\eta_i(x), \\ \rho_j(x)&=\Big[\sum_{r=1}^{m}\eta_r(x)+\sum_{s=1}^{k}\mu_s(x)\Big]^{-1}\mu_j(x), \qquad x\in\bar\Omega. \end{aligned} \tag{4.22}$$

Clearly σ_i, $\rho_j\in C^2(\bar\Omega)$ for $i=1,\dots,m$; $j=1,\dots,k$, and

$$\begin{cases} \sum_{i=1}^{m}\sigma_i(x)+\sum_{j=1}^{k}\rho_j(x)\equiv 1 \text{ in } \bar\Omega \\ \sigma_i\equiv 0 \text{ in } \bar\Omega-U_i',\ i=1,\dots,m, \\ \rho_j\equiv 0 \text{ in } \bar\Omega-V_j',\ j=1,\dots,k; \end{cases} \tag{4.23}$$

in addition, by (4.21) it follows that

$$(\beta(x)\,|\,D\rho_j(x))_n=0 \qquad \forall x\in\partial\Omega,\ j=1,\dots,k. \tag{4.24}$$

Step 1 is finished.

<u>Step 2</u>. Here we reduce our problem to the cases (a) and (b)

mentioned at the beginning of the proof. Let

$$f\in\begin{cases} C^{0,2\theta}(\overline{\Omega}) \ \text{(resp. } h^{0,2\theta}(\overline{\Omega})) \text{ if } \theta\in]0,1/2[\\ C^{*,1}_{\beta}(\overline{\Omega}) \ \text{(resp. } h^{*,1}_{B}(\overline{\Omega})) \text{ if } \theta=1/2 \\ C^{1,2\theta-1}_{B}(\overline{\Omega}) \ \text{(resp. } h^{1,2\theta-1}_{B}(\overline{\Omega})) \text{ if } \theta\in]1/2,1[\ ; \end{cases}$$

we localize f by setting

$$F_i(x)=\begin{cases} \sigma_i(x)f(x) & \text{if } x\in U_i' \\ & ,i=1,\dots,m\ , \\ 0 & \text{if } x\in\overline{\Omega}-U_i' \end{cases} \tag{4.25}$$

$$G_j(\xi)=[\rho_j o\psi_j^{-1}(\xi)]\cdot[fo\psi_j^{-1}(\xi)]\ ,\quad \xi\in\mathbb{R}^n_+\ ,\ j=1,\dots,k\ . \tag{4.26}$$

Define

$$[\Lambda_j u](\xi)=-\frac{\alpha o\psi_j^{-1}(\xi)}{(\gamma_j)_n(\xi)}\,u(\xi)+(\lambda_j(\xi)|Du(\xi))_n,\ \ \xi\in\Sigma\ ,\ j=1,\dots,k; \tag{4.27}$$

then we have:

<u>PROPOSITION</u> 4.2 <u>For</u> i=1,...,m <u>and</u> j=1,...,k <u>we have</u>:

$$F_i\big|_{U_i'}\in\begin{cases} C_0^{0,2\theta}(\overline{U}_i) \ \underline{\text{(resp.}}\ h_0^{0,2\theta}(\overline{U}_i)) \ \underline{\text{if}}\ \theta\in]0,1/2[\\ C_0^{*,1}(\overline{U}_i) \ \underline{\text{(resp.}}\ h_0^{*,1}(\overline{U}_i)) \ \underline{\text{if}}\ \theta=1/2 \\ C_0^{1,2\theta-1}(\overline{U}_i) \ \underline{\text{(resp.}}\ h_0^{1,2\theta-1}(\overline{U}_i)) \ \underline{\text{if}}\ \theta\in]1/2,1[\ , \end{cases}$$

$$G_j\in\begin{cases} C^{0,2\theta}(\mathbb{R}^n_+) \ \underline{\text{(resp.}}\ h^{0,2\theta}(\mathbb{R}^n_+)) \ \underline{\text{if}}\ \theta\in]0,1/2[\\ C^{*,1}_{\Lambda_j}(\mathbb{R}^n_+) \ \underline{\text{(resp.}}\ h^{*,1}_{\Lambda_j}(\mathbb{R}^n_+)) \ \underline{\text{if}}\ \theta=1/2 \\ C^{1,2\theta-1}_{\Lambda_j}(\mathbb{R}^n_+) \ \underline{\text{(resp.}}\ h^{1,2\theta-1}_{\Lambda_j}(\mathbb{R}^n_+)) \ \underline{\text{if}}\ \theta\in]1/2,1[\ . \end{cases}$$

<u>Moreover</u>

$$\sum_{i=1}^{m}\|F_i\big|_{U_i}\|+\sum_{j=1}^{k}\|G_j\|\le c\,\|f\|$$

<u>in the corresponding norms</u>.

<u>Proof</u>. Obviously, $F_i\big|_{U_i}$ has compact support in U_i ,and its regularity follows easily (for the case θ=1/2 recall Lemma 1.18), as well as the required estimate.

About G_j, the result is obvious if θ∈]0,1/2[. Suppose now θ=1/2; by Lemmata 1.19 and 1.18 it is clear that $G_j\in C^{*,1}(\mathbb{R}^n_+)$ (resp. $G_j\in h^{*,1}(\mathbb{R}^n_+)$) and $\|G_j\|_{C^{*,1}(\mathbb{R}^n_+)}\le c\,\|f\|_{C^{*,1}_{\beta}(\overline{\Omega})}$; so we have only to show that

$$\sup\left\{\frac{|G_j(x-\sigma\lambda_j(\xi))-G_j(\xi)|}{\sigma};\xi\in\Sigma\ ,\sigma>0\right\}\le c\|f\|_{C^{*,1}_{\beta}(\overline{\Omega})}\quad\text{if } f\in C^{*,1}_{\beta}(\overline{\Omega}), \tag{4.28}$$

and

$$\lim_{\sigma\to 0^+}\frac{G_j(\xi-\sigma\lambda_j(\xi))-G_j(\xi)}{\sigma}=-\frac{\alpha o\psi_j^{-1}(\xi)}{(\gamma_j)_n(\xi)}\,G_j(\xi)\quad\forall\xi\in\Sigma\ \text{if } f\in h^{*,1}_{B}(\overline{\Omega}). \tag{4.29}$$

First, we observe that $\psi_j^{-1}\in C^3(\mathbb{R}^n)$ and that, by (4.16) and (4.18), we have

$$D\psi_j^{-1}(\xi)\cdot\lambda_j(\xi)=-\frac{1}{(\gamma_j)_n(\xi)}\,\beta(\psi_j^{-1}(\xi))\qquad\forall\xi\in\Sigma\ , \tag{4.30}$$

hence by Taylor's formula

$$\left|\psi_j^{-1}(\xi-\sigma\lambda_j(\xi))-\psi_j^{-1}(\xi)-\sigma\,\frac{\beta(\psi_j^{-1}(\xi))}{(\gamma_j)_n(\xi)}\right|\le\|D^2\psi_j^{-1}\|_\infty\|\lambda_j\|_\infty^2\sigma^2 \tag{4.31}$$

∀σ≥0, ∀ξ∈Σ .

