Nonlinear Analysis: Real World Applications 63 (2022) 103413

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Minimum energy with infinite horizon: From stationary to non-stationary states[☆]

P. Acquistapace[a], F. Gozzi[b,*]

[a] *Dipartimento di Matematica, Università di Pisa, Italy*
[b] *Dipartimento di Economia e Finanza, Università LUISS - Guido Carli, Roma, Italy*

A R T I C L E I N F O

Article history:
Received 31 January 2021
Received in revised form 8 August 2021
Accepted 16 August 2021
Available online 10 September 2021

Keywords:
Minimum energy
Null controllability
Landau–Ginzburg model
Optimal control with infinite horizon
Algebraic Riccati equation in infinite dimension
Value function as maximal solution

A B S T R A C T

We study a non-standard infinite horizon, infinite dimensional linear–quadratic control problem arising in the physics of non-stationary states (see e.g. Bertini et al. (2004, 2005)): finding the minimum energy to drive a given stationary state $\bar{x} = 0$ (at time $t = -\infty$) into an arbitrary non-stationary state x (at time $t = 0$). This is the opposite to what is commonly studied in the literature on null controllability (where one drives a generic state x into the equilibrium state $\bar{x} = 0$). Consequently, the Algebraic Riccati Equation (ARE) associated with this problem is non-standard since the sign of the linear part is opposite to the usual one and since its solution is intrinsically unbounded. Hence the standard theory of AREs does not apply. The analogous finite horizon problem has been studied in the companion paper (Acquistapace and Gozzi, 2017). Here, similarly to such paper, we prove that the linear selfadjoint operator associated with the value function is a solution of the above mentioned ARE. Moreover, differently to Acquistapace and Gozzi (2017), we prove that such solution is the maximal one. The first main result (Theorem 5.8) is proved by approximating the problem with suitable auxiliary finite horizon problems (which are different from the one studied in Acquistapace and Gozzi (2017)). Finally in the special case where the involved operators commute we characterize all solutions of the ARE (Theorem 6.5) and we apply this to the Landau–Ginzburg model.

1. Introduction

We study a non-standard infinite dimensional, infinite horizon, linear–quadratic control problem as follows. Take two real separable Hilbert spaces: the state space X and the control space U. Consider the linear controlled dynamical system

$$y'(t) = Ay(t) + Bu(t), \qquad t \leq 0, \tag{1}$$

☆ Fausto Gozzi and Paolo Acquistapace have been supported by the Italian Ministry of University and Research (MIUR), in the framework of PRIN projects 2015233N54 006 *Deterministic and stochastic evolution equations* and 2017FKHBA8 001 *The Time-Space Evolution of Economic Activities: Mathematical Models and Empirical Applications.*

* Corresponding author.
E-mail addresses: paolo.acquistapace@unipi.it (P. Acquistapace), fgozzi@luiss.it (F. Gozzi).

https://doi.org/10.1016/j.nonrwa.2021.103413
1468-1218/

where $y :]-\infty, 0] \to X$ is the state, $u :]-\infty, 0] \to U$ is the control, $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a C_0-semigroup and $B : U \to X$ is a linear bounded operator (the "control operator"). The goal is to find the minimum energy, i.e. the minimum of the functional

$$\int_{-\infty}^{0} \|u(t)\|_U^2 \, dt$$

among all controls u which drive a given stationary state $\bar{x} = 0$ (at time $t = -\infty$) into an arbitrary non-stationary state x (at time $t = 0$).

This kind of problems arises in the control representation of the rate function for a class of large deviation problems (see e.g. [1] and the references quoted therein; see also [2, Chapter 8] for an introduction to the subject). It is motivated by applications in the physics of non-equilibrium states and in this context it has been studied in various papers, see e.g. [3–8] (see Section 3 for a description of a model case).

