# UNIQUENESS FOR RICCATI EQUATIONS WITH UNBOUNDED OPERATOR COEFFICIENTS

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Abstract. In this article we address the issue of uniqueness for differential and algebraic operator Riccati equations, under a distinctive set of assumptions on their unbounded coefficients. The class of boundary control systems characterized by these assumptions encompasses diverse significant physical interactions, all modeled by systems of coupled hyperbolic-parabolic partial differential equations. The proofs of uniqueness provided tackle obstacles raised by the peculiar regularity properties of the composite dynamics. These results supplement the theories of the finite and infinite time horizon LQ-problem devised by the authors jointly with Lasiecka, as the unique solution to the Riccati equation enters the closed loop form of the optimal control.

## 1. Introduction

Well-posedness of Riccati equations is a fundamental question within control theory of Partial Differential Equations (PDE). While the issues of existence and uniqueness for the corresponding solutions are both natural to be addressed and significant in themselves, when seen in the context of linear-quadratic optimal control uniqueness proves particularly relevant, not exclusively from a theoretical perspective. This is because – having established existence – it brings about *in a univocal manner* the (optimal cost, or Riccati) operator which occurs in the feedback representation of the optimal control, thereby allowing its synthesis.

In the present work focus is on the differential and algebraic (i.e. time independent) Riccati equations arising from optimal control problems with quadratic functionals for the class of infinite dimensional abstract control systems dealt with in our earlier works [1] and [3] (joint with Lasiecka). The basic characteristics of these linear systems – which read as $y' = Ay + Bu$, according to a standard notation – are the following: the free dynamics operator $A$ is the infinitesimal generator of a $C_0$-semigroup $\{e^{At}\}_{t\ge 0}$ on the state space $Y$, while the control operator $B$ is *unbounded*. It is well known that the latter is an intrinsic feature of differential systems describing evolutionary PDE with boundary (and also point) control, already recognized by the end of the sixties in the pioneering work of Fattorini [16]. (More precisely, $B$ maps continuously the control space $U$ into a larger functional space than $Y$, that is the extrapolation space $[\mathcal{D}(A^*)]'$; see the basic Assumptions 2.1.)

We require in addition and more specifically that several assumptions on the operators $A, B$ are fulfilled, recorded in Section 2 as Assumptions 2.4. These are regularity properties that pertain to the operator $B^*e^{A^*t}$, with respective PDE counterparts. It is worth emphasizing here that not only the aforesaid control-theoretic properties do not prescribe analyticity of the semigroup $e^{At}$, in accordance with the fact that the class of systems under consideration – introduced by these authors with Lasiecka in [1] – is inspired by and tailored on systems of coupled hyperbolic-parabolic PDE, subjected to boundary/interface control. They are also weaker (and somewhat trickier) than the full *singular estimates* for $e^{At}B$ which are known to be equally effective for well-posedness of the Riccati equations; see [19], [20], [23, 24]. In this respect, we remark that it was first discovered in [9] that the very same thermoelastic system, subjected to boundary (thermal) control, may or may not yield a singular estimate for the corresponding operator $e^{At}B$, depending on which boundary conditions are taken into consideration.

Another distinguishing feature of the coefficients of the Riccati equations under study, which read as

$$(Px, Az)_Y + (Ax, Pz)_Y - (B^*Px, B^*Pz)_U + (Rx, Rz)_Z = 0\,, \qquad x, z \in \mathcal{D}(A)$$

when the unknown $P$ is time-independent, is that $R$ – that is the *observation* operator in the optimization problem – does not need to be smoothing.

Working in the framework described above, a theory for both the finite and infinite time horizon LQ-problem has been devised in [1] and [3], the latter under the Assumptions 2.9 (replacing Assumptions 2.4). The strenght of these theories is confirmed by the trace regularity results that have been established for the solutions to significant PDE systems comprising hyperbolic and parabolic components, over the years. Indeed, the novel class introduced in [1] has proven successful in describing a diverse range of physical interactions such as mechanical-thermal, acoustic-structure, fluid-elasticity ones. And above all, successful in order to attain solvability of the associated optimization problems; see [2], [7], [10, 11], [3], and the most recent [8].

Our aim with the present work is to complete the complex of findings of [1, 3]. Because in [1] and [3] we followed a variational approach, by using the optimality conditions a bounded operator – *viz.* $P(t)$ or $P$, in accordance with either a finite or infinite time interval – is constructed in terms of the optimal state and only subsequently shown to satisfy the corresponding Riccati equations. For this reason the works [1] and [3] provide existence for Riccati equations, but not uniqueness.

By contrast and as it is well-known, in the (so called) 'direct' approach, the well-posedness of the nonlinear Riccati equation is studied in a first step, posssibly independently from the minimization problem. For the Cauchy problems associated with the differential Riccati equations, existence is established along with uniqueness, since they are generally obtained via fixed point theorems and *a priori* estimates. Then, in order to achieve the actual feedback representation of the optimal control, that is the ultimate goal from a mathematical as well as a practical perspective, further steps are necessary. (And yet, this is not the case here.)

Our main results are Theorems 2.7 and 2.11. Besides being of intrinsic interest, they supplement earlier results concerning the LQ-problem established by these authors (jointly with Lasiecka) in [1] and [3], with the achievement of uniqueness for the corresponding differential and algebraic Riccati equations. Although often following classical arguments, the respective proofs face novel challenges, in comparison with similar results of the past literature, owing to the peculiar structure and weakness of the present assumptions on the operator coefficients of the Riccati equations.

1.1. **An insight into the mathematical proofs.** We find it useful to devote a separate section to several comments and points concerning our analysis. We provide two proofs of Theorem 2.7, that pertains to uniqueness for the differential

Riccati equations (2.13) (DRE, in short). This result is relevant for the optimal control problem on a finite time horizon (i.e. Problem 2.3 with $T < +\infty$), under the Assumptions 2.4. The first proof, given in Section 3, follows the method employed by Lasiecka and Triggiani in [25, Theorem 1.5.3.3] up to a certain point. The basic rationale is standard: one proceeds by contradiction, assuming there exists another solution $P_1(t)$ to the DRE, besides the Riccati operator $P(t)$. (That $P(t)$ solves the DRE has been proved in [1]; see the statement S6. of Theorem 2.6.) On the basis of the integral forms of the DRE derived in Lemma 3.1, one finds that the difference $Q(t) = P_1(t) - P(t)$ solves a suitable integral equation. It is in the estimates performed afterwards, that the paths diverge, with iiic) of the Assumptions 2.4 playing a major role (together with the distinctive class of operators $P_1$ belongs to), whereas the proof in [25, Theorem 1.5.3.3] fully exploits the enhanced regularity of the analytic semigroup that describes the free dynamics.

It is unlikely that the method of proof described above could be adjusted in order to establish uniqueness for the algebraic Riccati equation, relevant for Problem 2.3 with $T = +\infty$. This owing to the argument employed when $T < +\infty$: $Q(t)$ is shown to be zero on some subinterval of $[0, T]$, with the soughtafter goal attained in a finite number of steps. Other methods of proof are certainly worth to be explored. However, in this respect an obstacle seems to come from the fact that a *full* characterization of the domain of the generator $A_P$ of the optimal state semigroup is not available (see statement A5. of Theorem 2.10). This differs from what is seen in the case the free dynamics is governed by an analytic semigroup, where the said characterization results in another advantage of the Riccati theory for parabolic PDE, having a role, for instance, in the proof of uniqueness carried out in [25, Theorem 2.3.4]. In any case, we leave this question open.

Therefore, in order to prove Theorem 2.11 we choose to return to the dynamic programming approach to the LQ-problem, and borrow from it a key element in attaining that the optimal control admits a (pointwise in time) feedback representation. This element is fulfilled by the so called *fundamental identity*. In a direct approach, the fundamental identity builds a bridge between the nonlinear Riccati equation – whose well-posedness is studied in a first step, independently from the minimization problem, as recalled above – and the actual closed loop form of the optimal control. The latter goal (i.e. the feedback representation of the optimal control) was already attained in [1] and [3]; see the statements S4. of in Theorem 2.6 and A4. of Theorem 2.10, respectively. And yet, the identities we establish in Lemma 4.1 and Lemma 4.3 constitute a major (and technically nontrivial) step in our analysis, allowing to achieve uniqueness for both differential and algebraic Riccati equations, respectively. Theorem 2.7 and Theorem 2.11 are thus established, via methods of proof which are akin.

1.2. **Riccati equations: a brief historical synopsis.** Historically, the appearance of *matrix* Riccati equations – named after Jacopo Riccati (XVII century) – as a research subject is recognized to date back to the sixties, with the independent contributions of Kalman [18] in the USA and of Letov [28] in the former Soviet Union (although they certainly appeared before). Their study has already lasted for more than half a century, owing to its connections to a wide variety of topics such as stability and stabilization problems, the linear-quadratic (LQ) optimal control, differential games, just to name a few. We refer the reader to the introductory monograph [34], along with its references.

During the seventies, the theories of the LQ-problem and of Riccati equations is extended to the infinite dimensional setting. The abstract formulation of initial-boundary value problems (IBVP) for PDE in bounded domains, in the presence of *distributed* control, leads to the usual Cauchy problem for the control system $y' = Ay + Bu$, where $B$ is a *bounded* operator from the control space $U$ to the state space $Y$; see [6]. (The same framework encompasses also other kinds of differential problems such as, e.g., delay systems.) The first contributions to the study of Riccati equations with unbounded operator coefficients (still, with $B$ bounded) are due to the works of Da Prato [14], Tartar, Curtain and Pritchard, Zabczyk; see the introductory monograph [34, Part IV, Ch. 4]).

The proof performed in [14] paves the way for subsequent direct studies of well-posedness of Riccati equations with an *unbounded* operator $B$. Within the theory of the LQ-problem, these extensions were initially motivated by and focused on parabolic PDE. Hence, they could exploit the regularity properties brought about by the analytic semigroup that describes the free dynamics. Although well known to those acquainted with the subject, it is important to emphasise that the transition to infinite-dimensional systems describing PDE with *boundary* or *point* control (as opposed to PDE with distributed control), has drastically impacted on the mathematical analysis of the corresponding Riccati equations. It is here that the parabolic (parabolic-like) and the hyperbolic analyses split apart.

We refer the reader to [22] and [6], and to the in-depth monograph [25] for a complete overview of the subject. The reference section therein include the principal contributions to the Riccati theory for parabolic (and parabolic-like) PDEs by Balakrishnan (1977), Lasiecka and Triggiani (1983), Pritchard and Salamon (1984), Flandoli (1984), Da Prato and Ichikawa (1985). As for the Riccati theory for hyperbolic (and hyperbolic-like) PDE, we mention explicitly the latest works by Flandoli, Lasiecka and Triggiani (1988), Barbu, Lasiecka and Triggiani (2000); see also the second volume of [25].

The hyperbolic character of the dynamics – in the present work, of one component of the coupled PDE system –, combined with the unboundedness of the control operator, produces further mathematical challenges. It will suffice to highlight that under the *abstract trace regularity* assumption or *admissibility condition*, which is characteristic of hyperbolic-like dynamics,

- well-posedness of differential Riccati equations holds true provided the observation operator $R$ has appropriate *smoothing properties*;
- given the optimal cost operator $P$ for the infinite time horizon problem, it turns out the gain operator $B^*P$ that occurrs in the algebraic Riccati equation may be even *not densely defined*. (Then, appropriate extensions of $B^*P$ are called for.)

A simple illustration of the latter anomaly is given by Weiss and Zwart [33]. This work exhibits a first-order hyperbolic PDE (in one space dimension), with point control; given a certain quadratic functional, the optimal cost operator $P$ is computed explicitly, and then it is shown that $B^*$ is intrinsically not defined on $Py$, $y \in \mathcal{D}(A)$. In this connection we remark that, to the best of our knowledge, the question as to whether there actually exist examples of hyperbolic PDE with boundary control (rather than point control) which give rise to the same 'failure', is open.

Thus, we like to point out that in the case of systems of hyperbolic-parabolic PDEs (with boundary control) which fulfil the requirements of our Assumptions 2.4

(or Assumptions 2.9, when time varies in the positive half line), the highlighted difficulties are overcome, as the theory of the Riccati equations devised in [1, 3] and in the present work shows.

A technical discussion of the Bolza problem (where the quadratic functional to be minimized includes a penalization of the state at the final time $T$), drawing a picture of the evolution of the increasingly enhanced results that have been established, would require a space which falls outside the scope of the present work. We refer the reader to [25] and its references. We remark that the Bolza problem is not covered by the theory of the LQ-problem in [1]; it is indeed an open problem.

Last but not least, the case of time dependent (operator) coefficients in the Riccati equation is considerably more difficult, from a technical point of view. The major contributions, that all pertain to the parabolic case, are due to Acquistapace and Terreni, in part jointly with Flandoli; see [4], [5].