Next, by (4.30) and (4.24),

$$\begin{aligned}(D(\rho_j o\psi_j^{-1})(\xi)|\lambda_j(\xi))_n&=((D\rho_j)o\psi_j^{-1}(\xi)|(D\psi_j^{-1})(\xi)\cdot\lambda_j(\xi))_n=\\ &=-\frac{1}{(\gamma_j)_n(\xi)}((D\rho_j)o\psi_j^{-1}(\xi)|\beta o\psi_j^{-1}(\xi))_n=0\quad\forall\xi\in\Sigma.\end{aligned} \tag{4.32}$$

Now observe that, by (4.17) and Lemma 1.15, for each $\xi\in\Sigma$ the point $\psi_j^{-1}(\xi) + \frac{\sigma}{(\gamma_j)_n(\xi)}\beta(\psi_j^{-1}(\xi))$ certainly lies in $\bar\Omega$, provided σ is sufficiently small, say $\sigma\in]0,\sigma_0]$. Consequently we can write

$$\frac{G_j(\xi-\sigma\lambda_j(\xi))-G_j(\xi)}{\sigma} = \frac{1}{\sigma}[\rho_j o\psi_j^{-1}(\xi-\sigma\lambda_j(\xi))-\rho_j o\psi_j^{-1}(\xi)]\, fo\psi_j^{-1}(\xi-\sigma\lambda_j(\xi))+$$

$$+\rho_j o\psi_j^{-1}(\xi)\cdot\frac{1}{\sigma}[f(\psi_j^{-1}(\xi-\sigma\lambda_j(\xi)))-f(\psi_j^{-1}(\xi)+\frac{\sigma}{(\gamma_j)_n(\xi)}\beta(\psi_j^{-1}(\xi)))] \quad (4.33)$$

$$+\rho_j o\psi_j^{-1}(\xi)\cdot\frac{1}{\sigma}[f(\psi_j^{-1}(\xi)+\frac{\sigma}{(\gamma_j)_n(\xi)}\beta(\psi_j^{-1}(\xi)))-f(\psi_j^{-1}(\xi))].$$

Hence, if $f\in C_\beta^{*,1}(\bar\Omega)$ by (4.31) we easily deduce that

$$\left|\frac{G_j(\xi-\sigma\lambda_j(\xi))-G_j(\xi)}{\sigma}\right| \le c\{\|f\|_\infty+[f]_{1/2}+[f]_{1,\beta}\}\le c\|f\|_{C_\beta^{*,1}(\bar\Omega)}$$

$\forall\xi\in\Sigma$, $\forall\sigma\in]0,\sigma_0]$;

by Remark 1.4(iii), this yields (4.28), so that $G_j\in C_{\lambda_j}^{*,1}(\mathbb{R}_+^n)$ and the required estimate holds.

Suppose now, is addition, that $f\in h_B^{*,1}(\bar\Omega)$: then by (4.33), (4.32) and (4.31), recalling that in particular $f\in h^{0,1/2}(\bar\Omega)$, we get as $\sigma\to 0^+$

$$\frac{G_j(\xi-\sigma\lambda_j(\xi))-G_j(\xi)}{\sigma} = o(1)\cdot[fo\psi_j^{-1}(\xi)+o(1)]+\rho_j o\psi_j^{-1}(\xi)\cdot o(1)+$$

$$+\rho_j o\psi_j^{-1}(\xi)[-\frac{\alpha(\psi_j^{-1}(\xi))}{(\gamma_j)_n(\xi)}f(\psi_j^{-1}(\xi))+o(1)] = -\frac{\alpha(\psi_j^{-1}(\xi))}{(\gamma_j)_n(\xi)}G_j(\xi)+o(1)$$

$\forall\xi\in\Sigma$.

This proves (4.29), and hence $G_j\in h_{\Lambda_j}^{*,1}(\mathbb{R}_+^n)$. The proof for the case $\theta=1/2$ is complete.

Suppose finally $\theta\in]1/2,1[$; it is easy to see that $G_j\in C^{1,2\theta-1}(\mathbb{R}_+^n)$ (resp. $G_j\in h^{1,2\theta-1}(\mathbb{R}_+^n)$), and that

the required estimate holds. Hence it is enough to verify that $\Lambda_j G_j=0$, with Λ_j given by (4.27). Now if $\xi\in\Sigma$ we get, by (4.19), (4.24) and (4.32)

$$[\Lambda_j G_j](\xi) = -\frac{1}{(\gamma_j)_n(\xi)}\{[\alpha o\psi_j^{-1}(\xi) fo\psi_j^{-1}(\xi) -$$

$$-(\gamma_j)_n(\xi)(\lambda_j(\xi)|D(fo\psi_j^{-1})(\xi))_n]\rho_j o\psi_j^{-1}(\xi) -$$

$$-(\gamma_j)_n(\xi)(\lambda_j(\xi)|D(\rho_j o\psi_j^{-1})(\xi))_n fo\psi_j^{-1})\} =$$

$$= -\frac{1}{(\gamma_j)_n(\xi)}\{[\alpha o\psi_j^{-1}(\xi) fo\psi_j^{-1}(\xi)+(\beta o\psi_j^{-1}(\xi)|(Df)o\psi_j^{-1}(\xi))_n]\rho_j o\psi_j^{-1}(\xi)\}$$

$$= -\frac{1}{(\gamma_j)_n(\xi)}\{[Bf](x)\rho_j(x)\}_{x=\psi_j^{-1}(\xi)} = 0.$$

This shows that $G_j\in C_{\Lambda_j}^{1,2\theta-1}(\mathbb{R}_+^n)$ (resp. $G_j\in h_{\Lambda_j}^{1,2\theta-1}(\mathbb{R}_+^n)$), and the proof is complete. □

By a result of Lunardi ([13], Proposition 2.5) we have for $i=1,\ldots,m$

$F_i|_{U_i}\in(C_0^2(\bar U_i),C_0^0(\bar U_i))_{1-\theta,\infty}$ (resp. $F_i|_{U_i}\in(C_0^2(\bar U_i),C_0^0(\bar U_i))_{1-\theta}$),

while by Theorem 3.1 we get for $j=1,\ldots,k$

$G_j\in(C_{\Lambda_j}^2(\mathbb{R}_+^n),C^0(\mathbb{R}_+^n))_{1-\theta,\infty}$ (resp. $G_j\in(C_{\Lambda_j}^2(\mathbb{R}_+^n),C^0(\mathbb{R}_+^n))_{1-\theta}$),

and in addition

$$\sum_{i=1}^{m} \|F_i|_{U_i}\|_{(C_0^2(\bar U_i),\,C_0^0(\bar U))_{1-\theta,\infty}} + \sum_{j=1}^{k} \|G_j\|_{(C^2_{\Lambda_j}(\mathbb{R}^n_+),\,C^0(\mathbb{R}^n_+))_{1-\theta,\infty}} \le \begin{cases} c\|f\|_{C^{0,2\theta}(\bar\Omega)} & \text{if } \theta\in\,]0,1/2[\\ c\|f\|_{C^{*,1}_{\beta}(\bar\Omega)} & \text{if } \theta=1/2 \\ c\|f\|_{C^{1,2\theta-1}(\bar\Omega)} & \text{if } \theta\in\,]1/2,1[\,. \end{cases} \tag{4.34}$$

Hence by Definition 1.7 there exist functions z_i, v_j satisfying

$$\begin{cases} z_i\in C^0([0,1],C_0^0(\bar U_i)) \quad , \quad z_i(0)=F_i|_{U_i} \\ z_i\in C_{1-\theta}(]0,1],C_0^2(\bar U_i))\,(\text{resp. } z_i\in C_{1-\theta}([0,1],C_0^2(\bar U_i)))\quad i=1,\dots,m \\ z_i'\in C_{1-\theta}(]0,1],C_0^0(\bar U_i))\,(\text{resp. } z_i'\in C_{1-\theta}([0,1],C_0^0(\bar U_i))) \end{cases} \tag{4.35}$$

and

$$\begin{cases} v_j\in C^0([0,1],C^0(\mathbb{R}^n_+)) \ , \ v_j(0)=G_j \\ v_j\in C_{1-\theta}(]0,1],C^2_{\Lambda_j}(\mathbb{R}^n_+))\ (\text{resp. } v_j\in C_{1-\theta}([0,1],C^2_{\Lambda_j}(\mathbb{R}^n_+)))\ j=1,\dots,k \\ v_j'\in C_{1-\theta}(]0,1],C^0(\mathbb{R}^n_+))\ (\text{resp. } v_j'\in C_{1-\theta}([0,1],C^0(\mathbb{R}^n_+)))\ . \end{cases} \tag{4.36}$$