In such applications a departure point of the theory is to apply the dynamic programming approach to characterize the value function as the unique (or maximal/minimal) solution of the associated Hamilton–Jacobi–Bellman (HJB) equation, a problem left open e.g. in [6,8]. This problem is quite difficult since it deals with the opposite to what is commonly studied in the literature on null controllability (where one drives a generic state x into the equilibrium state $\bar{x} = 0$). For this reason we start studying here the simplest case, i.e. when the state equation is linear and the energy functional is purely quadratic: so the problem falls into the class of linear–quadratic optimal control problems, the value function is quadratic, and the associated HJB equation reduces to an Algebraic Riccati Equation (ARE), which can be formally written (with unknown R) as

$$0 = -\langle Ax, Ry \rangle_X - \langle Rx, Ay \rangle_X - \langle B^* Rx, B^* Ry \rangle_U, \qquad x, y \in \mathcal{D}(A). \tag{2}$$

The above feature (i.e. the fact we bring 0 to x instead of the opposite) implies that the ARE associated with this problem is non-standard for two main reasons: first, the sign of the linear part is opposite to the usual one; second, the set of "reachable states" (i.e. the set, which we will call H, of all $x \in X$ such that there exists at least a control $u(\cdot)$ steering the solution of (1) from 0 to x) is strictly smaller than the whole state space X, so that the solution R is intrinsically unbounded in X. The combination of these two difficulties does not allow to apply the standard theory of AREs (described e.g. in [9, pp. 390–394 and 479–486], see also [10, p. 1018]). Therefore we are driven to use a different approach, that exploits the structure of the problem; we partly borrow some ideas from [10] and from the literature about model reduction[1] (see e.g. [11] and [12]: indeed our results partly generalize Theorem 2.2 of [12], see Remark 5.5).

In the companion paper [13] we studied, as a first step, the associated finite horizon case. Here we partially exploit the results of such paper to deal with the more interesting infinite horizon case, which is the one that arises in the above mentioned papers in physics.

Our first main result (Theorem 5.8) shows that, under a null controllability assumption (after a given time $T_0 \geq 0$) and a condition on the range of the control operator B, the linear selfadjoint operator R associated with the value function is the maximal solution (over a slightly restricted class of solutions $\mathcal{Q}$, see Definition 4.8) of the above mentioned ARE.

The second main result (Theorem 6.5) looks at the case where A is selfadjoint and the operators A, BB^* commute, characterizing all solutions of the ARE without any null controllability assumption, any condition on the control operator and any restriction on the class of solutions. This allows to apply such result to the case of Landau–Ginzburg model.

This is only partially similar to what has been done in [13]. Indeed, concerning the first main result, Theorem 5.8, the proof that R is a solution of the above ARE is substantially similar to what is done

[1] We thank prof. R. Vinter for providing us these references.

in [13, Section 4.3]. On the other hand, while in [13, Section 4.4] we prove a partial uniqueness result (i.e. uniqueness in a suitable family of invertible operators), here we are able to prove, through a delicate comparison argument (based on a non-trivial approximation procedure), that R is the maximal solution of the associated ARE in a wider class which goes beyond the one of invertible operators. To prove the comparison argument (which is the content of the key Lemma 4.10) we need to introduce a family of auxiliary finite horizon problems (see Section 4), which are different from the one studied in [13], and to rewrite the unknown R of the ARE (2) as $R = QP$, where Q is a given unbounded operator in X, and P, the new unknown, is a bounded operator on the set H of reachable states, endowed with a suitable Hilbert structure.

Finally, concerning the second main result, Theorem 6.5, we characterize all solutions of the ARE (2) with a completely new approach that still uses the new unknown P introduced right above.

1.1. Plan of the paper

In Section 2 we present the setting of the problem, some preliminary results and the strategy to show the main results. It is divided in five subsections:

- the first one (Section 2.1) is devoted to introduce some notation;
- the second one (Section 2.2) presents the basic setting and the assumptions used in the paper;
- Section 2.3 provides few basic results on the state equation;
- Section 2.4 concerns the minimum energy problem and the associated ARE;
- Section 2.5 describes the properties of the "reachable space" H.

In Section 3 we present, as an example, a special case of the motivating problem given in [6] (the case of the so-called Landau–Ginzburg model): we show that it falls into the class of problems treated in this paper.

Section 4 concerns the auxiliary problem. After devoting the first part of the section to some basic results on it, we show, in Section 4.1, the comparison Lemma 4.10 which will be used to prove the maximality result in the infinite horizon case.

In Section 5 we state and prove the main maximality result.

In Section 6 we analyze the important case when the operators A and BB^* commute: this case is applied to the motivating example of Section 3.