The study of optimal boundary control problems for thermoelastic systems and then more generally PDE systems comprising two (or more) evolutionary PDE of different type, in the presence of boundary/interface control actions, motivated the introduction of a class characterized by certain (local in time) estimates for the kernel $e^{At}B$. These were named and are know in the literature as *singular estimates*. The corresponding theory of the optimal boundary control problem is well established: after the former works [19], [20], [23, 24], a great deal of attention has been devoted to the Bolza problem by Lasiecka and Tuffaha (see [26], [27], [32]). Apparently, the question of uniqueness of Riccati equations is addressed in none of the aforementioned works, with the exception of [32].

1.3. **Outline of the paper.** The structure of the paper is outlined readily. In the next Subsection 2 we give the statements of our main results, *viz.* Theorem 2.7 and Theorem 2.11, after having recalled the framework and the core statements of the theories of the LQ-problem devised in [1] and [3].

In Section 3 we present a first proof of Theorem 2.7. This is preceded by Lemma 3.1, which establishes two integral forms of the differential Riccati equation. An integral form of the algebraic Riccati equation is derived as well here, as Lemma 3.2.

Section 4 provides a second proof of Theorem 2.7 and the proof of Theorem 2.11. Instrumental to the proofs are Lemma 4.1 and Lemma 4.2 for the former result, and Lemma 4.3 along with Lemma 4.4 for the latter. These lemmas establish the fundamental identities and discuss certain built closed loop equations.

In Appendix A we gather several regularity results (some old, some new) which are used throughout the paper.

## 2. Abstract framework, theoretical results

2.1. **The LQ problem: abstract dynamics and setting.** Let $Y$ and $U$ be two separable Hilbert spaces, the *state* and *control* space, respectively. We consider the abstract (linear) control system $y' = Ay + Bu$ and the corresponding Cauchy problem

$$\begin{cases} y'(t) = Ay(t) + Bu(t)\,, & 0 \le t < T \\ y(0) = y_0 \in Y\,, \end{cases} \tag{2.1}$$

under the following basic Assumptions.

**Assumptions 2.1 (Basic Assumptions).** *Let $Y$, $U$ be separable complex Hilbert spaces.*

- *The closed linear operator $A : \mathcal{D}(A) \subset Y \to Y$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{At}\}_{t\ge 0}$ on $Y$;*
- *$B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$.*

**Remarks 2.2.** The apparent weakness of the basic assumptions on the pair $(A, B)$ which characterize the initial/boundary value problems described by the abstract equation in (2.1) – *viz.* the free dynamics operator $A$ and the control operator $B$ – is a reflection of the modeling of most boundary control problems for systems of coupled hyperbolic/parabolic PDEs. It is worth recalling its most prominent features: (i) first, the control operator $B$ will *not* be bounded from the control space $U$ into the state space $Y$; (ii) secondly, the semigroup $e^{At}$ will *not* be analytic.
We remind the reader that (i) is intrinsic to the mathematical modeling of control actions on the boundary of the domain (or on some part of it), as first shown in [16]. We also note that the presence of control actions concentrated on points (in 1-D) or on curves (in 2-D) in the interior of domain results in unboundedness of the control operator as well; illustrations of both situations are found in [22], [6] and [25]. As for the simple requirement in (ii), it is a natural feature of composite dynamics comprising solely a parabolic component: analiticity of the overall semigroup should not be expected.

Thus, given $y_0 \in Y$, the Cauchy problem (2.1) possesses a unique *mild* solution given by

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}Bu(s)\,ds\,, \qquad t \in [0, T)\,, \tag{2.2}$$

where

$$L\colon u(\cdot) \longrightarrow (Lu)(t) := \int_0^t e^{A(t-s)}Bu(s)\,ds$$

is the *input-to-state mapping*, that is the operator which associates to any control function $u(\cdot)$ the solution to the Cauchy problem (2.1) with $y_0 = 0$, and (2.2) makes sense at least in the extrapolation space $[\mathcal{D}(A^*)]'$; see [25, § 0.3, p. 6, and Remark 7.1.2, p. 646].

To the state equation (2.1) we associate the quadratic functional

$$J(u) = \int_0^T \big(\|Ry(t)\|_Z^2 + \|u(t)\|_U^2\big)\,dt\,, \tag{2.3}$$

where $Z$ is a third separable Hilbert space – the so called observation space (possibly, $Z \equiv Y$) – and at the outset the *observation* operator $R$ simply satisfies

$$R \in \mathcal{L}(Y, Z)\,. \tag{2.4}$$

The formulation of the optimal control problem under study is classical. The adjectives finite or infinite time horizon problem refer to the cases $T < +\infty$ or $T = +\infty$, respectively.

**Problem 2.3 (The optimal control problem).** *Given $y_0 \in Y$, seek a control function $u \in L^2(0, T; U)$ which minimizes the cost functional* (2.3)*, where $y(\cdot) = y(\cdot\,; y_0, u)$ is the solution to* (2.1) *corresponding to the control function $u(\cdot)$ (and with initial state $y_0$) given by* (2.2).

It is well known that aiming at solving Problem 2.3, certain principal facts need to be ascertained, beside the existence of a unique optimal pair $(\hat{u}(\cdot,s;y_0),\hat{y}(\cdot,s;y_0))$ (which is readily established by using classical variational arguments); namely,

- that the optimal control $\hat{u}(t)$ admits a (pointwise in time) *feedback* representation, in terms of the optimal state $\hat{y}(t)$;
- that the optimal cost operator $P(t)$ ($P$, when $T=+\infty$) solves the corresponding Differential (Algebraic) Riccati equation; thus, the issue of well-posedness of the DRE (ARE) arises, requiring
- that a meaning is given to the gain operator $B^*P(t)$ ($B^*P$) on the state space $Y$ (by means of extensions, or – and this will be the case here –, as a *bounded* operator on a dense subset of $Y$).

2.2. **Theoretical results: finite and infinite time horizon problems.** We begin by recalling the theory of the LQ-problem on a finite time interval developed in [1]. This theory pertains to the class of control systems – introduced in the very same [1] – whose dynamics, control and observation operators are subject to the following assumptions.

**Assumptions 2.4 (Finite time horizon case).** *Let $Y$, $U$ and $Z$ be separable complex Hilbert spaces, and let $T>0$ be given. The pair $(A,B)$ (which describes the state equation* (2.1)*) fulfils Assumptions 2.1, with the additional property $A^{-1}\in\mathcal{L}(Y)$, while the observation operator $R$ (which occurs in the cost functional* (2.3)*) satisfies the basic condition* (2.4)*.*

*The operator $B^*e^{A^*t}$ can be decomposed as*

$$B^*e^{A^*t}x=F(t)x+G(t)x\,,\qquad 0\le t\le T\,,\; x\in\mathcal{D}(A^*)\,,\tag{2.5}$$

*where $F(t)\colon Y\longrightarrow U$ and $G(t)\colon \mathcal{D}(A^*)\longrightarrow U$, $t>0$, are bounded linear operators satisfying the following assumptions:*

i) *there exist constants $\gamma\in(0,1)$ and $N>0$ such that*

$$\|F(t)\|_{\mathcal{L}(Y,U)}\le N\,t^{-\gamma}\,,\qquad 0<t\le T\,;\tag{2.6}$$

ii) *the operator $G(\cdot)$ belongs to $\mathcal{L}(Y,L^p(0,T;U))$ for all $p\in[1,\infty)$;*

iii) *there exists $\epsilon>0$ such that:*

a) *the operator $G(\cdot)A^{*-\epsilon}$ belongs to $\mathcal{L}(Y,C([0,T];U))$, with*

$$\sup_{t\in[0,T]}\|G(t)A^{*-\epsilon}\|_{\mathcal{L}(Y,U)}<\infty\,;$$

b) *the operator $R^*R$ belongs to $\mathcal{L}(\mathcal{D}(A^\epsilon),\mathcal{D}(A^{*\epsilon}))$, i.e.*

$$\|A^{*\epsilon}R^*RA^{-\epsilon}\|_{\mathcal{L}(Y)}\le c<\infty\,;\tag{2.7}$$

c) *there exists $q\in(1,2)$ (depending, in general, on $\epsilon$) such that the map $x\longmapsto B^*e^{A^*t}A^{*\epsilon}x$ has an extension which belongs to $\mathcal{L}(Y,L^q(0,T;U))$.*

**Remarks 2.5.** 1. We note that it is assumed at the very outset that $0\in\rho(A)$, i.e. the dynamics operator $A$ is boundedly invertible on $Y$. It is important to emphasize that this property happens to hold true for an ample variety of composite systems of hyperbolic-parabolic PDE, such as e.g. thermoelastic systems, structural-acoustics models, fluid-elasticity interactions; see [2], [7], [10, 11], [8]. This allows in particular to define the fractional powers $(-A)^\alpha$, $\alpha\in(0,1)$; see [31, § 1.15.1-2], [30], [29,

§ 2.2.2]. (In order to make the notation lighter, we wrote $A^\alpha$ instead of $(-A)^\alpha$; the same will happen throughout the paper.)

On the other hand, when $\lambda = 0$ is not in the resolvent set of $A$, one can find $\omega_0 > 0$ – the *type* of the semigroup – such that the translation $\hat{A} := \omega - A$ is a positive operator for any $\omega > \omega_0$; then $\hat{A}$ is boundedly invertible, and the fractional powers $\hat{A}^\theta$ of $\hat{A}$ are well-defined. The extension of the present theory to the case of unstable semigroups $e^{At}$ is particularly relevant in the infinite time horizon case ($T = +\infty$). It would certainly require a tedious series of technical changes and *is so far lacking*.

2. If the singular estimate (2.6) for the component $F$ (*cf.* i) of Assumptions 2.4) is shown to hold true in an arbitrarily small right neighbourhood of $t = 0$, then it usually extends to all $t \in (0, T]$, by using semigroup theory.

3. We note that iiia) of the Assumptions 2.4 tells us that the 'basic' (time) regularity of the $G$ component, that is $G(\cdot)y \in L^p(0, T; U)$ for $y \in Y$ and all *finite* summability exponents $p \ge 1$, improves to $G(\cdot)y \in C([0, T]; U)$, when $y \in \mathcal{D}(A^{*\epsilon})$.

4. The findings of the work [1], summarized in the next Theorem 2.6, were actually established under the *weaker* regularity assumption

iiic)' there exists $q \in (1, 2)$ such that the map $x \longmapsto B^* e^{A^* t} R^* R A^\epsilon x$ has an extension which belongs to $\mathcal{L}(Y, L^q(0, T; U))$.

Indeed, iiic)' of Assumptions 2.4, combined with iiib), implies readily iiic), as already pointed out in [1, p. 1401] (with a reversed notation, though).

However, on one side the present iiic) – more precisely, the boundary regularity result that (case by case) the control-theoretic condition iiic) translates to – has been shown over the years to hold true in the case of distinct PDE systems studied in the aforementioned references (*viz.* [2], [7], [10, 11], [8]). On the other side, uniqueness of solutions to the Riccati equations appears to be in need of it: both within the first proof of Theorem 2.6 given in the next section (specifically to perform the estimates which bring about (3.7)), and also to show Lemma 4.2, instrumental to the distinct proof of the same result proposed in Section 4. Furthermore, the stronger (A.3) – which is central to the proof of Lemma 4.4 relevant to the infinite time horizon case – is based upon iiic) of Assumptions 2.4.

Under the listed Assumptions 2.4, a full solution to the optimal control Problem 2.3, as detailed by the complex of statements S1.–S6. collected in Theorem 2.6 below, was obtained in [1]. These include, in particular, two specific novel features over the *parabolic* or *hyperbolic* cases ([22]):

- the lack of continuity (in time) of the optimal control $\hat{u}(\cdot)$ (*viz.* S1.), and
- that the gain operator $B^*P(t)$ is *bounded* only on a certain dense subset of $Y$, yet not preventing well-posedness of the Differential Riccati Equations corresponding to the LQ problem.