Finally we extend the functions $z_i(t,\cdot)$ to the whole $\bar\Omega$ by defining

$$u_i(t,x)=\begin{cases} \tau_i(x)z_i(t,x) & \text{if } t\in[0,1],\,x\in U_i \\ 0 & \text{if } t\in[0,1],\,x\in\bar\Omega-U_i \end{cases}\ ,\ i=1,\dots,m$$

where for $i=1,\dots,m$, $\tau_i\in C^\infty(\bar\Omega,\mathbb{R})$ is a function with support contained in U_i and such that $\tau_i\equiv 1$ in U_i'. (Compare with (4.23) and (4.25)). It is clear that

$$\begin{cases} u_i\in C^0([0,1],C_0^0(\bar\Omega)),\ u_i(0)=F_i \\ u_i\in C_{1-\theta}(]0,1],C_0^2(\bar\Omega))\ \ (\text{resp. } u_i\in C_{1-\theta}([0,1],C_0^2(\bar\Omega))) \\ u_i'\in C_{1-\theta}(]0,1],C_0^0(\bar\Omega))\ \ (\text{resp. } u_i'\in C_{1-\theta}([0,1],C_0^0(\bar\Omega))) \\ u_i(t,\cdot)|_{\partial\Omega}=0,\ Du_i(t,\cdot)|_{\partial\Omega}=0\quad \forall t\in\,]0,1]\,. \end{cases} \tag{4.37}$$

This concludes Step 2.

<u>Step</u> 3. In order to construct a function $w(t)$ satisfying (4.2) we just glue together the functions (4.37) and (4.36), setting

$$w(t,x)=\sum_{i=1}^{m} u_i(t,x)+\sum_{j=1}^{k} v_j(t,\psi_j(x)),\quad t\in[0,1],\ x\in\bar\Omega\,. \tag{4.38}$$

We have to show that (4.2) holds. By (4.25), (4.26), (4.37) and (4.36), recalling (4.22) and (4.23), it is clear that $w\in C^0([0,1]\times\bar\Omega)$ and $w(0,x)=f(x)\ \forall x\in\bar\Omega$.
By (4.38) it is also clear that $w_t, D^2w\in C_{1-\theta}(]0,1],C^0(\bar\Omega))$ (resp. $C_{1-\theta}([0,1],C^0(\bar\Omega))$) and, by (4.34)

$$\sup_{t\in]0,1]} t^{1-\theta}\|w_t(t,\cdot)\|_{C^0(\bar\Omega)} + \sup_{t\in]0,1]} t^{1-\theta}\|D^2w(t,\cdot)\|_{C^0(\bar\Omega)} \le \begin{cases} c\|f\|_{C^{0,2\theta}(\bar\Omega)} & \text{if } \theta\in\,]0,1/2[\\ c\|f\|_{C^{*,1}_{\beta}(\bar\Omega)} & \text{if } \theta=1/2 \\ c\|f\|_{C^{1,2\theta-1}(\bar\Omega)} & \text{if } \theta\in\,]1/2,1[\,. \end{cases} \tag{4.39}$$

It remains to verify that $Bw(t,\cdot)=0\ \forall t\in\,]0,1]$. Now, for each $x\in\partial\Omega$ we get

$$[Bw(t,\cdot)](x)=\alpha(x)w(t,x)+(\beta(x)|Dw(x))_n =$$

$$=\sum_{j=1}^{k}\rho_j(x)[\alpha(x)v_j(t,\psi_j(x))+(\beta(x)|D[v_j(t,\psi_j(x))])_n]+$$

$$+\sum_{j=1}^{k}(\beta(x)|D\rho_j(x))_n\cdot v_j(t,\psi_j(x));$$

and recalling (4.36), (4.24) and (4.27)

$$[Bw(t,\cdot)](x)=\sum_{j=1}^{k}\rho_j(x)[\alpha(x)\cdot v_j(t,\psi_j(x))-(\gamma_j)_n o\psi_j(x)\cdot$$

$$\cdot(\lambda_j o\psi_j(x)|(Dv_j)(t,\psi_j(x)))_n]=$$

$$=-\sum_{j=1}^{k}[(\gamma_j)_n o\psi_j(x)]\rho_j(x)[\Lambda_j v_j(t,\cdot)]o\psi_j(x)=0$$

since $v_j(t,\cdot)\in C^2_{\Lambda_j}(\mathbb{R}^n_+)$.

Thus we have shown that the function w defined in (4.37) satisfies (4.2). Hence $f\in(C^2_B(\bar\Omega),C^0(\bar\Omega))_{1-\theta,\infty}$ (resp. $f\in(C^2_B(\bar\Omega),C^0(\bar\Omega))_{1-\theta}$), and

$$\|f\|_{(C^2_B(\bar\Omega),C^0(\bar\Omega))_{1-\theta,\infty}} \le \begin{cases} c\|f\|_{C^{0,2\theta}(\bar\Omega)} & \text{if } \theta\in]0,1/2[\\ c\|f\|_{C^{*,1}_{\beta}(\bar\Omega)} & \text{if } \theta=1/2 \\ c\|f\|_{C^{1,2\theta-1}(\bar\Omega)} & \text{if } \theta\in]1/2,1[. \end{cases}$$

Theorem 4.1 is proved. □

5. THE REVERSE INCLUSION

Let Ω be a bounded open set of $\mathbb{R}^n$, $n\ge 1$, with boundary $\partial\Omega$ of class C^3. Consider the differential operator with complex-valued coefficients, defined by

$$[Eu](x) = \sum_{ij=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}(x)+\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x)+c(x)u(x), \quad (5.1)$$

$x\in\bar\Omega$,

under the following assumptions:

(A.1) - (Strong uniform ellipticity). There exists $\nu>0$ such that

$$\mathrm{Re}\sum_{ij=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2 \qquad \forall\xi\in\mathbb{R}^n,\ \forall x\in\bar\Omega.$$

(A.2) - The coefficients a_{ij}, b_i, c belong to $C^0(\bar\Omega,\mathbb{C})$ for $i,j=1,\dots,n$.

Consider also the boundary differential operator with real-valued coefficients, defined by

$$[Bu](x) = \alpha(x)u(x) + (\beta(x)|Du(x))_n \quad ,x\in\partial\Omega \qquad (5.2)$$

under the following assumptions:

(B.1) - Denote by $\nu(x)$ the unit exterior normal vector at $x\in\partial\Omega$; there exists $\delta>0$ such that

$$\alpha(x)\ge 0,\quad (\beta(x)|\nu(x))_n \ge \delta \qquad \forall x\in\partial\Omega.$$

(B.2) - The coefficients α, β_i belong to $C^2(\partial\Omega,\mathbb{R})$ for $i=1,\dots,n$.