2. Setting and preliminary material

2.1. Notation

- Given any two Banach spaces Y and Z, we denote by $\mathcal{L}(Y, Z)$ the set of all linear bounded operators from Y to Z, writing $\mathcal{L}(Y)$ when $Z = Y$. The adjoint of an operator T will be denoted by T^*. We denote by I_Y the identity operator on Y. When Y is a Hilbert space we denote by $\mathcal{S}_+(Y)$ the set of all elements of $\mathcal{L}(Y)$ which are selfadjoint and nonnegative. The domain of an operator T will be denoted by $\mathcal{D}(T)$ and its range by $\mathcal{R}(T)$.
- Given a possibly unbounded linear operator $T : \mathcal{D}(T) \subseteq Y \to Y$ and λ in the resolvent set $\rho(T)$, we call $R(\lambda, T) = (\lambda I_Y - T)^{-1}$ the associated resolvent operator.
- Given a linear operator $F : X \to Y$, where X and Y are Hilbert spaces, we define, as in [14, p. 209] (see also [15, p. 429]), the pseudoinverse $F^\dagger$ of F as the linear operator
$$F^\dagger : \mathcal{D}(F^\dagger) \subseteq Y \to X,$$
with domain $\mathcal{D}(F^\dagger) = \mathcal{R}(F)$, where $F^\dagger y$ is the element of the fiber $F^{-1}(\{y\})$ with minimal norm. Note that $\mathcal{R}(F^\dagger) = [\ker F]^\perp$. Moreover
$$FF^\dagger y = y, \qquad \forall y \in \mathcal{R}(F) \subseteq Y,$$

while
$$F^\dagger F x = \Pi x \qquad \forall x \in X,$$
where $\Pi : X \to X$ is the orthogonal projection onto the subspace $[\ker F]^\perp$.

2.2. Basic setting and assumptions

Let $-\infty \le s < t < +\infty$. Consider the abstract linear equation
$$\begin{cases} y'(r) = Ay(r) + Bu(r), \quad r \in \,]s,t], \\ y(s) = x \in X \end{cases} \tag{3}$$
(here, when $s = -\infty$ the initial condition is meant to be $\lim_{t\to-\infty} y(t) = x$), under the following assumption, which will be always in force from now on, without mentioning it.

Hypothesis 2.1.

(i) X, the state space, and U, the control space, are real separable Hilbert spaces;

(ii) $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a C_0-semigroup on X such that
$$\|e^{tA}\|_{\mathcal{L}(X)} \le M e^{-\omega t}, \qquad t \ge 0, \tag{4}$$
for given constants $M \ge 1$ and $\omega > 0$;

(iii) $B : U \to X$ is a bounded linear operator;

(iv) u, the control strategy, belongs to $L^2(s,t;U)$.

To prove our main results (Theorems 5.8 and 6.5) we will also need three more assumptions: the first and the second only for Theorem 5.8, while the third only for Theorem 6.5.

The first one is the following null controllability assumption.

Hypothesis 2.2. There exists $T_0 \ge 0$ such that
$$\mathcal{R}(e^{T_0 A}) \subseteq \mathcal{R}(Q_{T_0}^{1/2}). \tag{5}$$
where the so-called controllability operator is given by
$$Q_t = \int_0^t e^{sA} B B^* e^{sA^*} ds, \qquad t \in [0, +\infty].$$

The second one is an assumption on the image of the operator B.

Hypothesis 2.3. It holds
$$\mathcal{R}(BB^*) = \overline{\mathcal{R}(BB^*)} = \overline{\mathcal{R}(Q_\infty)}. \tag{6}$$

The third one concerns the special commuting case.

Hypothesis 2.4. The operator A is selfadjoint and invertible, and commutes with BB^*, i.e. for every $x \in \mathcal{D}(A)$ we have $BB^*x \in \mathcal{D}(A)$ and $ABB^*x = BB^*Ax$.

Since the last three assumptions will not be used everywhere, we will explicitly mention them whenever they are used. Note, in particular, that the first equality of Hypothesis 2.3 is always true when X is finite dimensional, while the second one is always true in the commuting case of Hypothesis 2.4 (see [13, Proposition C.1-(iii)]).

2.3. The state equation

We recall the following well known result, pointed out e.g. in [13, Proposition 2.2].