**Theorem 2.6** (Finite time horizon theory; cf. [1], Theorem 2.3)**.** *With reference to the control problem* (2.1)–(2.3)*, under the Assumptions 2.4, the following statements are valid for each* $s \in [0, T)$*.*

**S1.** *For each* $x \in Y$ *the optimal pair* $(\hat{u}(\cdot, s; x), \hat{y}(\cdot, s; x))$ *satisfies*

$$\hat{y}(\cdot, s; x) \in C([s, T]; Y), \quad \hat{u}(\cdot, s; x) \in \bigcap_{1 \le p < \infty} L^p(s, T; U).$$

**S2.** *The linear bounded (on $Y$) operator $\Phi(t,s)$, defined by*

$$\Phi(t,s)x = \hat{y}(t,s;x) = e^{A(t-s)}x + [L_s\hat{u}(\cdot,s;x)](t)\,, \quad s \le t \le T\,,\ x \in Y\,, \tag{2.8}$$

*is an evolution operator, i.e.*

$$\Phi(t,t) = I_Y\,, \qquad \Phi(t,s) = \Phi(t,\sigma)\Phi(\sigma,s) \quad \text{for } s \le \sigma \le t \le T.$$

**S3.** *For each $t \in [0,T]$ the operator $P(t) \in \mathcal{L}(Y)$, defined by*

$$P(t)x = \int_t^T e^{A^*(\tau-t)}R^*R\Phi(\tau,t)x\;d\tau\,, \qquad x \in Y, \tag{2.9}$$

*is self-adjoint and positive; it belongs to $\mathcal{L}(Y,C([0,T];Y))$ and is such that*

$$(P(s)x,x)_Y = J_s(\hat{u}(\cdot,s;x),\hat{y}(\cdot,s;x)) \qquad \forall s \in [0,T]\,.$$

**S4.** *The gain operator $B^*P(\cdot)$ belongs to $\mathcal{L}(\mathcal{D}(A^\varepsilon),C([0,T];U))$ and the optimal pair satisfies for $s \le t \le T$*

$$\hat{u}(t,s;x) = -B^*P(t)\hat{y}(t,s;x) \qquad \forall x \in Y. \tag{2.10}$$

**S5.** *The operator $\Phi(t,s)$ defined in* (2.8) *satisfies for $s < t \le T$:*

$$\frac{\partial\Phi}{\partial s}(t,s)x = -\Phi(t,s)(A - BB^*P(t))x \in L^{1/\gamma}(s,T;[\mathcal{D}(A^{*\varepsilon})]') \tag{2.11}$$

*for all $x \in \mathcal{D}(A)$, and*

$$\frac{\partial\Phi}{\partial t}(t,s)x = (A - BB^*P(t))\Phi(t,s)x \in C([s,T],[\mathcal{D}(A^*)]') \tag{2.12}$$

*for all $x \in \mathcal{D}(A^\varepsilon)$.*

**S6.** *The operator $P(t)$ defined by* (2.9) *satisfies the following (differential) Riccati equation in* $[0,T)$*:*

$$\begin{cases} \frac{d}{dt}\,(P(t)x,y)_Y + (P(t)x,Ay)_Y + (Ax,P(t)y)_Y + (Rx,Ry)_Z \\ \qquad\qquad\qquad -(B^*P(t)x,B^*P(t)y)_U = 0 \qquad\qquad \forall\, x,y \in \mathcal{D}(A) \\ P(T) = 0\,. \end{cases} \tag{2.13}$$

Among the fundamental conclusions of Theorem 2.6 is assertion S6., namely the property that the optimal cost operator $P(\cdot)$ defined in (2.9) does solve the differential Riccati equation (DRE) corresponding to Problem (2.1)-(2.3). That $P(\cdot)$ is actually the *unique* solution to the DRE (2.13), at least within an appropriate class of operators, is an issue which was not explicitly dealt with in the paper [1].

Thus, in order to render the finite time horizon theory devised in [1] complete, we complement assertion S6. of Theorem 2.6 about existence of solutions to the DRE (2.13) with the (novel) achievement of *uniqueness*, thereby concluding the proof of well-posedness of the DRE. As we will see, uniqueness is meant within a suitable class – that is class $\mathcal{Q}_T$ in (2.14) below – of linear, bounded, self-adjoint operators also meeting an additional requirement, which is consistent with the regularity property displayed by the gain operator in assertion S4. above.

**Theorem 2.7 (Uniqueness for the DRE).** *With reference to the control problem* (2.1)–(2.3)*, let the Assumptions 2.4 hold. Then,*

**S7.** *the differential Riccati equation* (2.13) *has a unique solution within the class*

$$\begin{aligned}\mathcal{Q}_T = \big\{Q \in C([0,T];\mathcal{L}(Y))\colon\ & Q(t)=Q(t)^* \geq 0\,,\ Q(T)=0\,,\\ & B^*Q(\cdot)\in \mathcal{L}(\mathcal{D}(A^\epsilon),C([0,T];U))\big\}\,.\end{aligned} \tag{2.14}$$

*The optimal cost operator* $P(\cdot)$ *defined by* (2.9) *is consequently that solution.*

**Remark 2.8.** We give two distinct proofs of Theorem 2.7: a first one in Section 3 and a second one in Section 4. Both proofs utilize the link between the differential and the *integral* forms of the Riccati equation, as clarified in the beginning of the next section. The standing Assumptions 2.4 play a key role in both proofs, as expected, with the trickier iiic) influencing them in a decisive and crucial way. A major technical step in the longer second proof of uniqueness is the derivation of a *fundamental identity* which is classical in control theory, stated as Lemma 4.1.

In the infinite time horizon case – i.e., when $T=+\infty$ in (2.3) – appropriate requirements on the decay (as $t\to+\infty$) of the semigroup $e^{At}$ as well as of the component $F(t)$ involved in the decomposition of the operator $B^*e^{A^*t}$ are introduced, which both appear very natural; see (2.15) and (2.16) (the latter being (i)' of Assumptions 2.9) below, respectively.

Interestingly, as a consequence of the aforesaid asymptotic behaviour, the requirements on the $L^p$ (in time) regularity of the component $G(\cdot)$ as well as of the operator $B^*e^{A^*\cdot}A^{*\epsilon}$ will need to hold only on a certain bounded interval $[0,T]$, rather than on the entire half-line $(0,\infty)$.
For the sake of completeness and the reader's convenience, the hypotheses pertaining to the infinite time horizon are wholly recorded below.

**Assumptions 2.9 (Infinite time horizon case).** *Let $Y$, $U$ and $Z$ be separable complex Hilbert spaces, and let the basic Assumptions 2.1 be valid, with the additional property that the $C_0$-semigroup $e^{At}$ is exponentially stable on $Y$, $t\geq 0$; namely, there exist constants $M\geq 1$ and $\omega>0$ such that*

$$\|e^{At}\|_{\mathcal{L}(Y)} \leq M\,e^{-\omega t} \qquad \forall t\geq 0\,. \tag{2.15}$$

*Then in particular, $A^{-1}\in\mathcal{L}(Y)$.*

*The operator $B^*e^{A^*t}$ admits the decomposition* (2.5)*, where $F(t)\colon Y\longrightarrow U$, $t\geq 0$, is a bounded linear operator such that*

i)' *there exist constants $\gamma\in(0,1)$ and $N,\eta>0$ such that*

$$\|F(t)\|_{\mathcal{L}(Y,U)} \leq N\,t^{-\gamma}\,e^{-\eta t} \qquad \forall t>0\,, \tag{2.16}$$

*while ii)-iiia)-iiib)-iiic) of the Assumptions 2.4 on the (linear, bounded) component $G(t)\colon\mathcal{D}(A^*)\longrightarrow U$, $t\geq 0$, hold true for some $T>0$.*

We note that the functional (2.3) with $T=+\infty$ makes sense at least for $u\equiv 0$. This again in view of the exponential stability of the semigroup $e^{At}$ ((2.15) of Assumptions 2.9), which combined with (2.4) ensures $Ry(\cdot,y_0;0)\in L^2(0,\infty;Y)$.
(The analysis carried out in the present paper easily extends to more general quadratic functionals, like

$$J(u)=\int_0^\infty \big(\|Ry(t)\|_Z^2+\|\tilde{R}u(t)\|_U^2\big)dt\,,$$

provided $\tilde{R}$ is a coercive operator in $U$. We take $\tilde{R}=I$ just for the sake of simplicity and yet without loss of generality.)

**Theorem 2.10** (Infinite time horizon theory; cf. [3], Theorem 1.5)**.** *Under the Assumptions 2.9, the following statements are valid.*

**A1.** *For any $y_0 \in Y$ there exists a unique optimal pair $(\hat{u}(\cdot), \hat{y}(\cdot))$ for Problem* (2.1)-(2.3)*, which satisfies the following regularity properties*

$$\begin{aligned} &\hat{u} \in \bigcap_{2\le p<\infty} L^p(0,\infty;U)\,, \\ &\hat{y} \in C_b([0,\infty);Y) \cap \Big[ \bigcap_{2\le p<\infty} L^p(0,\infty;Y)\Big]\,. \end{aligned}$$

**A2.** *The family of operators $\Phi(t)$, $t \ge 0$, defined by*

$$\Phi(t)y_0 := \hat{y}(t) = y(t,y_0;\hat{u}) \tag{2.17}$$

*is a $C_0$-semigroup on $Y$, $t \ge 0$, which is exponentially stable.*

**A3.** *The operator $P \in \mathcal{L}(Y)$ defined by*

$$Py_0 := \int_0^\infty e^{A^*t} R^* R \Phi(t) y_0 \, dt \qquad x \in Y \tag{2.18}$$

*is the* optimal cost operator*; $P$ is (self-adjoint and) non-negative.*

**A4.** *The following (*pointwise in time*) feedback representation of the optimal control is valid for any initial state $y_0 \in Y$:*

$$\hat{u}(t) = -B^* P \hat{y}(t) \qquad \text{for a.e. } t \in (0,\infty),$$

*where the gain operator satisfies $B^*P \in \mathcal{L}(\mathcal{D}(A^\epsilon), U)$ (that is, it is just densely defined on $Y$ and yet it is bounded on $\mathcal{D}(A^\epsilon)$).*

**A5.** *The infinitesimal generator $A_P$ of the (optimal state) semigroup $\Phi(t)$ defined in* (2.17) *coincides with the operator $A(I - A^{-1}BB^*P)$; more precisely,*

$$\begin{aligned} &A_P \equiv A(I - A^{-1}BB^*P)\,, \\ &\mathcal{D}(A_P) \subset \big\{ x \in Y : \; x - A^{-1}B\,B^*Px \in \mathcal{D}(A)\big\}\,. \end{aligned}$$

**A6.** *The operator $e^{At}B$, defined in $U$ and a priori with values in $[\mathcal{D}(A^*)]'$, is such that*

$$e^{\delta\cdot} e^{A\cdot} B \in \mathcal{L}(U, L^p(0,\infty;[\mathcal{D}(A^{*\epsilon})]') \quad \forall p \in [1, 1/\gamma) \tag{2.19}$$

*for all $\delta \in [0, \omega \wedge \eta)$; almost the very same regularity is inherited by the operator $\Phi(t)B$:*

$$e^{\delta\cdot}\Phi(\cdot)B \in \mathcal{L}(U, L^p(0,\infty;[\mathcal{D}(A^{*\epsilon})]') \quad \forall p \in [1, 1/\gamma)\,,$$

*with $\delta > 0$ sufficiently small.*

**A7.** *The optimal cost operator $P$ defined in* (2.18) *is a solution to the algebraic Riccati equation (ARE) corresponding to Problem* (2.1)-(2.3)*, that is*

$$\begin{aligned} (Px, Az)_Y + (Ax, Pz)_Y - (B^*Px, B^*Pz)_U + (Rx, Rz)_Z = 0 \\ \text{for any } x, z \in \mathcal{D}(A) \text{ (or } x, z \in \mathcal{D}(A_P)). \end{aligned} \tag{2.20}$$

In order to render the infinite time horizon theory devised in [3] complete, we complement assertion A7. of Theorem 2.10 about existence of solutions to the ARE (2.20) corresponding to Problem (2.1)-(2.3), with the achievement of *uniqueness*, thereby concluding the proof of well-posedness of the ARE.

**Theorem 2.11** (**Uniqueness for the ARE).** *Consider the optimal control problem* (2.1)-(2.3)*, with $T = +\infty$, under the Assumptions 2.9. Then,*

**A8.** *the algebraic Riccati equation* (2.20) *has a unique solution $P$ within the class $\mathcal{Q}$ defined as follows:*

$$\mathcal{Q} := \big\{ Q \in \mathcal{L}(Y) \colon Q = Q^* \ge 0\,,\ B^*Q \in \mathcal{L}(\mathcal{D}(A^\epsilon), U) \big\}\,. \tag{2.21}$$

*The optimal cost operator $P$ defined by* (2.18) *is consequently that solution.*

**Remark 2.12.** As in the finite time horizon case, the (linear, bounded, self-adjoint) operators that belong to the class $\mathcal{Q}$ in (2.21) are characterized by a requirement that is consistent with the regularity property displayed by the gain operator in assertion A4. of Theorem 2.10.

## 3. A first proof of uniqueness for the DRE

In this Section we derive integral forms of both the differential and algebraic Riccati equations, and present a first proof of Theorem 2.7. The argument employed in this proof is pretty standard: the difference between another possible solution $P_1(t)$ to the DRE and the Riccati operator $P(t)$ is shown – in a series of steps – to be identically zero on the interval $[0, T]$. The final goal is attained fully exploiting the Assumptions 2.4 and more specifically iiic), having taken as a starting point the integral form of the DRE.