Under these hypotheses the pair $\{E, B\}$ is a very special example of the situation considered by Stewart [20]. Hence if we set

$$\begin{cases} D(A)=\{u\in\bigcap_{q\in[1,\infty[} H^{2,q}(\Omega):Eu\in C^0(\bar\Omega),Bu=0\} \\ Au=[Eu](\cdot), \end{cases} \qquad (5.3)$$

by Theorem 1 of [20] we get that A is the infinitesimal generator of an analytic semi-group in the space $C^0(\bar\Omega)$; more precisely we have:

<u>LEMMA</u> 5.1 - <u>Let Ω be a bounded connected open set of $\mathbb{R}^n$, with</u>

$\partial\Omega$ of class C^3; suppose that the operators E,B defined by (5.1) and (5.2) satisfy (A.1), (A.2), (B.1) and (B.2). There exists $\lambda_0>0$ such that $[\lambda_0,+\infty[$ is contained in the resolvent set of the operator A defined by (5.3); moreover if $q>n$ there exists c_0, $k_0>0$ such that for each $\lambda\geq\lambda_0$ the following estimate holds:

$$\lambda\|u\|_{C^0(\overline{\Omega})}+\lambda^{1/2}\|Du\|_{C^0(\overline{\Omega})}+\lambda^{n/2q}\sup_{x_0\in\overline{\Omega}}\|D^2u\|_{L^q(\Omega(x_0,r_\lambda))}\leq$$

$$\leq k_0\|g\|_{C^0(\overline{\Omega})}\qquad\forall g\in C^0(\overline{\Omega}),$$

where $u=R(\lambda,A)g$, $\Omega(x_0,r_\lambda)=\{x\in\Omega:|x-x_0|<r_\lambda\}$ and $r_\lambda=c_0\lambda^{-1/2}$.

Proof. See Theorem 1 of [20]; see also Acquistapace-Terreni [1], Lemma 7.7. □

By Remark 1.9 it follows that a norm in the intermediate space $D_A(\theta,\infty)=(D(A),C^0(\overline{\Omega}))_{1-\theta,\infty}$ is given by

$$\|f\|_\theta=\|f\|_{C^0(\overline{\Omega})}+|f|_\theta\ ,$$

where

$$|f|_\theta=\sup_{\lambda\geq\lambda_0}\|\lambda^\theta AR(\lambda,A)f\|_{C^0(\overline{\Omega})}.$$

In addition if we set

$$|f|_{\theta,M}=\sup_{\lambda\geq M}\|\lambda^\theta AR(\lambda,A)f\|_{C^0(\overline{\Omega})}\ ,\qquad M>\lambda_0\ ,$$

then

$$D_A(\theta)=(D(A),C^0(\overline{\Omega}))_{1-\theta}=\{f\in D_A(\theta,\infty):\lim_{M\to\infty}|f|_{\theta,M}=0\}.$$

Obviously we have also

$$|f|_{\theta,M}\leq|f|_\theta\leq\|f\|_\theta\qquad\forall M\geq\lambda_0,\ \forall f\in D_A(\theta,\infty).$$

This section is concerned with the proof of the following result:

THEOREM 5.2 - Under the assumptions of Lemma 5.1, if $\theta\in]0,1[$ the following continuous inclusions hold:

$$D_A(\theta,\infty)\hookrightarrow\begin{cases}C^{0,2\theta}(\overline{\Omega}) & \text{if } \theta\in]0,1/2[\\ C_B^{*,1}(\overline{\Omega}) & \text{if } \theta=1/2\\ C_B^{1,2\theta-1}(\overline{\Omega}) & \text{if } \theta\in]1/2,1[\ ;\end{cases}$$

$$D_A(\theta)\hookrightarrow\begin{cases}h^{0,2\theta}(\overline{\Omega}) & \text{if } \theta\in]0,1/2[\\ h_B^{*,1}(\overline{\Omega}) & \text{if } \theta=1/2\\ h_B^{1,2\theta-1}(\overline{\Omega}) & \text{if } \theta\in]1/2,1[\ .\end{cases}$$

Proof. Let $f\in D_A(\theta,\infty)$ and set $t_0=\dfrac{c_0^2\wedge 1}{\lambda_0}$, where c_0 and λ_0 are defined in Lemma 5.1.

For each $t\in[0,t_0]$ let us consider the auxiliary function defined by

$$u(t)=\begin{cases}\dfrac{1}{t}R(\dfrac{1}{t},A)f & \text{if } t\in]0,t_0]\\ f & \text{if } t=0.\end{cases}\tag{5.4}$$

In the next proposition we list the main properties of the function (5.4).

PROPOSITION 5.3 - Under the assumptions of Lemma 5.1, let $f\in D_A(\theta,\infty)$, $\theta\in]0,1[$, and let u be defined by (5.4). Then:

(i) $u\in C^0([0,t_0],C^0(\overline{\Omega}))$ and $u(t)\in D(A)$ $\forall t\in]0,t_0]$.

(ii) $u:]0,t_0]\to D(A)$ is continuously differentiable and

$$u_t(t)=R(\tfrac{1}{t},A)[\tfrac{1}{t^2}AR(\tfrac{1}{t},A)f]\qquad\forall t\in]0,t_0].$$

(iii) If $q>n$ there exists $k_1>0$ such that

$$\|u(t)\|_{C^0(\bar\Omega)}+t^{1/2}\|Du(t)\|_{C^0(\bar\Omega)}+t^{1-n/2q}\sup_{x_0\in\bar\Omega}\|D^2u(t)\|_{L^q(\Omega(x_0,t^{1/2}))}\le$$
$$\le k_1\|f\|_{C^0(\bar\Omega)}\quad \forall t\in[0,t_0].$$

(iv) If $q>n$ there exists $k_2>0$ such that

$$t^{1-\theta}\|u_t(t)\|_{C^0(\bar\Omega)}+t^{3/2-\theta}\|Du_t(t)\|_{C^0(\bar\Omega)}+$$
$$+t^{2-\theta-n/2q}\sup_{x_0\in\bar\Omega}\|D^2u_t(t)\|_{L^q(\Omega(x_0,t^{1/2}))}\le k_2|f|_{\theta,t^{-1}}\quad \forall t\in]0,t_0].$$

(v) $u\in C^{0,\theta}([0,t_0],C^0(\bar\Omega))$ and there exists $k_3>0$ such that

$$\|u(t)-u(r)\|_{C^0(\bar\Omega)}\le\frac{k_3}{\theta}(t-r)^\theta|f|_{\theta,t^{-1}},\quad 0\le r<t\le t_0.$$

(vi) If $\theta\in]0,1/2[$, for each $\varepsilon\in]0,t_0]$ we have

$$t^{1/2-\theta}\|Du(t)\|_{C^0(\bar\Omega)}\le\frac{k_1}{\varepsilon^{1/2}}t^{1/2-\theta}\|f\|_{C^0(\bar\Omega)}+\frac{k_2}{\frac12-\theta}|f|_{\theta,\varepsilon^{-1}}\quad \forall t\in]0,\varepsilon].$$

(vii) If $\theta\in]1/2,1[$, $Du\in C^{0,\theta-1/2}([0,t_0],C^0(\bar\Omega))$ and there exists $k_4>0$ such that

$$\|Du(t)-Du(r))\|_{C^0(\bar\Omega)}\le\frac{k_4}{\theta-1/2}(t-r)^{\theta-1/2}|f|_{\theta,t^{-1}},\quad 0\le r<t\le t_0.$$

(viii) If $q>\frac12 n(1-\theta)^{-1}$, for each $\varepsilon\in]0,t_0]$ we have

$$t^{1-\theta-n/2q}\sup_{x_0\in\bar\Omega}\|D^2u(t)\|_{L^q(\Omega(x_0,t^{1/2}))}\le$$
$$\le\frac{k_1}{\varepsilon^{1-n/2q}}t^{1-\theta-n/2q}\|f\|_{C^0(\bar\Omega)}+\frac{k_2}{1-\theta-n/2q}|f|_{\theta,\varepsilon^{-1}}\quad \forall t\in]0,\varepsilon].$$