Proposition 2.5. *For $-\infty < s < t < +\infty$, $x \in X$ and $u \in L^2(s,t;U)$, the mild solution of (3), defined by*

$$y(r;s,x,u) = e^{(r-s)A}x + \int_s^r e^{(r-\sigma)A}Bu(\sigma)\,\mathrm{d}\sigma, \quad r \in [s,t], \tag{7}$$

is in $C([s,t],X)$.

We now consider the state equation in the half-line $]-\infty,t]$:

$$\begin{cases} y'(r) = Ay(r) + Bu(r), & r \in \,]-\infty,t], \\ \lim\limits_{s\to-\infty} y(s) = 0. \end{cases} \tag{8}$$

Since (8) is not completely standard we introduce the following definition of solution.

Definition 2.6. Given $u \in L^2(-\infty,t;U)$, we say that $y \in C(\,]-\infty,t];X)$ is a solution of (8) if for every $-\infty < r_1 \le r_2 \le t$ we have

$$y(r_2) = e^{(r_2-r_1)A}y(r_1) + \int_{r_1}^{r_2} e^{(r_2-\tau)A}Bu(\tau)\mathrm{d}\tau. \tag{9}$$

and

$$\lim_{s\to-\infty} y(s) = 0. \tag{10}$$

Lemma 2.7. *Given any $u \in L^2(-\infty,t;U)$, there exists a unique solution of the Cauchy problem (8) and it is given by*

$$y(r;-\infty,0,u) := \int_{-\infty}^r e^{(r-\tau)A}Bu(\tau)\,\mathrm{d}\tau, \qquad r \le t. \tag{11}$$

Proof. We prove first that the function $y(\cdot;-\infty,0,u)$ given by (11) is continuous. Fixed $r_1 < r_2 \le t$, we have

$$\begin{aligned} &y(r_2;-\infty,0,u) - y(r_1,-\infty,0,u) \\ &\quad = \int_{-\infty}^{r_2} e^{(r_2-\tau)A}Bu(\tau)\,\mathrm{d}\tau - \int_{-\infty}^{r_1} e^{(r_1-\tau)A}Bu(\tau)\,\mathrm{d}\tau \\ &\quad = \int_{-\infty}^{r_1} \left(e^{(r_2-r_1)A} - I\right) e^{(r_1-\tau)A}Bu(\tau)\,\mathrm{d}\tau + \int_{r_1}^{r_2} e^{(r_2-\tau)A}Bu(\tau)\,\mathrm{d}\tau, \end{aligned}$$

and then continuity follows by standard arguments. We now prove that (9) holds. For $-\infty < r_1 \le r_2 \le t$, we have

$$\begin{aligned} y(r_2;-\infty,0,u) &= \int_{-\infty}^{r_2} e^{(r_2-\tau)A}Bu(\tau)\,\mathrm{d}\tau \\ &= e^{(r_2-r_1)A}\int_{-\infty}^{r_1} e^{(r_1-\tau)A}Bu(\tau)\,\mathrm{d}\tau + \int_{r_1}^{r_2} e^{(r_2-\tau)A}Bu(\tau)\,\mathrm{d}\tau \\ &= e^{(r_2-r_1)A}y(r_1;-\infty,0,u) + \int_{r_1}^{r_2} e^{(r_2-\tau)A}Bu(\tau)\,\mathrm{d}\tau, \end{aligned}$$

so (9) is satisfied. Moreover letting $r \to -\infty$, since $u \in L^2(-\infty,t;U)$ and thanks to inequality (4), we have $y(r;-\infty,x,u) \to 0$ as $r \to -\infty$.