By contrast, in the next Section 4 a unified approach and method of proof will prove effective in showing uniqueness for both cases.

### 3.1. **Finite time interval, differential Riccati equations.**

In this subsection we make reference to the optimal control problem (2.1)–(2.3), with $T < +\infty$. We address the issue of uniqueness of solutions to the Cauchy problem (2.13) for the Riccati equation corresponding to problem (2.1)–(2.3), under the Assumptions 2.4.

We begin by relating the differential form (2.13) of the Riccati equation to an integral form of it, which in turn can be further interpreted.

**Lemma 3.1** (Integral forms of the Riccati equation)**.** *Let $\mathcal{Q}_T$ be the class defined in* (2.14)*, and let $Q(\cdot) \in \mathcal{Q}_T$ be a solution to the DRE* (2.13)*. Then the following assertions hold true.*

**1.** *$Q(\cdot)$ solves the integral Riccati equation (in short, IRE), that is*

$$\begin{aligned}\big(Q(t)e^{A(t-s)}x, e^{A(t-s)}y\big)_Y &= (Q(s)x,y)_Y - \int_s^t \big(Re^{A(r-s)}x, Re^{A(r-s)}y\big)_Z\,dr \\ &\quad + \int_s^t \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)e^{A(r-s)}y\big)_U\,dr\,,\end{aligned} \tag{3.1}$$

*with $0 \le s \le t \le T$ and $x, y \in \mathcal{D}(A^\epsilon)$.*

**2.** $B^*Q(\cdot)e^{A(\cdot-s)} \in \mathcal{L}(Y, L^2(s,T;U))$.

**3.** *The IRE* (3.1) *can be rewritten in the form*

$$\begin{aligned}\big(e^{A^*(t-s)}Q(t)e^{A(t-s)}x, y\big)_Y &= (Q(s)x,y)_Y - \int_s^t \big(e^{A^*(r-s)}R^*Re^{A(r-s)}x, y\big)_Y\,dr \\ &\quad + \int_s^t \big(e^{A^*(r-s)}Q(r)BB^*Q(r)e^{A(r-s)}x, y\big)_Y\,dr\,,\end{aligned} \tag{3.2}$$

*valid for any $x, y \in Y$ and with $0 \le s \le t \le T$.*

*Proof.* **1.** Let $x, y \in \mathcal{D}(A)$: then $e^{A\cdot}x$, $e^{A\cdot}y$ are differentiable, and therefore, using (2.13), there exists

$$\begin{aligned}
&\frac{d}{dr}\big(Q(r)e^{A(r-s)}x, e^{A(r-s)}y\big)_Y = \\
&\qquad = -\big(Q(r)e^{A(r-s)}x, Ae^{A(r-s)}y\big)_Y - \big(Ae^{A(r-s)}x, Q(r)e^{A(r-s)}y\big)_Y - \\
&\qquad - \big(Re^{A(r-s)}x, Re^{A(r-s)}y\big)_Z + \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)e^{A(r-s)}y\big)_U + \\
&\qquad + \big(Q(r)Ae^{A(r-s)}x, e^{A(r-s)}y\big)_Y + \big(Q(r)e^{A(r-s)}x, Ae^{A(r-s)}y\big)_Y = \\
&\qquad - \big(Re^{A(r-s)}x, Re^{A(r-s)}y\big)_Z + \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)e^{A(r-s)}y\big)_U .
\end{aligned}$$

Integrating the above identity in $r \in [s,t]$, one readily obtains the IRE (3.1), valid for $x, y \in \mathcal{D}(A)$. In view of Lemma A.5, the validity of the IRE is extended to all $x, y \in \mathcal{D}(A^\epsilon)$ by density.

**2.** By taking now in (3.1) $t = T$, $x = y \in \mathcal{D}(A^\epsilon)$, since $P(T) = 0$ we establish

$$\int_s^T \big\|B^*P(r)e^{A(r-s)}x\big\|_U^2\,dr \le \int_s^T \big\|Re^{A(r-s)}x\big\|_Z^2\,dr \le C\|x\|_Y^2$$

by density.

**3.** The equivalent form (3.2) of the IRE follows in view of **2.** and by density. $\square$

*A first proof of Theorem 2.7.* We follow the proof of Theorem 1.5.3.3 in [25], up to a point. The subsequent arguments and estimates are driven by the distinctive assumptions on the adjoint of the kernel $e^{At}B$, as well as by the different class of regularity the solutions to the DRE are sought.

We know already that the optimal cost operator $P(\cdot)$ defined by (2.9) solves (the Cauchy problem (2.13) for) the differential Riccati equation, as well as that $P \in \mathcal{Q}_T$. Assume there exists another operator in $\mathcal{Q}_T$, say $P_1(\cdot)$, which solves (2.13), and set $Q(t) := P_1(t) - P(t)$, $t \in [0,T]$; we aim to prove that $Q(t) \equiv 0$. By construction $Q(\cdot) \in \mathcal{Q}_T$. By Lemma 3.1, both $P_1(\cdot)$ and $P(\cdot)$ satisfy the IRE (3.1). Then, taking in particular $t = T$, we find that $Q(s)$ satisfies

$$\begin{aligned}
(Q(s)x, y)_Y = &-\int_s^T (B^*Q(r)e^{A(r-s)}x, B^*P_1(r)e^{A(r-s)}y)_U\,dr \\
&-\int_s^T (B^*P(r)e^{A(r-s)}x, B^*Q(r)e^{A(r-s)}y)_U\,dr\,,
\end{aligned} \tag{3.3}$$

for any $x, y \in D(A^\epsilon)$. To render the computations cleaner, set $V(r) := B^*Q(r)$ (that $r$ belongs to $[s,T]$ is omitted here and below, as clear from the context). Because $Q(\cdot) \in \mathcal{Q}_T$, it holds $V(r)^* \in \mathcal{L}(U, [\mathcal{D}(A^\epsilon)]')$, along with

$$\|V(r)^*\|_{\mathcal{L}(U,[\mathcal{D}(A^\epsilon)]')} = \|V(r)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)} \le \|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U)} =: c\,.$$

We see that

$$\big|\langle V(r)^*w, y\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\big| \le c\,\|w\|_U\|y\|_{D(A^\epsilon)}$$

consequently, as well as that $\big[A^{*-\epsilon}V(r)\big]^* \in \mathcal{L}(U,Y)$, with

$$\big|\big(\big[A^{*-\epsilon}V(r)\big]^*w,x\big)_Y\big| = \big|\langle [V(r)^*w, A^{-\epsilon}x\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\big| \le c\,\|w\|_U\|x\|_Y\,.$$

The same observations apply to $[B^*P_1(r)]^*$ and $[B^*P(r)]^*$, bringing about analogous estimates.

We may now rewrite (3.3) as

$$\begin{aligned}(Q(s)x,y)_Y = &-\int_s^T (e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)}x, A^\epsilon y)_Y\,dr\\ &-\int_s^T (e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}x, A^\epsilon y)_Y\,dr\,,\end{aligned}$$

which tells us that

$$\begin{aligned}A^{*\epsilon}\int_s^T \Big[&e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)}\\ &+e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}\Big]x\,dr\,,\end{aligned}$$

a priori an element of $[\mathcal{D}(A^\epsilon)]'$, in fact coincides with $-Q(s)x \in Y$ by the very definition of adjoint operator. We deduce

$$\begin{aligned}Q(s)x = -A^{*\epsilon}\int_s^T \Big[&e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)}\\ &+e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}\Big]x\,dr\,,\end{aligned}\tag{3.4}$$

valid for every $x \in D(A^\epsilon)$, where, as pointed out above, the right hand side is an element of $Y$. As $x \in \mathcal{D}(A^\epsilon)$, $B^*Q(s)x$ is meaningful, and we are allowed to apply $B^*$ to both sides of (3.4), thus obtaining

$$\begin{aligned}V(s)x = -B^*(A^*)^\epsilon\int_s^T \Big[&e^{A^*(r-s)}A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)}\\ &+e^{A^*(r-s)}A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}\Big]x\,dr\,.\end{aligned}\tag{3.5}$$

It is here where iiic) of Assumptions 2.4, that is

$$\exists\, q \in (1,2)\,,\ C = C(T) > 0\colon\quad \|B^*e^{A^*(\cdot-s)}A^{*\epsilon}x\|_{L^q(s,T;U)} \le C\,\|x\|_Y \qquad \forall x \in Y\,,$$

becomes crucially important: indeed, it yields as well

$$\|[B^*e^{A^*(\cdot-s)}A^{*\epsilon}]^*g(\cdot)\|_Y \le C\,\|g\|_{L^{q'}(s,T;U)}$$

($q'$ denotes the conjugate exponent of $q$), so that in particular

$$\|[B^*e^{A^*(\cdot-s)}A^{*\epsilon}]^*w\|_Y \le C\,(T-s)^{1/q'}\|w\|_U \qquad \forall w \in U\,. \tag{3.6}$$

We return to (3.5), and highlight a few blocks within its right hand side, as follows:

$$\begin{aligned}V(s)x = &-\int_s^T \big[B^*e^{A^*(r-s)}A^{*\epsilon}\big]\,A^{*-\epsilon}[B^*P_1(r)]^*V(r)e^{A(r-s)}x\,dr\\ &-\int_s^T \big[B^*e^{A^*(r-s)}A^{*\epsilon}\big]A^{*-\epsilon}V(r)^*B^*P(r)e^{A(r-s)}x\,dr\,;\end{aligned}$$

multiply next both members by $w \in U$, to find

$$(V(s)x, w)_U = -\int_s^T \big(V(r)e^{A(r-s)}x, [B^*P_1(r)A^{-\epsilon}]\,\big[B^*e^{A^*(r-s)}A^{*\epsilon}\big]^*w\big)_U\,dr \\ - \int_s^T \big(B^*P(r)e^{A(r-s)}x, [V(r)A^{-\epsilon}]\,\big[B^*e^{A^*(r-s)}A^{*\epsilon}\big]^*w\big)_U\,dr\,.$$

We now proceed to estimate either summand in the right hand side, making use of (3.6); this leads to

$$\begin{aligned}&|(V(s)x, w)_U| \\ &\le M\,\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U))}\|x\|_{\mathcal{D}(A^\epsilon)}\|B^*P_1(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U))}\|w\|_U(T-s)^{1/q'} \\ &\quad + M\,\|B^*P(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U))}\|x\|_{\mathcal{D}(A^\epsilon)}\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U))}\|w\|_U(T-s)^{1/q'}\,.\end{aligned}$$

Therefore, there exists a positive constant $C$ (depending on $P$ and $P_1$) such that

$$\big|(V(s)x, w)_U\big| \le C\,\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U))}(T-s)^{1/q'}\|w\|_U\|x\|_{\mathcal{D}(A^\epsilon)}\,,$$

which establishes

$$\|V(s)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)} \le C\,\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s,T];U))}(T-s)^{1/q'}\,, \tag{3.7}$$

for any $s \in [0,T)$.

The argument is now pretty standard: set $s_0$ such that $(T-s_0)^{1/q'} < 1/C$; since the estimate (3.7) holds true in particular for any $s \in [s_0,T)$, we have

$$\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s_0,T];U))} \le C\,(T-s_0)^{1/q'}\,\|V(\cdot)\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),C([s_0,T];U))}$$

which is impossible unless $V(\cdot) \equiv 0$ on $[s_0,T]$. Iterating the same argument, in a finite number of steps we obtain $V(s) \equiv 0$ on $[0,T]$. This in turn implies, by (3.3),

$$(Q(s)x, y)_Y = 0 \qquad \forall s \in [0,T]\,,\ \forall x, y \in \mathcal{D}(A^\epsilon)\,;$$

by density we obtain $(Q(s)x = 0$ for any $x \in Y$ first, and then) $Q(\cdot) \equiv 0$, that is $P_1(\cdot) \equiv P(\cdot)$, as desired.

□

3.2. **Infinite time interval. Preparatory material.** We turn now our attention to the optimal control problem (2.1)–(2.3), with $T = +\infty$. In order to establish a uniqueness result for the corresponding algebraic Riccati equation (2.20), we will employ a different method of proof than the one utilized in the previous subsection for the DRE. Still, an integral form of the ARE will prove more effective (than its algebraic form) to accomplish this goal, just like the integral forms of the DRE in Lemma 3.1 provide fundamental tools for both proofs of Theorem 2.7. This is the reason why we derive the said integral form of the ARE here. The following Lemma contributes to the preparatory material for the forthcoming analysis in Section 4. Its proof is not difficult, yet it is explicitly given for the reader's convenience.