Proof. (i) For each $t\in]0,t_0]$ we have $\frac1t\ge\lambda_0$, so that $R(1/t,A)$ is defined; as $f\in D_A(\theta,\infty)\subseteq\overline{D(A)}$, the result is an obvious consequence of the properties of the resolvent.
(ii) It follows easily by a straightforward computation.
(iii) Set $\lambda=c_0^2/t$, $t\in]0,t_0]$; then $\lambda\in[\lambda_0,+\infty[$ and the estimate follows easily by Lemma 5.1 with $g=t^{-1}\cdot f$.
(iv) Set again $\lambda=c_0^2/t$; by applying Lemma 5.1 with $g=t^{-2}AR(1/t,A)f$ we obtain

$$t^{-1}\|u_t(t)\|_{C^0(\bar\Omega)}+t^{-1/2}\|Du_t(t)\|_{C^0(\bar\Omega)}+$$
$$+t^{-n/2q}\sup_{x_0\in\bar\Omega}\|D^2u_t(t)\|_{L^q(\Omega(x_0,t^{1/2}))}\le k_0\|t^{-2}AR(1/t,A)f\|_{C^0(\bar\Omega)}$$
$$\forall t\in]0,t_0]$$

and the result follows.
(v) For each $x\in\bar\Omega$ we get by (iii)

$$|u(t,x)-u(r,x)|=\left|\int_r^t u_s(s,x)ds\right|\le\int_r^t\frac{k_1}{s^{1-\theta}}|f|_{\theta,s^{-1}}ds\le$$
$$\le\frac{k_1}{\theta}(t-r)^\theta|f|_{\theta,t^{-1}},\quad 0<r<t\le t_0,$$

and (v) follows.
(vi) Let $t\in]0,\varepsilon]$, $x\in\bar\Omega$. By (iv)

$$|Du(t,x)-Du(\varepsilon,x)|=\left|\int_t^\varepsilon Du_s(s,x)ds\right|\le\int_t^\varepsilon\frac{k_2}{s^{3/2-\theta}}|f|_{\theta,s^{-1}}ds\le$$
$$\le k_2\frac{t^{\theta-1/2}}{1/2-\theta}|f|_{\theta,\varepsilon^{-1}},$$

hence, by (iii)

$$|Du(t,x)|\le|Du(\varepsilon,x)|+|Du(t,x)-Du(\varepsilon,x)|\le k_1\varepsilon^{-1/2}\|f\|_{C^0(\bar\Omega)}+$$
$$+\frac{k_2}{1/2-\theta}t^{\theta-1/2}|f|_{\theta,\varepsilon^{-1}}$$

which implies (vi).

(vii) We have by (iv) for $0<r<t\le t_0$

$$\|Du(t)-Du(r)\|_{C^0(\overline{\Omega})} \le \int_r^t \|Du_s(s)\|_{C^0(\overline{\Omega})}\, ds \le$$

$$\le k_2 |f|_{\theta,t^{-1}} \int_r^t \frac{ds}{s^{3/2-\theta}} \le \frac{k_2}{\theta-1/2}\,(t-r)^{\theta-1/2} |f|_{\theta,\varepsilon^{-1}}$$

and (vii) is proved.

(viii) Let $t\in]0,\varepsilon]$. We have

$$D^2u(t,x)-D^2u(\varepsilon,x)= \int_t^{\varepsilon} D^2u_s(s,x)\,ds \quad \text{for almost all } x\in\Omega,$$

and therefore by (iii)

$$\|D^2u(t)-D^2u(\varepsilon)\|_{L^q(\Omega(x_0,t^{1/2}))} \le \int_t^{\varepsilon} \|D^2u_s(s)\|_{L^q(\Omega(x_0,t^{1/2}))}\, ds \le$$

$$\le k_2 |f|_{\theta,\varepsilon^{-1}} \int_t^{\varepsilon} \frac{ds}{s^{2-\theta-n/2q}} \le \frac{k_2}{1-\theta-n/2q}\, t^{\theta-1+n/2q} |f|_{\theta,\varepsilon^{-1}} ,$$

and the result is proved. □

Now we are ready to prove Theorem 5.2. We distinguish three cases: $\theta<1/2$, $\theta=1/2$ and $\theta>1/2$.

Case 1 ($\theta\in]0,1/2[$). Let $f\in D_A(\theta,\infty)$; choose $\varepsilon>0$ such that $\varepsilon<t_0\wedge(M_1^2\sigma_1^2)$, where σ_1, M_1 are the numbers defined in Lemma 1.16, and take $x,y\in\overline{\Omega}$ with $|x-y|\le \frac{\varepsilon^{1/2}}{M_1}$. Then the points x,y satisfy the assumptions of Lemma 1.16, and hence there exists a continuously differentiable path $\Gamma:[0,1]\to\overline{\Omega}$ such that

$$\Gamma(0)=x, \quad \Gamma(1)=y, \quad \ell(\Gamma)= \int_0^1 |\Gamma'(\lambda)|\,d\lambda \le M_1|x-y|.$$

(Clearly if Ω is convex we can take as Γ the segment joining x and y; in this case we have to require only $|x-y|\le \varepsilon^{1/2}\le t_0^{1/2}$).

Set $t=M_1^2|x-y|^2$, so that $t\in]0,\varepsilon]$. If u is the function (5.4),

by Proposition 5.3 (v)-(vi) we have

$$|f(x)-f(y)|\le|f(x)-u(t,x)|+|u(t,x)-u(t,y)|+|u(t,y)-f(y)|$$

$$\le \frac{2k_3}{\theta}\, t^{\theta}|f|_{\theta,t^{-1}} + \left|\int_0^1 (Du(t,\Gamma(\lambda))|\Gamma'(\lambda))_n\, d\lambda\right|$$

$$\le \frac{2k_3}{\theta}\, t^{\theta}|f|_{\theta,\varepsilon^{-1}}+\{k_1\varepsilon^{-1/2}\|f\|_\infty + \frac{k_2}{1/2-\theta}\, t^{\theta-1/2}|f|_{\theta,\varepsilon^{-1}}\}\,\ell(\Gamma) \tag{5.5}$$

$$\le |x-y|^{2\theta}\{c(\varepsilon)|x-y|^{1-2\theta}\|f\|_\infty + c|f|_{\theta,\varepsilon^{-1}}\}.$$

Hence $f\in C^{0,2\theta}(\overline{\Omega})$ and $\|f\|_{C^{0,2\theta}(\overline{\Omega})} \le c\|f\|_\theta$; if in addition $f\in D_A(\theta)$, choosing in (5.5) $t=M_1^2|x-y|^2\le r<\varepsilon$, we deduce that

$$[f]_{2\theta,\overline{\Omega(x,r^{1/2}M_1^{-1})}} \le c(\varepsilon)\, r^{1/2-\theta}\|f\|_\infty + c|f|_{\theta,\varepsilon^{-1}} , \quad 0<r<\varepsilon, \ x\in\overline{\Omega}.$$

Consequently

$$\limsup_{r\to 0^+} \ \sup_{x\in\Omega} [f]_{2\theta,\overline{\Omega(x,r^{1/2}M_1^{-1})}} \le c\, [f]_{\theta,\varepsilon^{-1}} ,$$

and since ε is arbitrarily small, we conclude that $f\in h^{0,2\theta}(\overline{\Omega})$.