In order to prove uniqueness, consider two solutions $y_1(\cdot)$ and $y_2(\cdot)$ and a point $r \in\ ]-\infty, t[$. Since $y_1(\cdot)$ and $y_2(\cdot)$ satisfy (9), for their difference we have, for $r_0 < r < t$,

$$\|y_1(r) - y_2(r)\|_X = \|e^{(r-r_0)A}(y_1(r_0) - y_2(r_0))\|_X \leq M\, e^{-(r-r_0)\omega}\|(y_1(r_0) - y_2(r_0))\|_X.$$

As $y_1(\cdot)$ and $y_2(\cdot)$ satisfy (10), letting $r_0 \to -\infty$ above we get $y_1(r) = y_2(r)$ for every $r < t$. $\square$

Remark 2.8. Notice that, if the initial condition (10) is not zero, then the above equation cannot have any solution. Indeed any solution $y(\cdot; -\infty, x, u)$ of the state equation (8), with 0 replaced by $x \in X \setminus \{0\}$ in (10), must satisfy (9) and $\lim_{s\to-\infty} y(s) = x$. But, as $r_1 \to -\infty$, (9) implies, as in (11), that

$$y(r_2; -\infty, x, u) := \int_{-\infty}^{r_2} e^{(r_2-\tau)A} Bu(\tau)\mathrm{d}\tau, \qquad r_2 \leq t. \tag{12}$$

Taking the limit as $r_2 \to -\infty$ we get $x = 0$, a contradiction. $\square$

2.4. Minimum energy problems with infinite horizon and associated ARE

We now give a precise formulation of our minimum energy problem putting together the finite and the infinite horizon case (see [13, Section 2.2 and Remark 2.8]).

We take the Hilbert spaces X (state space) and U (control space), as well as the operators A and B, as in Hypothesis 2.1. Given $-\infty \leq s < t < +\infty$, an initial state $z \in X$ and a control $u \in L^2(s, t; U)$ we consider the state equation (3), which we rewrite here:

$$\begin{cases} y'(r) = Ay(r) + Bu(r), & r \in\ ]s, t], \\ y(s) = z \end{cases} \tag{13}$$

(when $s = -\infty$ we agree that $z = 0$). Denote by $y(\cdot; s, z, u)$ the mild solution of (13) as in Proposition 2.5 (for $s > -\infty$) and Lemma 2.7 (for $s = -\infty$). We define the class of controls $u(\cdot)$ bringing the state $y(\cdot)$ from a fixed $z \in X$ at time s ($z = 0$ when $s = -\infty$) to a given target $x \in X$ at time t:

$$\mathcal{U}_{[s,t]}(z, x) \stackrel{\text{def}}{=} \left\{u \in L^2(s, t; U)\ :\ y(t; s, z, u) = x\right\}. \tag{14}$$

Consider the quadratic functional (the energy)

$$J_{[s,t]}(u) = \frac{1}{2}\int_s^t \|u(r)\|_U^2\, \mathrm{d}r. \tag{15}$$

The minimum energy problem at $(s, t; z, x)$ is the problem of minimizing the functional $J_{[s,t]}(u)$ over all $u \in \mathcal{U}_{[s,t]}(z, x)$. The value function of this control problem (the *minimum energy*) is

$$V_1(s, t; z, x) \stackrel{\text{def}}{=} \inf_{u\in\mathcal{U}_{[s,t]}(z,x)} J_{[s,t]}(u), \tag{16}$$

with the agreement that the infimum over the emptyset is $+\infty$. Similarly to what we did in [13, Section 2.2], given any $z \in X$ we define the *reachable set* in the interval $[s, t]$, starting from z, as

$$\mathbf{R}^z_{[s,t]} \stackrel{\text{def}}{=} \left\{x \in X :\ \mathcal{U}_{[s,t]}(z, x) \neq \emptyset\right\}. \tag{17}$$

and set

$$\bar{\mathbf{R}}_{[s,t]} \stackrel{\text{def}}{=} \bigcup_{z\in X} \mathbf{R}^z_{[s,t]}. \tag{18}$$

Define the operators

$$\mathcal{L}_{s,t} : L^2(s,t;U) \to X, \qquad \mathcal{L}_{s,t}u = \int_s^t e^{(t-\tau)A}Bu(\tau)\,\mathrm{d}\tau, \qquad -\infty \le s < t < +\infty \tag{19}$$

$$\mathcal{L}_t : L^2(0,t;U) \to X, \qquad \mathcal{L}_t u = \int_0^t e^{(t-\tau)A}Bu(\tau)\,\mathrm{d}\tau, \qquad t \in [0,+\infty] \tag{20}$$

and

$$Q_t x = \int_0^t e^{rA}BB^* e^{rA^*}x\,\mathrm{d}r, \quad x \in X, \qquad t \in [0,+\infty]. \tag{21}$$

We have the following, mostly well known, result.