**Lemma 3.2** (Integral form of the ARE)**.** *Let $\mathcal{Q}$ be the class defined in* (2.21)*, and let $P_1 \in \mathcal{Q}$ be a solution to the algebraic Riccati equation* (2.20)*. Then, $P_1$ solves* *the following integral form of the ARE valid for all* $x, y \in \mathcal{D}(A^\epsilon)$*:*

$$\begin{aligned}\big(P_1 e^{A(t-s)}x, e^{A(t-s)}y\big)_Y = (P_1 x, y)_Y + \int_s^t \big(B^* P_1 e^{A(r-s)}x, B^* P_1 e^{A(r-s)}y\big)_U\, dr \\ - \int_s^t \big(R e^{A(r-s)}x, R e^{A(r-s)}y\big)_Z\, dr\,,\end{aligned} \tag{3.8}$$

*with* $0 \le s \le t$.

*Proof.* Let $P_1 \in \mathcal{Q}$ be a solution to the ARE (2.20), that is

$$(P_1 x, Ay)_Y + (Ax, P_1 y)_Y - (B^* P_1 x, B^* P_1 y)_U + (Rx, Ry)_Z = 0\,, \quad x, y \in \mathcal{D}(A)\,.$$

With $e^{A(t-s)}x$, $e^{A(t-s)}y \in \mathcal{D}(A)$ in place of $x, y$, and with $0 \le s \le t$, the equation becomes

$$\begin{aligned}&(P_1 e^{A(t-s)}x, Ae^{A(t-s)}y)_Y + (Ae^{A(t-s)}x, P_1 e^{A(t-s)}y)_Y \\ &\qquad - (B^* P_1 e^{A(t-s)}x, B^* P_1 e^{A(t-s)}y)_U + (Re^{A(t-s)}x, Re^{A(t-s)}y)_Z = 0\,,\end{aligned}$$

that is nothing but

$$\begin{aligned}\frac{d}{dt}\big(P_1 e^{A(t-s)}x, e^{A(t-s)}y\big)_Y = \big(B^* P_1 e^{A(t-s)}x, B^* P_1 e^{A(t-s)}y\big)_U \\ - \big(Re^{A(t-s)}x, Re^{A(t-s)}y\big)_Z\,, \quad x, y \in \mathcal{D}(A)\,.\end{aligned} \tag{3.9}$$

Integrating both sides of (3.9) between $s$ and $t$ we attain (3.8), initially for any $x, y \in \mathcal{D}(A)$. Its validity is then extended to all $x, y \in \mathcal{D}(A^\epsilon)$ by density, since $P_1 \in \mathcal{Q}$. □

While the integral form (3.8) of the ARE will constitute the starting point for the proof of Theorem 2.11, it is important to emphasize the central role of the distinguishing (and improved) regularity properties of the operator $B^* e^{A^* \cdot} A^{*\epsilon}$. We refer the reader to Appendix A, where we collected and highlighted several instrumental results, with the aim of displaying their statements in a clear sequence and framework. See, more specifically, Proposition A.6 therein.

## 4. A unified method of proof of uniqueness for both DRE and ARE

In this Section we provide a second proof of Theorem 2.7 and then show Theorem 2.11, thereby settling the question of uniqueness for the differential and algebraic Riccati equations corresponding to the optimal control problem (2.2)–(2.3). We recall from Section 1.1 that the a crucial intermediate step to achieve either goal is an identity which is classical in control theory.

4.1. **Finite time interval, differential Riccati equations.** In this subsection we focus on the optimal control problem (2.1)–(2.3), with $T < +\infty$, along with the corresponding Riccati equation. In approaching the second proof of Theorem 2.7, we start by showing the above-mentioned fundamental identity. Despite being a standard element in classical optimal control theory, the identity should not be taken for granted in the absence of evident beneficial regularity properties of the kernel $e^{At}B$ – such as analiticity of the semigroup or more generally singular estimates. Achieving the said equality requires that the Assumptions 2.4 are fully exploited. The delicate, careful computations are carried out in the following Lemma.

**Lemma 4.1** (Fundamental identity). *Let $Q \in \mathcal{Q}_T$ be a solution to the integral Riccati equation* (3.1). *With $u \in L^2(s,T;U)$ and $x \in \mathcal{D}(A^\epsilon)$, let $y(\cdot)$ be the semigroup solution to the state equation in* (2.1) *corresponding to $u(\cdot)$, with $y(s) = x$, that is*

$$y(t) = e^{A(t-s)}x + \int_s^t e^{A(t-r)}Bu(r)\,dr =: e^{A(t-s)}x + L_s u(t)\,, \quad t \in [s,T]\,.$$

*Then, the following identity is valid: for $t \in [s,T]$*

$$\begin{aligned}(Q(t)y(t),y(t))_Y - (Q(s)x,x)_Y = &-\int_s^t \left[\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\right]\,dr \\ &+\int_s^t \|u(r)+B^*Q(r)y(r)\|_U^2\,dr\,.\end{aligned} \tag{4.1}$$

*Proof.* Assume initially that $u \in L^\infty(s,T;U)$. We examine the right hand side of the identity (4.1). For the first term we have

$$\begin{aligned}-\int_s^t \|Ry(r)\|_Z^2 = &-\int_s^t \left\|Re^{A(r-s)}x\right\|_Z^2 dr - \int_s^t \left\|RL_s u(r)\right\|_Z^2 dr \\ &- 2\mathrm{Re}\int_s^t \left(Re^{A(r-s)}x, RL_s u(r)\right)_Z dr =: \sum_{j=1}^3 R_j\,.\end{aligned}$$

We note that each summand $R_j$ makes sense, just considering the space regularity originally singled out in [1] and here recalled in Proposition A.1; more specifically, $u \in L^\infty(s,T;U)$ implies $L_s u \in C([s,T];Y)$ by its fourth assertion. We consider next the remainder

$$-\int_s^t \|u(r)\|_U^2\,dr + \int_s^t \|u(r)+B^*Q(r)y(r)\|_U^2\,dr\,.$$

Computing the square in the second integral, discarding additive inverses and replacing again the expression of $y(r)$, we get

$$\begin{aligned}&-\int_s^t \|u(r)\|_U^2\,dr + \int_s^t \|u(r)+B^*Q(r)y(r)\|_U^2\,dr \\ &= 2\mathrm{Re}\int_s^t \left(B^*Q(r)e^{A(r-s)}x, u(r)\right)_U dr + 2\mathrm{Re}\int_s^t \left(B^*Q(r)L_s u(r), u(r)\right)_U dr \\ &\quad + \int_s^t \left\|B^*Q(r)e^{A(r-s)}x\right\|_U^2 dr + 2\mathrm{Re}\int_s^t \left(B^*Q(r)e^{A(r-s)}x, B^*Q(r)L_s u(r)\right)_U dr \\ &\quad + \int_s^t \left\|B^*Q(r)L_s u(r)\right\|_U^2 dr =: \sum_{j=1}^5 C_j\,.\end{aligned}$$

That each summand $C_j$ makes sense as well is justified by the following observations: $B^*Q(\cdot)e^{A(\cdot-s)}x \in L^2(s,T;U)$ because of item 2. of Lemma 3.1; in addition, since $L^\infty(s,T;U) \subset L^{q'}(s,T;U)$, Lemma A.2 yields the improved regularity $L_s u \in C([s,T];\mathcal{D}(A^\epsilon))$, which in turn implies $B^*Q(\cdot)L_s u(\cdot) \in C([s,T];U)$, as shown in Lemma A.5.

By using the original form (3.1) of the integral Riccati equation (IRE), with $x=y$, we find that

$$\begin{aligned} R_1+C_3 &= -\int_s^t \|Re^{A(r-s)}x\|_Z^2\,dr + \int_s^t \big\|B^*Q(r)e^{A(r-s)}x\big\|_U^2\,dr \\ &= \big(Q(t)e^{A(t-s)}x, e^{A(t-s)}x\big)_Y - (Q(s)x,x)_Y\,. \end{aligned} \tag{4.2}$$

Next,

$$\begin{aligned} R_3+C_4+C_1 &= -2\mathrm{Re}\int_s^t \big(Re^{A(r-s)}x, RL_s u(r)\big)_Z\,dr \\ &\quad + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, B^*Q(r)L_s u(r)\big)_U\,dr \\ &\quad + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, u(r)\big)_U\,dr \\ &= -2\mathrm{Re}\int_s^t \Big(R^*Re^{A(r-s)}x, \int_s^r e^{A(r-\sigma)}Bu(\sigma)\,d\sigma\Big)_Y\,dr \\ &\quad + 2\mathrm{Re}\int_s^t \Big\langle Q(r)BB^*Q(r)e^{A(r-s)}x, \int_s^r e^{A(r-\sigma)}Bu(\sigma)\,d\sigma\Big\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\,dr \\ &\quad + 2\mathrm{Re}\int_s^t \big(B^*Q(r)e^{A(r-s)}x, u(r)\big)_U\,dr\,, \end{aligned}$$

where the duality in the penultimate term is based on the membership $Q(\cdot)\in\mathcal{Q}_T$ (along with the estimate (A.2)) which yields $Q(r)B\in\mathcal{L}(U,[\mathcal{D}(A^\epsilon)]')$, combined as before with $L_s u\in C([s,T];\mathcal{D}(A^\epsilon))$. The above leads to

$$\begin{aligned} R_3+C_4+C_1 &= -2\mathrm{Re}\int_s^t\int_s^r \big(B^*e^{A^*(r-\sigma)}R^*Re^{A(r-s)}x, u(\sigma)\big)_U\,d\sigma\,dr + \\ &\quad + 2\mathrm{Re}\int_s^t\int_s^r \big(B^*e^{A^*(r-\sigma)}Q(r)BB^*Q(r)e^{A(r-s)}x, u(\sigma)\big)_U\,d\sigma dr \\ &\quad + 2\mathrm{Re}\int_s^t (B^*Q(\sigma)e^{A(\sigma-s)}x, u(\sigma)\big)_U\,d\sigma\,, \end{aligned}$$

which can be rewritten, exchanging the order of integration, as follows:

$$\begin{aligned} R_3+C_4+C_1 = -\,2\mathrm{Re}\int_s^t \Big(B^*\Big\{ & \int_\sigma^t e^{A^*(r-\sigma)}R^*R\big[e^{A(r-s)}x\big]\,dr \\ & - \int_\sigma^t e^{A^*(r-\sigma)}Q(r)BB^*Q(r)e^{A(r-\sigma)}\big[e^{A(\sigma-s)}x\big]\,dr \\ & - Q(\sigma)\big[e^{A(\sigma-s)}x\big]\Big\}, u(\sigma)\Big)_U\,d\sigma\,. \end{aligned} \tag{4.3}$$

Let us focus on the expression inside the curly bracket. Because $Q(\cdot)$ solves the IRE (3.1), as well as its second form (3.2) valid for any pair $x,y\in Y$, then the following identity – a *strong* form of the IRE, when $Q(\cdot)$ is unknown – holds true:

$$e^{A^*(t-\sigma)}Q(t)e^{A(t-\sigma)}z = Q(\sigma)z - \int_\sigma^t e^{A^*(r-\sigma)}R^*Rz\,dr$$
$$+\int_\sigma^t e^{A^*(r-\sigma)}Q(r)BB^*Q(r)e^{A(r-\sigma)}z\,dr\,, \quad 0\le\sigma\le t\le T\,,\ z\in Y\,.$$

Thus, returning to (4.3) with this information and setting in particular $z = e^{A(\sigma-s)}x$, we find that $R_3 + C_4 + C_1$ simply reads as follows:

$$\begin{aligned} R_3 + C_4 + C_1 &= -2\mathrm{Re}\int_s^t \big(B^*e^{A^*(t-\sigma)}Q(t)e^{A(t-s)}x, u(\sigma)\big)_U\,d\sigma \\ &= -2\mathrm{Re}\big(Q(t)e^{A(t-s)}x, L_s u(t)\big)_Y\,. \end{aligned} \tag{4.4}$$

We examine next the sum

$$R_2 + C_5 = -\int_s^t \|RL_su(r)\|_Z^2\,dr + \int_s^t \big\|B^*Q(r)L_su(r)\big\|_U^2\,dr\,,$$

where, again, since $u\in L^\infty(s,T;U)\subset L^{q'}(s,T;U)$, we know from Lemma A.2 that $L_s u\in C([s,T];\mathcal{D}(A^\epsilon))$. Consequently, one gets

$$\begin{aligned} R_2 + C_5 &= -\int_s^t \big\langle \big[R^*R - Q(r)BB^*Q(r)\big]L_su(r), L_su(r)\big\rangle_{[\mathcal{D}(A^\epsilon)]',\mathcal{D}(A^\epsilon)}\,dr \\ &= -\mathrm{Re}\int_s^t \Big(A^{*-\epsilon}\big[R^*R - Q(r)BB^*Q(r)\big]A^{-\epsilon}\int_s^r A^\epsilon e^{A(r-\lambda)}Bu(\lambda)\,d\lambda, \\ &\qquad\qquad \int_s^r A^\epsilon e^{A(r-\mu)}Bu(\mu)\,d\mu\Big)_Y\,dr\,. \end{aligned} \tag{4.5}$$