Case 2. ($\theta=1/2$). This proof is a little more delicate. Let $f\in D_A(\frac{1}{2},\infty)$ and choose $\varepsilon>0$ such that $\varepsilon<t_0\wedge(\frac{1}{4}M_1^2\sigma_1^2)$, where, again, σ_1, M_1 are defined in Lemma 1.16. Take $x,y\in\overline{\Omega}$ such that $\frac{x+y}{2}\in\overline{\Omega}$ and $|x-y|\le \frac{2\varepsilon^{1/2}}{M_1}$: then the pairs $x,\frac{x+y}{2}$ and $y,\frac{x+y}{2}$ both satisfy the assumptions of Lemma 1.16, and consequently there exist two continuously differentiable paths $\phi:[0,1]\to\overline{\Omega}$, $\psi:[0,1]\to\overline{\Omega}$ such that

$$\phi(0)=x, \ \psi(0)=y, \ \phi(1)=\psi(1)= \frac{x+y}{2}, \ \ell(\phi)\le \frac{M_1}{2}|x-y|, \ \ell(\psi)\le \frac{M_1}{2}|x-y|$$

(again, for convex Ω the argument simplifies obviously). Choose

$t = \frac{1}{4} M_1^2 |x-y|^2$; then if u is the function (5.4) we have

$$|f(x)+f(y)-2f(\tfrac{x+y}{2})| \le |f(x)-u(t,x)|+|f(y)-u(t,y)|+2|f(\tfrac{x+y}{2})-u(t,\tfrac{x+y}{2})|+|u(t,x)+u(t,y)-2u(t,\tfrac{x+y}{2})|. \qquad (5.6)$$

By Proposition 5.3 (v)

$$|f(x)-u(t,x)|+|f(y)-u(t,y)|+2|u(t,\tfrac{x+y}{2})-f(\tfrac{x+y}{2})| \le 8k_3 t^{1/2}|f|_{1/2,t^{-1}} \le c|x-y||f|_{1/2,\varepsilon^{-1}}. \qquad (5.7)$$

On the other hand

$$u(t,x)+u(t,y)-2u(t,\tfrac{x+y}{2}) = -\int_0^1 \frac{d}{d\lambda}[u(t,\phi(\lambda))+u(t,\psi(\lambda))]\,d\lambda = -\int_0^1 [(Du(t,\phi(\lambda))|\phi'(\lambda))_n+(Du(t,\psi(\lambda))|\psi'(\lambda))_n]\,d\lambda; \qquad (5.8)$$

since

$$\int_0^1 (Du(t,\tfrac{x+y}{2})|[\phi'(\lambda)+\psi'(\lambda)])_n\,d\lambda = 0,$$

we can rewrite (5.8) as

$$u(t,x)+u(t,y)-2u(t,\tfrac{x+y}{2}) = -\int_0^1 \{[(Du(t,\phi(\lambda))-Du(t,\tfrac{x+y}{2}))]|\phi'(\lambda))_n + ([Du(t,\psi(\lambda))-Du(t,\tfrac{x+y}{2})]|\psi'(\lambda))_n\}\,d\lambda. \qquad (5.9)$$

Now recall that, by Lemma 1.20, Du(t) is Hölder continuous with any exponent $\alpha \in]0,1[$, and moreover if $p = n(1-\alpha)^{-1}$

$$[Du(t)]_{\alpha,\overline{\Omega(x_0,\sigma)}} \le c\|D^2u(t)\|_{L^p(\Omega(x_0,\sigma))} \quad \forall x_0 \in \bar\Omega, \forall \sigma > 0; \qquad (5.10)$$

hence for $\alpha \in]0,1[$ fixed, as $\frac{1}{2} M_1|x-y| = t^{1/2}$, by (5.9) we get

$$|u(t,x)+u(t,y)-2u(t,\tfrac{x+y}{2})| \le$$

$$\le [Du(t)]_{\alpha,\overline{\Omega(\frac{x+y}{2},t^{1/2})}} \int_0^1 \{|\phi(\lambda)-\phi(\tfrac{x+y}{2})|^\alpha|\phi'(\lambda)|+$$

$$+|\psi(\lambda)-\psi(\tfrac{x+y}{2})|^\alpha|\psi'(\lambda)|\}\,d\lambda \le c\|D^2u(t)\|_{L^p(\Omega(\frac{x+y}{2},t^{1/2}))}[\ell(\phi)^{1+\alpha}+\ell(\psi)^{1+\alpha}]$$

$$\le c\|D^2u(t)\|_{L^p(\Omega(\frac{x+y}{2},t^{1/2}))}|x-y|^{1+\alpha}.$$

As $\theta = 1/2$, $p = n(1-\alpha)^{-1} > n$, by Proposition 5.3(viii) we obtain

$$|u(t,x)+u(t,y)-2u(t,\tfrac{x+y}{2})| \le c|x-y|^{1+\alpha}\{c(\varepsilon)\|f\|_\infty + c\,t^{-\alpha/2}|f|_{1/2,\varepsilon^{-1}}\}$$

$$\le |x-y|\{c(\varepsilon)|x-y|^\alpha\|f\|_\infty + c|f|_{1/2,\varepsilon^{-1}}\}. \qquad (5.11)$$

By (5.6), (5.7) and (5.11) we get

$$|f(x)+f(y)-2f(\tfrac{x+y}{2})| \le |x-y|\{c(\varepsilon)|x-y|^\alpha\|f\|_\infty + c|f|_{1/2,\varepsilon^{-1}}\}. \qquad (5.12)$$

This proves that $f \in C^{*,1}(\bar\Omega)$ and

$$\|f\|_{C^{*,1}(\bar\Omega)} \le c\,\|f\|_{1/2}. \qquad (5.13)$$

If in addition $f \in D_A(\frac{1}{2})$, choosing in (5.11) $t = \frac{1}{4} M_1^2|x-y|^2 \le r < \varepsilon$, we get

$$[f]_{*,1,\overline{\Omega(x,2r^{1/2}M_1^{-1})}} \le c(\varepsilon)r^\alpha\|f\|_\infty + c|f|_{1/2,\varepsilon^{-1}}, \quad 0<r<\varepsilon,\ x \in \bar\Omega,$$

which yields, since ε is arbitrarily small,

$$\lim_{r\to 0^+} \sup_{x\in\Omega} [f]_{*,1,\overline{\Omega(x,2r^{1/2}M_1^{-1})}} = 0, \text{ i.e. } f \in h^{*,1}(\bar\Omega).$$

Next, we have to prove that $f \in C_\beta^{*,1}(\bar\Omega)$ (or $f \in h_B^{*,1}(\bar\Omega)$ if $f \in D_A(\frac{1}{2})$); thus, we have to estimate the quantity $\sigma^{-1}[f(x-\sigma\beta(x))-f(x)]$ when

$x\in\partial\Omega$, $\sigma>0$, $x-\sigma\beta(x)\in\overline{\Omega}$.

By Lemma 1.15, it suffices to consider small values of σ, say $\sigma\in]0,\sigma_0]$, with $\sigma_0\le\|\beta\|_\infty^{-1}t_0^{1/2}$. Choose $\varepsilon>0$ such that $\varepsilon<\sigma_0^2\|\beta\|_\infty^2$, and take $x\in\partial\Omega$; we can suppose that $x-\sigma\beta(x)\in\overline{\Omega}$ $\forall\sigma\in]0,\varepsilon^{1/2}\cdot\|\beta\|_\infty^{-1}]$.