Theorem 2.9. *Let $-\infty \le s < t < +\infty$ and let $z, x \in X$.*

(i) Let $s > -\infty$. The set $\mathcal{U}_{[s,t]}(z,x)$ is nonempty if and only if

$$x - e^{(t-s)A}z \in \mathcal{R}\left(\mathcal{L}_{s,t}\left(L^2(s,t;U)\right)\right) = \mathcal{R}\left(\mathcal{L}_{t-s}\left(L^2(0,t-s;U)\right)\right) = \mathcal{R}(Q_{t-s}^{1/2}).$$

In particular we have

$$\mathbf{R}^z_{[s,t]} = e^{(t-s)A}z + \mathcal{L}_{s,t}\left(L^2(s,t;U)\right) = e^{(t-s)A}z + \mathcal{R}(Q_{t-s}^{1/2}). \tag{22}$$

(ii) Let $s = -\infty$. The set $\mathcal{U}_{[-\infty,t]}(0,x)$ is nonempty if and only if

$$x \in \mathcal{R}\left(\mathcal{L}_{-\infty,t}\left(L^2(-\infty,t;U)\right)\right) = \mathcal{R}\left(\mathcal{L}_{\infty}\left(L^2(0,+\infty;U)\right)\right) = \mathcal{R}(Q_\infty^{1/2}).$$

In particular we have

$$\mathbf{R}^0_{[-\infty,t]} = \mathcal{L}_{-\infty,t}\left(L^2(s,t;U)\right) = \mathcal{R}(Q_\infty^{1/2}). \tag{23}$$

(iii) Let $s > -\infty$: then

$$V_1(s,t;z,x) = V_1(s-t,0;0,x-e^{(t-s)A}z) = V_1(0,t-s;0,x-e^{(t-s)A}z). \tag{24}$$

Hence from now on we set, for simplicity of notation,

$$V(t,x) = V_1(-t,0;0,x) = \inf_{u \in \mathcal{U}_{[-t,0]}(0,x)} J_{[-t,0]}(u), \qquad t \in \left]0,+\infty\right[, \quad x \in X. \tag{25}$$

(iv) Let $s = -\infty$: then

$$V_1(-\infty,t;0,x) = V_1(-\infty,0;0,x). \tag{26}$$

Hence from now on we set, for simplicity of notation,

$$V_\infty(x) = V_1(-\infty,0;0,x) = \inf_{u \in \mathcal{U}_{[-\infty,0]}(0,x)} J_{[-\infty,0]}(u), \qquad x \in X. \tag{27}$$

(v) If $t > 0$ and $x \in \mathcal{R}(Q_t^{1/2})$, there is exactly one minimizing strategy $\hat{u}_{t,x}$ for the functional $J_{[-t,0]}$ over $\mathcal{U}_{[-t,0]}(0,x)$, and moreover

$$V(t,x) = J_{[-t,0]}(\hat{u}_{t,x}) = \frac{1}{2}\|(Q_t^{1/2})^\dagger x\|_X^2\,, \tag{28}$$

where, for $t > 0$, $(Q_t^{1/2})^\dagger : \mathcal{R}(Q_t^{1/2}) \to [\ker Q_t^{1/2}]^\perp$ is the pseudoinverse of $Q_t^{1/2}$.

(vi) If $t > 0$ and $x \in \mathcal{R}(Q_t)$ then $V(t,x) = \frac{1}{2}\langle Q_t^\dagger x, x\rangle_X$, where $Q_t^\dagger : \mathcal{R}(Q_t) \to [\ker Q_t]^\perp$ is the pseudoinverse of Q_t.

Proof. Statements (i)–(iv)–(v) are classical, see e.g. [14, Theorem 2.3, p.210] or [13, Theorem 2.7]). Statement (iii) is given in [13, Proposition 2.6]).

We now look at statement (ii). The fact that $\mathcal{U}_{[-\infty,t]}(0,x)$ is nonempty if and only if

$$x \in \mathcal{R}\left(\mathcal{L}_{-\infty,t}\left(L^2(-\infty,t;U)\right)\right) = \mathcal{R}\left(\mathcal{L}_{\infty}\left(L^2(0,+\infty;U)\right)\right)$$

follows immediately from the form of the mild solution when $s=-\infty$ given in Lemma 2.7 and, for the second equality, by a standard change of variable. The last equality in (ii) follows from the fact that $Q_\infty = \mathcal{L}_\infty \mathcal{L}_\infty^*$ and from [15, Proposition B.1] (see also [13, Proposition A.1]).