It is important to emphasize that in going from the duality to the inner product in (4.5), two facts have been crucially used, besides $Q(\cdot)\in\mathcal{Q}_T$: the hypothesis (2.7) on the observation operator $R$ (that is iiib) of the Assumptions 2.4), and once again, Lemma A.2. Further handling of the right hand side of (4.5) leads to the triple integral

$$R_2 + C_5 = -\mathrm{Re}\int_s^t I(r,s)\,dr\,,$$

having set

$$I(r,s) = \int_s^r\int_s^r \big(B^*e^{A^*(r-\mu)}\,\big[R^*R - Q(r)BB^*Q(r)\big]e^{A(r-\lambda)}Bu(\lambda), u(\mu)\big)_U\,d\lambda\,d\mu\,.$$

Let us focus on the inner double integral $I(r,s)$. We note that this integral pertains to a symmetric function of $(\lambda,\mu)$, and hence the integral over the square $[s,r]\times[s,r]$ can be replaced by twice the integral over the triangle

$$\big\{(\lambda,\mu)\colon\, s\le\mu\le\lambda\le r\big\}\,.$$

It follows that

$$
\begin{aligned}
I(r,s) &= 2\int_s^r d\lambda \int_s^\lambda d\mu \left(B^* e^{A^*(r-\mu)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}Bu(\lambda), u(\mu)\right)_U \\
&= 2\int_s^r \Big[\int_s^\lambda \left(B^* e^{A^*(\lambda-\mu)}\, e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}Bu(\lambda), u(\mu)\right)_U d\mu\Big]\, d\lambda \\
&= 2\int_s^r \left( e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}Bu(\lambda), \int_s^\lambda e^{A(\lambda-\mu)}Bu(\mu)\,d\mu\right)_Y d\lambda \\
&= 2\int_s^r \left( e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}Bu(\lambda), L_s u(\lambda)\right)_Y d\lambda\,.
\end{aligned}
$$

Inserting the expression of $I(r,s)$ obtained above in the outer integral yields

$$
\begin{aligned}
&R_2 + C_5 \\
&\quad = -2\mathrm{Re}\int_s^t\!\!\int_s^r \left( e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}Bu(\lambda), L_s u(\lambda)\right)_Y d\lambda\,dr\,;
\end{aligned}
$$

next we exchange the order of integration and also move the first argument of the inner product, to achieve

$$
\begin{aligned}
&R_2 + C_5 \\
&= -2\mathrm{Re}\int_s^t\!\!\int_\lambda^t \left( e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}Bu(\lambda), L_s u(\lambda)\right)_Y dr\,d\lambda \\
&= -2\mathrm{Re}\int_s^t\!\!\int_\lambda^t \left(u(\lambda), B^* e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}L_s u(\lambda)\right)_Y dr\,d\lambda \\
&= -2\mathrm{Re}\int_s^t \left(u(\lambda), B^*\int_\lambda^t e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}\,L_s u(\lambda)\,dr\right)_U d\lambda\,.
\end{aligned}
\tag{4.6}
$$

It is apparent that the second form (3.2) of the IRE (with $\lambda$ in place of $s$) – in fact, a strong form of it – provides once more the tool, just like in deriving (4.4) from (4.3). With $z = L_s u(\lambda)$, replace the integral

$$
\int_\lambda^t e^{A^*(r-\lambda)}\left[R^*R - Q(r)BB^*Q(r)\right]e^{A(r-\lambda)}z\,dr
$$

by $[Q(\lambda) - e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}]z$, to find

$$
\begin{aligned}
R_2 + C_5 &= -2\mathrm{Re}\int_s^t \left(u(\lambda), B^*\left[Q(\lambda) - e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}\right]L_s u(\lambda)\right)_U d\lambda \\
&= -2\mathrm{Re}\int_s^t \left(B^*\left[Q(\lambda) - e^{A^*(t-\lambda)}Q(t)e^{A(t-\lambda)}\right]L_s u(\lambda), u(\lambda)\right)_U d\lambda\,.
\end{aligned}
\tag{4.7}
$$

Thus, adding $C_2$ to (4.7), we see that a useful simplification occurs, as detailed below:

$$\begin{aligned}
R_2 + C_5 + C_2 = &-2\mathrm{Re}\int_s^t \big(B^* Q(\lambda) L_s u(\lambda), u(\lambda)\big)_U \, d\lambda \\
&+ 2\mathrm{Re}\int_s^t \big(B^* e^{A^*(t-\lambda)} Q(t) e^{A(t-\lambda)} L_s u(\lambda), u(\lambda)\big)_U \, d\lambda \\
&+ 2\mathrm{Re}\int_s^t \big(B^* Q(r) L_s u(r), u(r)\big)_U \, dr \\
= &\; 2\mathrm{Re}\int_s^t\int_s^\lambda \big(Q(t) e^{A(t-\sigma)} B u(\sigma), e^{A(t-\lambda)} B u(\lambda)\big)_Y \, d\sigma\, d\lambda \,.
\end{aligned}$$

Owing to the simmetry of the latter integrand in $(\sigma,\lambda)$, we may replace twice the integral over the triangle $\{(\lambda,\sigma)\colon s \le \lambda \le \sigma \le t\}$ by the integral over the square $[s,t]\times[s,t]$, and finally get

$$\begin{aligned}
R_2 + C_5 + C_2 &= \mathrm{Re}\int_s^t\int_s^t \big(Q(t) e^{A(t-\sigma)} B u(\sigma), e^{A(t-\lambda)} B u(\lambda)\big)_Y \, d\lambda\, d\sigma \\
&= \mathrm{Re}\big(Q(t) L_s u(t), L_s u(t)\big)_Y = \big(Q(t) L_s u(t), L_s u(t)\big)_Y \,.
\end{aligned} \tag{4.8}$$

Combining (4.8) with (4.2) and (4.4), we finally obtain

$$\begin{aligned}
\sum_{i=1}^3 R_i + \sum_{j=1}^5 C_j &= \big(Q(t) e^{A(t-s)} x, e^{A(t-s)} x\big)_Y - (Q(s)x, x)_Y \\
&\quad + 2\mathrm{Re}(Q(t) e^{A(t-s)} x, L_s u(t))_Y + (Q(t) L_s u(t), L_s u(t))_Y \\
&= (Q(t) y(t), y(t))_Y - (Q(s)x, x)_Y \,,
\end{aligned}$$

which establishes the fundamental identity (4.1) in the case $u \in L^\infty(s,T;U)$. Finally, the identity extends to $u \in L^2(s,T;U)$ by density, which concludes the proof of Lemma 4.1.

□

We next introduce an integral equation that involves a given operator solution $Q(t)$ to the Riccati equation corresponding to optimal control problem (2.1)–(2.3). Once uniqueness for the DRE (2.13) is established, so that $Q(t)$ must coincide with the Riccati operator $P(t)$, then it will be clear that the said integral equation (*viz.* (4.9) below) is nothing but the well known *closed-loop equation*, of central importance for the synthesis of the optimal control. (This justifies the use of the term "closed-loop equation" for (4.9)).

As we shall see, the following Lemma 4.2 and (the independent) Lemma 4.1 constitute the core elements for the proof of Theorem 2.7.

**Lemma 4.2.** *Let $\epsilon$ be as in iii) of Assumptions 2.4. Let $Q \in \mathcal{Q}_T$, where $\mathcal{Q}_T$ is the class defined by* (2.14)*. Then, for every $x \in \mathcal{D}(A^\epsilon)$, the closed loop equation*

$$y(t) = e^{At}x - \int_0^t e^{A(t-s)} B B^* Q(\sigma) y(\sigma)\, d\sigma \,, \qquad t \in [0,T] \,, \tag{4.9}$$

*has a unique solution in the space*

$$X=\Big\{y\in C([0,T];\mathcal{D}(A^\epsilon))\colon \sup_{t\in[0,T]}\big(e^{-rt}\|y(t)\|_{\mathcal{D}(A^\epsilon)}\big)<\infty\Big\} \tag{4.10}$$

*endowed with the norm*

$$\|y\|_{X,r}=\sup_{t\in[0,T]} e^{-rt}\|y(t)\|_{\mathcal{D}(A^\epsilon)}\,, \qquad y\in X\,,$$

*provided $r>0$ is chosen sufficiently large.*

*Proof.* With $x\in\mathcal{D}(A^\epsilon)$, we set $E(t)=e^{At}x$. By semigroup theory we know that $E(\cdot)\in C([0,T];\mathcal{D}(A^\epsilon))$; even more, since $e^{At}$ is exponentially stable, it holds $E(\cdot)\in X$ provided $r$ is sufficiently large. As the integral equation (4.9) has the clear form

$$y(t)+\big[LB^*Q(\cdot)y(\cdot)\big](t)=E(t)\,,\quad t\in[0,T]\,,$$

we appeal to a classical argument of functional analysis: we will prove that $LB^*Q(\cdot)$ is a contraction mapping in $X$, having chosen $r$ sufficiently large. This will in turn imply that $I+LB^*Q(\cdot)$ is invertible in $X$, thus providing the sought unique solution to (4.9).

For each $y\in X$, $z\in\mathcal{D}(A^{*\epsilon})$, $t\in[0,T]$, we have by Lemma A.5

$$\begin{aligned}
&\big|(e^{-rt}LB^*Q(\cdot)y(\cdot),A^{*\epsilon}z)_Y\big|\\
&\quad=\Big|\int_0^t e^{-r(t-s)}\,\big(B^*Q(s)e^{-rs}y(s),B^*e^{A^*(t-s)}A^{*\epsilon}z\big)_U\,ds\Big|\\
&\quad\le\int_0^t e^{-r(t-s)}\|B^*Q(\cdot)e^{-r\cdot}y(\cdot)\|_{C([0,T];U)}\,\big\|B^*e^{(t-s)A^*}A^{*\epsilon}z\big\|_U\,ds\\
&\quad\le\|B^*Q(\cdot)\|_{C([0,T];\mathcal{D}(A^\epsilon)),C([0,T];U))}\,\|y\|_{X,r}\int_0^t e^{-r\sigma}\|B^*e^{A^*\sigma}A^{*\epsilon}z\|_U\,d\sigma\\
&\quad\le\|B^*Q(\cdot)\|_{C([0,T];\mathcal{D}(A^\epsilon)),C([0,T];U))}\,\|y\|_{X,r}\,\Big[\int_0^t e^{-r\sigma q'}\,d\sigma\Big]^{1/q'}\|B^*e^{\cdot A^*}A^{*\epsilon}z\|_{L^q(0,T;U)}\\
&\quad\le\|B^*Q(\cdot)\|_{C([0,T];\mathcal{D}(A^\epsilon)),C([0,T];U))}\,\frac{1}{(rq')^{1/q'}}\,\|B^*e^{A^*\cdot}A^{*\epsilon}\|_{\mathcal{L}(Y,L^q(0,T;U))}\,\|z\|_Y\,\|y\|_{X,r}\,.
\end{aligned}$$

We note that in going from the antepenultimate to the penultimate estimate we used iiic) of the Assumptions 2.4. Therefore, there exist positive constants $c,c'$ such that

$$\big\|e^{-rt}\big[L\,B^*Q(\cdot)y(\cdot)\big](t)\big\|_{\mathcal{D}(A^\epsilon)}\le\frac{c}{(rq')^{1/q'}}\,\|y\|_{X,r}\le\frac{c'}{r^{1/q'}}\,\|y\|_{X,r}\,,$$

so that by taking a sufficiently large $r$ we see that $LB^*Q(\cdot)$ is a contraction mapping in $X$. The conclusion of the Lemma follows. □

Uniqueness for the DRE is now a consequence of Lemmas 4.1 and 4.2, its proof following a somewhat familiar path.