Set $t=\sigma^2\|\beta\|_\infty^2$; then $t\in]0,\varepsilon]$ and

$$|f(x-\sigma\beta(x))-f(x)|\le|f(x-\sigma\beta(x))-u(t,x-\sigma\beta(x))|+|u(t,x)-f(x)| + |u(t,x-\sigma\beta(x))-u(t,x)|. \tag{5.14}$$

By Proposition 5.3(v)

$$|f(x-\sigma\beta(x))-u(t,x-\sigma\beta(x))|+|u(t,x)-f(x)|\le 4k_2|f|_{1/2,t^{-1}}\cdot t^{1/2}\le \le c|f|_{1/2,\varepsilon^{-1}}\cdot\sigma; \tag{5.15}$$

on the other hand

$$u(t,x-\sigma\beta(x))-u(t,x)=\int_0^\sigma\frac{d}{ds}[u(t,x-s\beta(x))]ds = =-\int_0^\sigma(Du(t,x-s\beta(x))|\beta(x))_n ds=-\int_0^\sigma(Du(t,x-s\beta(x))- \tag{5.16}$$

$$-Du(t,x)|\beta(x))_n ds-\sigma(Du(t,x)|\beta(x))_n.$$

Now recall that $u(t)$ satisfies

$$\alpha(x)u(t,x)+(\beta(x)|Du(t,x))_n=0\qquad\forall x\in\partial\Omega; \tag{5.17}$$

so (5.16) can be rewritten as

$$u(t,x-\sigma\beta(x))-u(t,x)=-\int_0^\sigma(Du(t,x-s\beta(x))-Du(t,x)|\beta(x))_n ds + +\sigma\cdot\alpha(x)u(t,x). \tag{5.18}$$

Now we fix $\alpha\in]0,1[$ and set $p=n(1-\alpha)^{-1}$. By (5.10) and Proposition 5.3 (viii)

$$\left|\int_0^\sigma(Du(t,x-s\beta(x))-Du(t,x)|\beta(x))_n ds\right|\le[Du(t)]_{\alpha,\overline{\Omega(x,t^{1/2})}}\sigma^{1+\alpha}\|\beta\|_\infty^\alpha$$

$$\le c\|D^2u(t)\|_{L^p(\Omega(x,t^{1/2}))}\cdot\sigma^{1+\alpha}\le\sigma\{c(\varepsilon)\sigma^\alpha\|f\|_\infty+c\sigma^\alpha t^{-\alpha/2}|f|_{1/2,\varepsilon^{-1}}\} \tag{5.19}$$

$$\le\sigma\{c(\varepsilon)\sigma^\alpha\|f\|_\infty+c|f|_{1/2,\varepsilon^{-1}}\},$$

while by Proposition 5.3 (iii)

$$\sigma|\alpha(x)u(t,x)|\le c\,\sigma\|f\|_\infty. \tag{5.20}$$

By (5.14), (5.15), (5.18), (5.19) and (5.20) we obtain $f\in C_\beta^{*,1}(\overline{\Omega})$ and, recalling (5.13), we have $\|f\|_{C_\beta^{*,1}(\overline{\Omega})}\le c\|f\|_{1/2}$; on the other hand, if in addition $f\in D_A(\frac{1}{2})$, we obtain similarly

$$\left|\frac{f(x-\sigma\beta(x))-f(x)}{\sigma}-\alpha(x)f(x)\right|\le c|f|_{1/2,\varepsilon^{-1}}+c(\varepsilon)\sigma^\alpha\|f\|_\infty + +\alpha(x)|u(t,x)-f(x)|\le c|f|_{1/2,\varepsilon^{-1}}+c(\varepsilon)\sigma^\alpha\|f\|_\infty. \tag{5.21}$$

As ε is arbitrarily small, the left hand side of (5.21) tends to 0 as $\sigma\to0^+$: this shows that $f\in h_B^{*,1}(\overline{\Omega})$.

<u>Case 3</u> ($\theta\in]1/2,1[$). Let $f\in D_A(\theta,\infty)$. First note that, if u is the function (5.4), then by Proposition 5.3(i)-(vii)

$$u(t)\to f,\quad Du(t)\to Du(0)\quad\text{in}\quad C^0(\overline{\Omega})\quad\text{as } t\to0^+,$$

which means $f\in C^1(\overline{\Omega})$ and $Df(x)=Du(0,x)$ $\forall x\in\overline{\Omega}$. In particular, as (5.17) holds for small positive t, we get

$$\alpha(x)f(x)+(\beta(x)|Df(x))_n=0\qquad\forall x\in\partial\Omega,$$

i.e. $f\in C_B^1(\overline{\Omega})$; in addition by Proposition 5.3(vii)-(iii)

$$\|Du(t)\|_{C^0(\overline{\Omega})}\le\|Du(t)-Du(t_0)\|_{C^0(\overline{\Omega})}+\|Du(t_0)\|_{C^0(\overline{\Omega})}\le$$

$$\le c|f|_\theta+c\|f\|_\infty\le c\|f\|_\theta$$

and as $t\to 0^+$ we clearly obtain

$$\|f\|_{C^1(\bar\Omega)} \le c\,\|f\|_\theta . \qquad (5.22)$$

Thus we have only to show that $Df\in C^{0,2\theta-1}(\bar\Omega)$ (or $Df\in h^{0,\theta-1/2}(\bar\Omega)$ if $f\in D_A(\theta)$). Choose $\varepsilon\in\,]0,t_0]$, and take $x,y\in\bar\Omega$ with $|x-y|\le\varepsilon^{1/2}$. Set $t=|x-y|^2$, so that $t\in\,]0,\varepsilon]$; then if $p=\frac{1}{2}n(1-\theta)^{-1}$ and $q>p$, by Lemma 1.20 and Proposition 5.3(vii)-(viii) we have

$$\begin{aligned}
|Df(x)-Df(y)| &\le |Df(x)-Du(t,x)|+|Du(t,x)-Du(t,y)|+|Du(t,y)-f(y)| \\
&\le 2c\, t^{\theta-1/2}|f|_{\theta,t^{-1}}+[Du(t)]_{2\theta-1,\overline{\Omega(x,t^{1/2})}}\,|x-y|^{2\theta-1} \\
&\le c\, t^{\theta-1/2}|f|_{\theta,t^{-1}}+c\|D^2u(t)\|_{L^q(\Omega(x_0,t^{1/2}))}\, t^{\frac{n}{2qp}(q-p)}\cdot|x-y|^{2\theta-1} \\
&\le |x-y|^{2\theta-1}\Big\{c|f|_{\theta,\varepsilon^{-1}}+c|f|_{\theta,\varepsilon^{-1}}\, t^{\frac{n}{2qp}(q-p)-1+\theta+\frac{n}{2q}} + \\
&\qquad +c(\varepsilon)\,\|f\|_\infty\, t^{\frac{n}{2qp}(q-p)}\Big\}.
\end{aligned} \qquad (5.23)$$

As $\frac{n}{2qp}(q-p)-1+\theta+\frac{n}{2q}=\frac{n}{2p}-1+\theta=0$, we get

$$|Df(x)-Df(y)|\le|x-y|^{2\theta-1}\Big\{c|f|_{\theta,\varepsilon^{-1}}+c(\varepsilon)|x-y|^{\frac{n}{p}-\frac{n}{q}}\|f\|_\infty\Big\}.$$

Hence $Df\in C^{0,2\theta-1}(\bar\Omega)$ and, recalling (5.22), $\|f\|_{C^{1,2\theta-1}(\bar\Omega)}\le c\|f\|_\theta$.