We finally look at statement (iv). Recalling that

$$\mathcal{U}_{[-\infty,t]}(0,x) = \left\{u \in L^2(-\infty,t;U) \;:\; y(t;-\infty,0,u) = x\right\}, \tag{29}$$

by a simple change of variable we get

$$u(\cdot) \in \mathcal{U}_{[-\infty,t]}(0,x) \qquad \Longleftrightarrow \qquad u(\cdot - t) \in \mathcal{U}_{[-\infty,0]}(0,x). \tag{30}$$

Hence

$$V_1(-\infty,s;0,x) = \inf_{u\in\mathcal{U}_{[-\infty,s]}(0,x)} J_{[-\infty,s]}(u) = \inf_{u\in\mathcal{U}_{[-\infty,0]}(0,x)} J_{[-\infty,0]}(u) = V_1(-\infty,0;0,x),$$

with the agreement that the infimum over the empty set is $+\infty$. This implies (26). $\square$

We take now $s=-\infty$ and, based on the result above, $t=0$. The peculiarity of this problem with respect to the most studied minimum energy problems in Hilbert spaces (see e.g. [1,10,16–19], and the general surveys [9,14,20–23]) is the "time reversal" of the formulation. If we apply the dynamic programming principle, we find the following Algebraic Riccati Equation (ARE from now on) in the state space X, with unknown R:

$$0 = -\langle Ax, Ry\rangle_X - \langle Rx, Ay\rangle_X - \langle B^*Rx, B^*Ry\rangle_U\,, \quad x,y \in \mathcal{D}(A)\cap\mathcal{D}(R). \tag{31}$$

or, in operator form,

$$0 = -RA - A^*R - RBB^*R.$$

The time reversal of the problem is reflected in two main features of the above ARE which, to our knowledge, prevent the application to it of the standard theory developed in the current literature:

- the 'wrong' sign[2] of the linear term $RA + A^*R$;
- the fact that the value function V_∞ is finite only in the reachable set $\mathbf{R}^0_{[-\infty,t]} = \mathcal{R}(Q_\infty^{1/2})$, which in general is not closed and is properly contained in X (see e.g. the diagonal example in [13, p. 29] where $[\ker Q_\infty]^\perp = X$ and $\mathcal{R}(Q_\infty^{1/2}) = \mathcal{D}((-A)^{1/2})$ is strictly contained in X).

To deal with such issues we find convenient to endow the space $\mathcal{R}(Q_\infty^{1/2})$ (which we will call H) with a suitable Hilbert structure which is defined in next Section 2.5, together with some useful lemmas. Later, in Sections 5 and 6, we will exploit such structure to study (31) in a more convenient form.

[2] Evidently the linear and the quadratic terms in Eq. (31) have the same sign, while in the standard case they do not. We infer that the 'wrong' sign is in the linear term looking at the corresponding finite horizon problem in [13].

2.5. The space H and its properties

We define the already announced space H:

$$H = \mathcal{R}(Q_\infty^{1/2}). \tag{32}$$

Of course it holds

$$H \subseteq \overline{\mathcal{R}(Q_\infty^{1/2})} = [\ker Q_\infty^{1/2}]^\perp = [\ker Q_\infty]^\perp.$$

As shown right above, the inclusion is in general proper.

Define in H the inner product

$$\langle x, y\rangle_H = \langle (Q_\infty^{1/2})^\dagger x, (Q_\infty^{1/2})^\dagger y\rangle_X\,, \qquad x, y \in H, \tag{33}$$

and, consequently, the norm

$$\|x\|_H = \|(Q_\infty^{1/2})^\dagger x\|_X\,, \qquad x \in H. \tag{34}$$

Note that (33) implies

$$\langle Q_\infty^{1/2} z, Q_\infty^{1/2} w\rangle_H = \langle z, w\rangle_X\,, \qquad z, w \in [\ker Q_\infty]^\perp. \tag{35}$$

Some useful results on the space H, which form the ground for our main results and are partly proved in [13], are recalled in the remainder of this subsection, together with some new results.