*Proof of Theorem 2.7.* For the optimal pair $(\hat y,\hat u)$ corresponding to the initial state $x\in Y$ it holds

$$(P(s)x,x)_Y=J(\hat u)=\int_s^T\Big(\|R\hat y(r)\|_Z^2+\|\hat u(r)\|_U^2\Big)\,dr\,,\quad 0\le s\le T\,,$$

where $P(\cdot)$ is the Riccati operator defined in (2.9), i.e.

$$P(t)x = \int_t^T e^{A^*(r-t)} R^* R \Phi(r,t)x \, dr\,, \qquad x \in Y\,,$$

while $\Phi(r,t)$ denotes the evolution operator

$$\Phi(r,t)x = e^{A(r-t)}x + L_t\hat{u}(r)\,, \qquad r \in [t,T]\,.$$

Let $Q(\cdot) \in \mathcal{Q}_T$ be another solution to the DRE (2.13): by Lemma 3.1 $Q(\cdot)$ solves the IRE (3.1) as well; then, with $u \in L^2(s,T;U)$ and $x \in \mathcal{D}(A^\epsilon)$, the identity (4.1) holds true by Lemma 4.1. With $t = T$, since $Q(T) = 0$, from (4.1) we see that

$$\begin{aligned}
(Q(s)x,x)_Y &= \int_s^T \left[\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\right] dr - \int_s^T \|u(r) + B^*Q(r)y(r)\|_U^2 \, dr \\
&\le \int_s^T \left[\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\right] dr = J(u)\,.
\end{aligned}$$

In particular, when $u = \hat{u}$, we establish

$$(Q(s)x,x)_Y \le J(\hat{u}) = (P(s)x,x)_Y \qquad \forall s \in [0,T]\,,\ \forall x \in \mathcal{D}(A^\epsilon)\,. \tag{4.11}$$

Conversely, let $y(\cdot)$ be the solution to the closed-loop equation (4.9) corresponding to $x \in \mathcal{D}(A^\epsilon)$, guaranteed by Lemma 4.2, and let $u(\cdot) = -B^*Q(\cdot)y(\cdot)$. By construction $u \in L^2(s,T;U)$, and the fundamental identity becomes

$$(Q(s)x,x)_Y = \int_s^t \Big(\|Ry(r)\|_Z^2 + \|u(r)\|_U^2\Big) dr + (Q(t)y(t),y(t))_Y\,,$$

which in turn gives, for $t = T$,

$$(Q(s)x,x)_Y = J(u) \ge J(\hat{u}) = (P(s)x,x)_Y \qquad \forall s \in [0,T]\,,\ \forall x \in \mathcal{D}(A^\epsilon)\,. \tag{4.12}$$

The inequality (4.12), combined with (4.11), establishes – via the usual polarization (first) and density (next) arguments – $Q(s) \equiv P(s)$ on $[0,T]$, as desired.

□

4.2. **Infinite time interval, algebraic Riccati equations.** In this Section we prove our second main result, that is Theorem 2.11, which pertains to uniqueness for the algebraic Riccati equation (2.20), under the standing Assumptions 2.9. Instrumental results are the counterparts of Lemmas 4.1 and 4.2, along with the integral form (3.8) of the ARE, already obtained in Section 3; see Lemma 3.2 therein.

The first Lemma is the infinite time horizon version of the fundamental identity established in Lemma 4.1.

**Lemma 4.3** (Fundamental identity ($T = +\infty$))**.** *Recall the class $\mathcal{Q}$ defined in* (2.21)*. Let $Q \in \mathcal{Q}$ be a solution to the integral Riccati equation* (3.8)*. With $u \in L^2_{loc}(0,\infty;U)$ and $x \in \mathcal{D}(A^\epsilon)$, let $y(\cdot)$ be the semigroup solution to the state equation* (2.1) *corresponding to $u(\cdot)$, with initial state $x$, given by* (2.2)*. Then, the following identity holds true, for any $t \ge 0$:*

$$\begin{aligned}
&(Qy(t),y(t))_Y - (Qx,x)_Y \\
&\qquad = -\int_0^t \Big(\|Ry(s)\|_Z^2 + \|u(s)\|_U^2\Big) ds + \int_0^t \big\|u(s) + B^*Qy(s)\big\|_U^2 \, ds\,.
\end{aligned} \tag{4.13}$$

*Proof.* It suffices to proceed along the lines of the proof of Lemma 4.1, replacing the interval $[s,t]$ by $[0,t]$ and assuming initially $u \in L^\infty_{\rm loc}(0,\infty;U)$; the proof is actually slightly simpler, since here $Q$ is independent of $t$. The details are omitted for the sake of conciseness. $\square$

The next Lemma is the infinite time horizon version of Lemma 4.2, dealing with an integral equation which – once uniquenss for the ARE is ascertained – will turn out to be the closed-loop equation.

**Lemma 4.4.** *Let $\epsilon$ be as in the Assumptions 2.9. Recall the class $\mathcal{Q}$ defined by* (2.21)*, and let $Q \in \mathcal{Q}$. For every $x \in \mathcal{D}(A^\epsilon)$ and a suitably large $r > 0$ there exists a unique solution $y(\cdot)$ to the closed loop equation*

$$y(t) = e^{At}x - \int_0^t e^{A(t-s)}BB^*Qy(s)\,ds\,, \qquad t > 0\,, \tag{4.14}$$

*in the space*

$$X = \big\{y \in C([0,\infty);\mathcal{D}(A^\epsilon))\colon \quad \sup_{t\ge 0} e^{-rt}\|y(t)\|_{\mathcal{D}(A^\epsilon)} < \infty\big\} \tag{4.15}$$

*endowed with the norm*

$$\|y\|_{X,r} = \sup_{t>0} e^{-rt}\|y(t)\|_{D(A^\epsilon)} \quad \forall y \in X\,, \quad r > 0\,.$$

*Proof.* The argument is pretty much the same employed in the proof of Lemma 4.2. A technically decisive (distinct) element here comes from the extended (and enhanced) regularity in time of the operator $B^*e^{A^*\cdot}A^{*\epsilon}$ over the half line $[0,\infty)$, which is guaranteed by [3, Proposition 3.2], recalled here as Proposition A.6. The computation is included for the reader's convenience.

Let $x \in \mathcal{D}(A^\epsilon)$ be given. By setting $E(t) = e^{At}x$, and recalling the input-to-state map $L$, the integral equation (4.14) reads as $\big([I + LB^*Q]y(\cdot)\big)(t) = E(t)$, in short. For any function $y(\cdot) \in X$ and any $z \in \mathcal{D}(A^{*\epsilon})$, we have

$$\begin{aligned}
|(e^{-rt}LB^*Qy(t), A^{*\epsilon}z)_Y &= \Big|\int_0^t e^{-r(t-s)}(B^*Qy(s)e^{-rs}, B^*e^{A^*(t-s)}A^{*\epsilon}z)_Y\Big| \\
&\le \int_0^t e^{-r(t-s)}\|B^*Q\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)}\|y\|_{X,r}\, e^{-\delta(t-s)}\, \|e^{\delta(t-s)}B^*e^{A^*(t-s)}A^{*\epsilon}z\|_U\,ds \\
&\le \|B^*Q\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)}\|y\|_{X,r}\Big(\int_0^t e^{-(r+\delta)(t-s)q'}\,ds\Big)^{1/q'}\cdot \\
&\qquad\qquad \cdot\Big(\int_0^t \|e^{\delta(t-s)}B^*e^{A^*(t-s)}A^{*\epsilon}z\|_U^q\,ds\Big)^{1/q} \\
&\le \frac{1}{[(r+\delta)q']^{1/q'}}\,\|B^*Q\|_{\mathcal{L}(\mathcal{D}(A^\epsilon),U)}\,\|e^{\delta\cdot}B^*e^{A^*\cdot}A^{*\epsilon}\|_{\mathcal{L}(Y,L^q(0,\infty;U))}\,\|y\|_{X,r}\,\|z\|_Y\,,
\end{aligned}$$

where $\delta$ belongs to the interval $(0,\omega\wedge\eta)$ ($\omega$ and $\eta$ being like in the Assumptions 2.9). The above estimate implies readily that there exists a constant $C > 0$ such that

$$\|LB^*Qy\|_{X,r} \le \frac{C}{(r+\delta)^{1/q'}}\,\|y\|_{X,r}\|e^{\delta\cdot}B^*e^{A^*\cdot}A^{*\epsilon}\|_{\mathcal{L}(Y,L^q(0,\infty;U))}$$

so that

$$\|LB^*Qy\|_{X,r} \le \frac{1}{2}\|y\|_{X,r}\,,$$

provided $r$ is sufficiently large. The conclusive argument is standard. □

*Proof of Theorem 2.11.* Let $y_0 \in Y$, and let $(\hat{y}, \hat{u})$ the optimal pair of the optimal control problem (2.1)-(2.3) (with $T = +\infty$), corresponding to the initial state $y_0$. Recall that

$$(Py_0, y_0)_Y = J(\hat{u}) = \int_0^\infty \|R\hat{y}(s)\|_Z^2\, ds + \int_0^\infty \|\hat{u}(s)\|_U^2\, ds\,,$$

where the (optimal cost) operator $P$ is defined in terms of the evolution map $\Phi(t)x := \hat{y}(t)$ via

$$Py_0 = \int_0^\infty e^{A^*t} R^* R \Phi(t) y_0\, dt, \quad y_0 \in Y\,.$$

In addition, $P$ belongs to the class $\mathcal{Q}$ and solves the ARE (2.20); consequently, by Lemma 3.2 $P$ solves the integral form (3.8) of the ARE.

Let now $Q \in \mathcal{Q}$ be another solution to the ARE. By Lemma 4.3, we know that for any given $y_0 \in \mathcal{D}(A^\epsilon)$, and any admissible control $u(\cdot)$, the identity (4.13) holds true (with $x$ replaced by $y_0$), where $y(\cdot)$ is the solution to the state equation corresponding to the control $u$ and the initial state $y_0$. Consequently,

$$(Qy_0, y_0)_Y \le (Qy(t), y(t))_Y + J(u) \qquad \forall u \in L^2_{\text{loc}}(0,\infty;U)\,,\ \forall t > 0\,;$$

by choosing in particular the admissible pair $(y_T, u_T)$ defined as follows,

$$u_T = \hat{u} \cdot \chi_{[0,T]}\,, \qquad y_T(t) = \begin{cases} \hat{y}(t) & \text{if } t \le T \\ e^{At} y_0 + e^{A(t-T)} L\hat{u}(T) & \text{if } t > T, \end{cases}$$

we find $(Qy_0, y_0)_Y \le (Qy_T(t), y_T(t))_Y + J(u_T)$, valid for arbitrary $t \ge T > 0$. By letting $t \to +\infty$ in the previous inequality, one obtains readily

$$(Qy_0, y_0)_Y \le J(u_T) \qquad \forall y_0 \in \mathcal{D}(A^\epsilon) \quad \forall T > 0\,, \tag{4.16}$$

in view of the fact that the semigroup $e^{At}$ decays exponentially, and thus $\|y_T(t)\|_Y \to 0$.

Observe now that

$$J(u_T) = \int_0^\infty \|Ry_T(s)\|_Z^2\, ds + \int_0^\infty \|u_T(s)\|_U^2\, ds = \int_0^T \|R\hat{y}(s)\|_Z^2\, ds +$$
$$+ \int_T^\infty \left\|R\big(e^{sA} y_0 + e^{(s-T)A} L\hat{u}(T)\big)\right\|^2 ds + \int_0^T \|\hat{u}(s)\|_U^2\, ds\,,$$

so that $J(u_T) \longrightarrow J(\hat{u})$, as $T \to +\infty$. Keeping this in mind, return to (4.16) and let $T \to +\infty$ to find

$$(Qy_0, y_0)_Y \le J(\hat{u}) = (Py_0, y_0)_Y \qquad \forall y_0 \in \mathcal{D}(A^\epsilon)\,. \tag{4.17}$$

On the other hand, given $y_0 \in \mathcal{D}(A^\epsilon)$ (and still with $Q \in \mathcal{Q}$ another solution to the ARE), let $y(\cdot)$ be the solution to the closed loop equation guaranteed by Lemma 4.4; by construction $y \in L^2_{\text{loc}}(0,\infty;Y)$. Take now the control

$u(\cdot) = -B^*Qy(\cdot)$, which belongs to $L^2_{\rm loc}(0,\infty;U)$. Then, the identity (4.13) holds true for any positive $t$, that is

$$\begin{aligned}(Qy_0,y_0)_Y &= (Qy(t),y(t))_Y + \int_0^t \Big(\|Ry(s)\|_Z^2 + \|u(s)\|_U^2\Big)\,ds \\ &\quad - \cancel{\int_0^t \big\|u(s)+B^*Qy(s)\big\|_U^2\,ds} \ge \int_0^t \|Ry(s)\|_Z^2\,ds + \int_0^t \|u(s)\|_U^2\,ds\,.\end{aligned} \tag{4.18}$$

As $t \to +\infty$, this shows that $Ry \in L^2(0,\infty;Z)$, $u \in L^2(0,\infty;U)$, as well as $(Qy_0,y_0)_Y \ge J(u)$. By minimality we find

$$(Qy_0,y_0)_Y \ge J(u) \ge J(\hat u) = (Py_0,y_0)_Y \qquad \forall y_0 \in \mathcal{D}(A^\epsilon)\,. \tag{4.19}$$

On the basis of (4.17) and (4.19), a standard polarization (first) and density (next) argument confirms that $Q=P$, thereby concluding the proof of Theorem 2.11. □

## Appendix A. Instrumental results

In this appendix we gather several results (some old, some new) which single out certain regularity properties – in time and space – of

- the input-to-state map $L$,
- the operator $B^*Q(\cdot)$, when $Q(t) \in \mathcal{Q}_T$,
- the operator $B^*e^{A^*t}A^{*\epsilon}$ and its adjoint.