If in addition $f\in D_A(\theta)$, choosing in (5.23) $t=|x-y|^2\le r<\varepsilon$ we obtain

$$[Df]_{2\theta-1,\overline{\Omega(x,r^{1/2})}}\le c|f|_{\theta,\varepsilon^{-1}}+c(\varepsilon)\, r^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_\infty ,\quad x\in\bar\Omega,$$

and since ε is arbitrarily small, we deduce that

$$\lim_{r\to0^+}\ \sup_{x\in\Omega}\ [Df]_{2\theta-1,\overline{\Omega(x,r^{1/2})}}=0$$ i.e. $f\in h_B^{1,2\theta-1}(\bar\Omega)$. Theorem 5.2 is completely proved. □

6. CONCLUSIONS

Collecting the results of the preceding sections, we have proved the following result:

THEOREM 6.1. *Let Ω be a bounded connected open set of $\mathbb{R}^n$, $n\ge1$, with boundary $\partial\Omega$ of class C^3. Let E, B be the differential operators respectively defined by (5.1) and (5.2), and suppose that conditions* (A.1), (A.2), (B.1), (B.2) *of Section 5 hold. Let A be the abstract operator defined by (5.3) in the space $C^0(\bar\Omega)$. Then the following equalities hold (with equivalent norms):*

$$D_A(\theta,\infty)=(C_B^2(\bar\Omega),C^0(\bar\Omega))_{1-\theta,\infty}=\begin{cases} C^{0,2\theta}(\bar\Omega) & \text{if } \theta\in\,]0,1/2[\\ C_B^{*,1}(\bar\Omega) & \text{if } \theta=1/2 \\ C_B^{1,2\theta-1}(\bar\Omega) & \text{if } \theta\in\,]1/2,1[\ ; \end{cases}$$

$$D_A(\theta)=(C_B^2(\bar\Omega),C^0(\bar\Omega))_{1-\theta}=\begin{cases} h^{0,2\theta}(\bar\Omega) & \text{if } \theta\in\,]0,1/2[\\ h_B^{*,1}(\bar\Omega) & \text{if } \theta=1/2 \\ h_B^{1,2\theta-1}(\bar\Omega) & \text{if } \theta\in\,]1/2,1[\,. \end{cases}$$ □

REMARK 6.2. The case of Dirichlet boundary conditions, i.e. $\beta(x)\equiv0$ in (5.2), can also be studied with our method; however the extension procedure given by (2.3) in the case $\Omega=\mathbb{R}^n_+$ has to be replaced by the odd extension method. It can be seen that in this case the treatment developed in Sections 2,3,4 still works; on the other hand the reverse inclusion of Section 5 can be proved in the same way, by applying the estimates of Stewart [19] instead of [20]. Hence we find again a known result of Lunardi, which had been proved in [13] with a slight strengthening of assumptions. Namely, we have:

THEOREM 6.3. Let Ω be a bounded, connected open set of $\mathbb{R}^n$, $n\geq 1$, with boundary $\partial\Omega$ of class C^2. Let E be the differential operator defined by (5.1), and suppose that conditions (A.1) and (A.2) of Section 5 hold. Let A be the abstract operator defined in the space $C^0(\bar{\Omega})$ by

$$\begin{cases} D(A)=\{u\in \bigcap_{p\in[1,\infty[} H^{2,p}(\Omega): Eu\in C^0(\bar{\Omega}),\ u|_{\partial\Omega}=0\} \\ Au=[Eu](\cdot). \end{cases}$$

Then the following equalities hold (with equivalent norms):

$$D_A(\theta,\infty)=(C_0^2(\bar{\Omega}),C_0^0(\bar{\Omega}))_{1-\theta,\infty} = \begin{cases} C_0^{0,2\theta}(\bar{\Omega}) & \text{if } \theta\in]0,1/2[\\ C_0^{*,1}(\bar{\Omega}) & \text{if } \theta=1/2 \\ C_0^{1,2\theta-1}(\bar{\Omega}) & \text{if } \theta\in]1/2,1[\ , \end{cases}$$

$$D_A(\theta) = (C_0^2(\bar{\Omega}),\ C_0^0(\bar{\Omega}))_{1-\theta} = \begin{cases} h_0^{0,2\theta}(\bar{\Omega}) & \text{if } \theta\in]0,1/2[\\ h_0^{*,1}(\bar{\Omega}) & \text{if } \theta=1/2 \\ h_0^{1,2\theta-1}(\bar{\Omega}) & \text{if } \theta\in]1/2,1[\ . \end{cases} \quad \square$$

REFERENCES

[1] P. ACQUISTAPACE, B. TERRENI, Some existence and regularity results for abstract non-autonomous parabolic equations, J. Math. Anal. Appl. 99 (1984) 9-64.

[2] P. ACQUISTAPACE, B. TERRENI, On the abstract non-autonomous parabolic Cauchy problem in the case of constant domains, Ann. Mat. Pura Appl. (4) 140 (1985) 1-55.

[3] P. ACQUISTAPACE, B. TERRENI, Maximal space regularity for abstract non-autonomous parabolic equations, J. Funct. Anal. 60 (1985) 168-210.

[4] R.A. ADAMS, "Sobolev Spaces", Academic Press, New York, 1975.

[5] A. ARDITO, P. RICCIARDI, Existence and regularity for linear delay partial differential equations, Nonlinear Anal. 4 (1980) 411-414.

[6] J.B. BAILLON, Caractére borné de certains générateurs de semi-groupes linéaires dans les espaces de Banach, C.R. Acad. Sci. Paris 290 (1980) 757-760.

[7] P.L. BUTZER, H. BERENS, "Semi-groups of Operators and Approximation", Springer Verlag, New York/Berlin, 1967.

[8] G. DA PRATO, P. GRISVARD, Equations d'evolution abstraites non-linéaires de type parabolyque, Ann. Mat. Pura Appl. (4) 120 (1979) 329-396.

[9] G. DA PRATO, E. SINESTRARI, Hölder regularity for non-autonomous abstract parabolic equations, Israel J. Math. 42 (1982) 1-19.

[10] J.L. LIONS, Théorèmes de trace et d'interpolation, I, Ann. Sc. Norm. Sup. Pisa (3) 13 (1959) 389-403.

[11] J.L. LIONS, J. PEETRE, Sur une classe d'espaces d'interpolation. Inst. Hautes Etudes Sci. Publ. Math. 19 (1964) 5-68.

[12] A. LUNARDI, Analiticity of the maximal solution of an abstract nonlinear parabolic equation, Nonlinear Anal. 6 (1982) 503-521.

[13] A. LUNARDI, Interpolation spaces between domains of elliptic operators and spaces of continuous functions with applications to nonlinear parabolic equations, Math. Nachr. 121 (1985) 295-318.

[14] A. LUNARDI, Abstract quasi-linear parabolic equations, Math. Ann. 267 (1984) 395-415.

[15] A. LUNARDI, Global solutions of abstract quasi-linear parabolic equations, J. Diff. Eq. 58 (1985) 228-242.

[16] E. SINESTRARI, Continuous interpolation spaces and spatial regularity in nonlinear Volterra integro-differential equations, J. Int. Eq. 5 (1983) 287-308.

[17] E. SINESTRARI, On the abstract Cauchy problem of parabolic type in spaces of continuous functions, J. Math. Anal. Appl. 107 (1985) 16-66.

[18] E.M. STEIN, "Singular Integrals and Differentiability Properties of Functions", Princeton University Press, Princeton 1970.

[19] H.B. STEWART, Generation of analytic semi-groups by strongly elliptic operators, Trans. Amer. Math. Soc. 199 (1974) 141-162.

[20] H.B. STEWART, Generation of analytic semi-groups by strongly elliptic operators under general boundary conditions, Trans. Amer. Math. Soc. 259 (1980) 299-310.

[21] M.H. TAIBLESON, On the theory of Lipschitz spaces of distributions on Euclidean spaces. I-Principal properties, J. Math. Mech. 13 (1964) 407-479.

[22] C.C. TRAVIS, Differentiability of strict solutions to an abstract inhomogeneous differential equation, Proc. Amer. Math. Soc. 82 (1981) 425-430.

[23] H. TRIEBEL, "Interpolation Theory, Function Spaces, Differential Operators", North-Holland, Amsterdam/New York/Oxford 1978.

[24] A. ZYGMUND, Smooth functions, Duke Math. J. 12 (1945) 47-76.

[25] A. ZYGMUND, "Trigonometric Series", vol. I, Cambridge University Press, Cambridge 1959.