All of them stem from the Assumptions 2.4 or 2.9 on the (dynamics and control) operators $A$ and $B$. The role played by the assertions of the novel Lemma A.2 and Lemma A.5 in the proofs of our uniqueness results is absolutely critical.

Initially, it is useful to recall from [1] and [3] the basic regularity properties of the input-to-state map $L$. The first result pertains to the finite time horizon problem. The reader is referred to [1, Appendix B] for the details of the computations leading to the various statements in the following Proposition.

**Proposition A.1** ([1], Proposition B.3)**.** *Let $L_s$ be the operator defined by*

$$L_s\colon u(\cdot) \longrightarrow (L_s u)(t) := \int_s^t e^{A(t-r}Bu(r)\,dr\,, \qquad 0 \le s \le t \le T\,. \tag{A.1}$$

*Under the Assumptions 2.4, the following regularity results hold true.*

1. *If $p=1$, then $L_s \in \mathcal{L}(L^1(s,T;U), L^{1/\gamma}(s,T;[\mathcal{D}(A^{*\epsilon})]'))$;*
2. *if $1<p<\frac{1}{1-\gamma}$, then $L_s \in \mathcal{L}(L^p(s,T;U), L^r(s,T;Y))$, with $r=\frac{p}{1-(1-\gamma)p}$;*
3. *if $p=\frac{1}{1-\gamma}$, then $L_s \in \mathcal{L}(L^p(s,T;U), L^r(s,T;Y))$ for all $r\in[1,\infty)$;*
4. *if $p>\frac{1}{1-\gamma}$, then $L_s \in \mathcal{L}(L^p(s,T;U), C([s,T];Y))$.*

*Moreover, in all cases the norm of $L_s$ does not depend on $s$.*

The space regularity in the last assertion can be actually enhanced. To be more precise, $L_s$ maps control functions $u(\cdot)$ which belong to $L^{q'}(s,T;U)$ into functions which take values in $\mathcal{D}(A^\epsilon)$ ($q'$ being the conjugate exponent of $q$ in the Assumptions 2.4). We highlight this property – appparently left out of the work [1] – as a separate result, since it will be used throughout in the paper. The proof is omitted, as it is akin to (and somewhat simpler than) the one carried out to establish assertion (v) of the subsequent Proposition A.3.

**Lemma A.2.** *Let $\epsilon$ and $q$ be as in (iii) of the Assumptions 2.4. Then, for the operator $L_s$ defined in* (A.1) *we have*

$$L_s \in \mathcal{L}(L^{q'}(s,T;U), C([s,T];\mathcal{D}(A^\epsilon)))\,.$$

A counterpart of Proposition A.1 specific for the infinite time horizon problem was proved in [3, Proposition 3.6]. The collection of findings on the regularity of the input-to-state map $L$ is recorded here for the reader's convenience.

**Proposition A.3** ([3], Proposition 3.6)**.** *Let $L$ be the operator defined by*

$$L\colon u(\cdot) \longrightarrow (Lu)(t) := \int_0^t e^{A(t-r}Bu(r)\,dr\,, \qquad t\ge 0\,.$$

*Under the Assumptions 2.9, the following regularity results hold true.*

(i) $L \in \mathcal{L}(L^1(0,\infty;U), L^r(0,\infty;[\mathcal{D}(A^{*\epsilon})]')$, *for any $r\in[1,1/\gamma)$;*
(ii) $L \in \mathcal{L}(L^p(0,\infty;U), L^r(0,\infty;Y))$, *for any $p\in(1,1/(1-\gamma))$ and any $r\in[p,p/(1-(1-\gamma)p)]$;*
(iii) $L \in \mathcal{L}(L^{\frac{1}{1-\gamma}}(0,\infty;U), L^r(0,\infty;Y))$, *for any $r\in[1/(1-\gamma),\infty)$;*
(iv) $L \in \mathcal{L}(L^p(0,\infty;U), L^r(0,\infty;Y)\cap C_b([0,\infty);Y))$, *for any $p\in(1/(1-\gamma),\infty)$ and any $r\in[p,\infty)$;*
(v) $L \in \mathcal{L}(L^r(0,\infty;U), C_b([0,\infty);\mathcal{D}(A^\epsilon))$, *for any $r\in[q',\infty]$.*

Because they occur in the present work, besides being central to the analysis of [3], we need to recall the $L^p$-spaces with weights. Set

$$L^p_g(0,\infty;X) := \big\{f\colon (0,\infty)\longrightarrow X\,,\ g(\cdot)f(\cdot)\in L^p(0,\infty;X)\big\}\,,$$

where $g\colon(0,\infty)\longrightarrow\mathbb{R}$ is a given (*weight*) function. We will use more specifically the exponential weights $g(t)=e^{\delta t}$, along with the following (simplified) notation:

$$L^p_\delta(0,\infty;X) := \big\{f\colon (0,\infty)\longrightarrow X\,,\ e^{\delta\cdot}f(\cdot)\in L^p(0,\infty;X)\big\}\,.$$

**Remark A.4.** As pointed out in [3, Remark 3.8], all the regularity results provided by the statements contained in the Propositions A.1 and A.3 extend readily to natural analogues involving $L^p_\delta$ spaces (rather than $L^p$ ones), maintaining the respective summability exponents $p$.

We now move on to a result which clarifies the regularity of the operator $B^*Q(\cdot)$, $Q(t)\in\mathcal{Q}_T$, when acting upon *functions* (with values in $\mathcal{D}(A^\epsilon)$) rather than on *vectors* – namely, on elements of the space $\mathcal{D}(A^\epsilon)$.

**Lemma A.5.** *Let $\epsilon$ be as in (iii) of the Assumptions 2.4. If $Q(\cdot)\in\mathcal{Q}_T$ and $f\in C([0,T];\mathcal{D}(A^\epsilon))$, then*

$$B^*Q(\cdot)f(\cdot)\in C([0,T];U)\,.$$

*Proof.* We proceed along the lines of the proof of [1, Lemma A.3]. Let $Q(\cdot)\in\mathcal{Q}_T$ and let $t_0\in[0,T]$. By the definition of $\mathcal{Q}_T$, there exists $c_1>0$ such that

$$\|B^*Q(t)z\|_U \le c_1\|z\|_{\mathcal{D}(A^\epsilon)} \qquad \forall t\in[0,T]\,,\ \forall z\in\mathcal{D}(A^\epsilon)\,. \tag{A.2}$$

Since $f(t_0)\in\mathcal{D}(A^\epsilon)$, then $B^*Q(\cdot)f(t_0)\in C([0,T];U)$. Then

$$\begin{aligned}
&\|B^*Q(t)f(t)-B^*Q(t_0)f(t_0)\|_U\\
&\quad\le \|B^*Q(t)[f(t)-f(t_0)]\|_U + \|B^*Q(t)f(t_0)-B^*Q(t_0)f(t_0)\|_U\\
&\quad\le c_1\|f(t)-f(t_0)\|_{\mathcal{D}(A^\epsilon)} + \|[B^*Q(t)-B^*Q(t_0)]f(t_0)\|_U = o(1)\,, \quad t\longrightarrow t_0\,.
\end{aligned}$$

□

Essential as well in this work, and more specifically in the proof of Theorem 2.11, is a stronger property of the operator $B^*e^{A^*\cdot}A^{*\epsilon}$, namely, (A.3) below, which holds true for appropriate $\delta$, under the Assumptions 2.9. Originally devised in [3], this result reveals that once the validity of iiic) of Assumptions 2.4 is ascertained on some bounded interval $[0,T]$, then the very same regularity estimate extends to the half line, along with an enhanced summability of the function $B^*e^{A^*\cdot}A^{*\epsilon}x$, $x\in Y$. The key to this is the exponential stability of the semigroup, i.e. (2.15); see [3, Proposition 3.2].

**Proposition A.6** ([3], Proposition 3.2). *Let $\omega$, $\eta$ and $\epsilon$ like in the Assumptions 2.9. For each $\delta\in(0,\omega\wedge\eta)$ the map*

$$t\longrightarrow e^{\delta t}B^*e^{A^*t}A^{*\epsilon}$$

*has an extension which belongs to $\mathcal{L}(Y,L^q(0,\infty;U))$. In short,*

$$B^*e^{A^*\cdot}A^{*\epsilon}\in\mathcal{L}(Y,L^q_\delta(0,\infty;U))\,. \tag{A.3}$$

We conclude providing a result that takes a more in-depth glance at the regularity of the operator $B^*e^{A^*t}A^{*\epsilon}$ and its adjoint.

**Lemma A.7.** *Under the Assumptions 2.9, the following regularity results are valid, for any $\delta\in(0,\omega\wedge\eta)$:*

$$\begin{aligned}&a)\quad e^{\delta\cdot}A^{\epsilon}e^{A\cdot}B\in\mathcal{L}(L^{q'}(0,\infty;U),Y)\,,\\&b)\quad e^{\delta\cdot}B^*e^{A^*\cdot}A^{*-\epsilon}\in\mathcal{L}(L^r(0,\infty;Y),U)\qquad\forall r>\frac{1}{1-\gamma}\,.\end{aligned}\tag{A.4}$$

*The respective actions of the operators in* (A.4) *are made explicit by* (A.5) *and* (A.6).

*Proof.* The regularity results in (A.4) are, in essence, dual properties of the regularity result in Proposition A.6 and of assertion A6. in Theorem 2.10, respectively. To infer (a), we introduce the notation $S$ for the mapping from $Y$ into $L^q(0,\infty;U)$ defined by

$$Y\ni z\longrightarrow[Sz](t):=e^{\delta t}B^*e^{A^*t}A^{*\epsilon}z\,,\qquad t>0\,.$$

For any $z\in Y$ and any $h\in L^{q'}(0,\infty;U)$, it must be $S^*\in\mathcal{L}(L^{q'}(0,\infty;U),Y)$ and more precisely,

$$\begin{aligned}\langle S^*h,z\rangle_Y&=\langle h,Sz\rangle_{L^{q'}(0,\infty;U),L^q(0,\infty;U)}=\int_0^\infty\big\langle h(t),e^{\delta t}B^*e^{A^*t}A^{*\epsilon}z\big\rangle_U\,dt\\&=\Big\langle\int_0^\infty e^{\delta t}A^\epsilon e^{At}Bh(t)\,dt,z\Big\rangle_Y\,.\end{aligned}$$

Therefore,

$$S^*h=\int_0^\infty e^{\delta t}A^\epsilon e^{At}Bh(t)\,dt\,,\qquad h\in L^{q'}(0,\infty;Y)\,. \tag{A.5}$$

To achieve (b) of (A.4), we recall instead the assertion A6. in Theorem 2.10, which further tells us that

$$e^{\delta\cdot}A^{-\epsilon}e^{A\cdot}B\in\mathcal{L}(U,L^p(0,\infty;Y))\,,\quad\text{for any } p \text{ such that } 1\le p<\frac{1}{\gamma}.$$

Similarly as above, we introduce the notation $T$ for the mapping from $U$ into $L^p(0,\infty;Y)$ defined by

$$U \ni w \longrightarrow [Tw](t) := e^{\delta t} A^{-\epsilon} e^{At} Bw\,, \qquad t>0\,;$$

by construction, $T^* \in \mathcal{L}(L^{p'}(0,\infty;Y),U)$ for all $p'>1/(1-\gamma)$. More precisely, for any $w\in U$ and any $g\in L^{p'}(0,\infty;Y)$ we have

$$\begin{aligned}\langle T^*g,w\rangle_U &= \langle g,Tw\rangle_{L^{p'}(0,\infty;Y),L^p(0,\infty;Y)} = \int_0^\infty \big\langle g(t), e^{\delta t}A^{-\epsilon}e^{At}Bw\big\rangle_Y\,dt \\ &= \Big\langle \int_0^\infty e^{\delta t}B^*e^{A^*t}A^{*-\epsilon}g(t)\,dt, w\Big\rangle_U\,,\end{aligned}$$

which establishes

$$T^*g = \int_0^\infty e^{\delta t}B^*e^{A^*t}A^{*-\epsilon}g(t)\,dt\,. \qquad \forall g\in L^{p'}(0,\infty;Y)\,. \tag{A.6}$$

The integrals in (A.5) and (A.6) are the sought respective representations of the adjoint operators in (A.4). □

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