Next Lemma 2.10 is exactly [13, Lemma 4.2], except for the statement (v): indeed here we need a slight modification which can be proved exactly in the same way.

Lemma 2.10.

(i) The space H is a Hilbert space continuously embedded into X.
(ii) The space $\mathcal{R}(Q_\infty)$ is dense in H.
(iii) The operator $(Q_\infty^{1/2})^\dagger$ is an isometric isomorphism from H to $[\ker Q_\infty^{1/2}]^\perp$.
(iv) We have $Q_\infty^{1/2} \in \mathcal{L}(H)$ and

$$\|Q_\infty^{1/2}\|_{\mathcal{L}(X)} = \|Q_\infty^{1/2}\|_{\mathcal{L}(H)}.$$

(v) Let Z be another real separable Hilbert space. For every $F \in \mathcal{L}(Z, X)$ such that $\mathcal{R}(F) \subseteq H$ we have $(Q_\infty^{1/2})^\dagger F \in \mathcal{L}(Z, X)$, so that $F \in \mathcal{L}(Z, H)$.

Next Lemma 2.11 is an extension of [13, Lemma 4.3].

Lemma 2.11.

(i) Let Hypothesis 2.2 hold. Then, for every $t \in [T_0, +\infty]$, the space $Q_t(\mathcal{D}(A^*))$ is dense in H and contained in $\mathcal{D}(A)$.
(ii) Let Hypothesis 2.4 hold. Then, for every $t \in\ ]0, +\infty]$, the space $Q_t(\mathcal{D}(A^*))$ is dense in H and contained in $\mathcal{D}(A)$.

In particular $\mathcal{D}(A) \cap H$ is dense in H.

Proof. Part (i) and the last statement are proved in [13, Lemma 4.3].

Part (ii) can be proved in the same way, simply changing the last part as follows. First of all, when Hypothesis 2.4 holds, by [13, Proposition C.2-(iii)] one has $\mathcal{R}(Q_t^{1/2}) = \mathcal{R}(Q_\infty^{1/2})$ for all $t > 0$. Hence, by the closed graph theorem, the operator $(Q_\infty^{1/2})^\dagger Q_t^{1/2} : X \to X$ is bounded.

Now fix $x \in H$ and $t \in \,]0,+\infty]$. Then there is a unique $z \in [\ker Q_\infty]^\perp = [\ker Q_t]^\perp$ such that $Q_t^{1/2} z = x$. Recalling that

$$[\ker Q_\infty]^\perp = \overline{R(Q_\infty)} = \overline{R(Q_\infty^{1/2})} = \overline{R(Q_t^{1/2})},$$

there exists $\{z_n\} \subset X$ such that $Q_t^{1/2} z_n \to z$ in X. Since $D(A^*)$ is dense in X, for each $n \in \mathbb{N} \setminus \{0\}$ we can find $y_n \in D(A^*)$ such that $\|y_n - z_n\|_X < 1/n$, so that $Q_t^{1/2} y_n \to z$ in X, too. Hence, as $n \to \infty$,

$$\begin{aligned}\|Q_t y_n - x\|_H &= \|(Q_\infty^{1/2})^\dagger Q_t y_n - (Q_\infty^{1/2})^\dagger x\|_X = \|(Q_\infty^{1/2})^\dagger Q_t y_n - (Q_\infty^{1/2})^\dagger Q_t^{1/2} z\|_X \\ &\le \|(Q_\infty^{1/2})^\dagger Q_t^{1/2}\|_{\mathcal{L}(X)} \|Q_t^{1/2} y_n - z\|_X \to 0,\end{aligned}$$

i.e. x belongs to the closure of $Q_t(D(A^*))$ in H. $\square$

Remark 2.12. The above lemma immediately implies that, for every $t \in [T_0,+\infty]$ (when Hypothesis 2.2 holds) or for every $t \in \,]0,\infty]$ (when Hypothesis 2.4 holds), $Q_t(\mathcal{D}(A^*))$ is dense in $[\ker Q_\infty]^\perp$ with the topology inherited by X, since the inclusion of H into $[\ker Q_\infty]^\perp$ is continuous.

Now we state and prove four very useful lemmas.

Lemma 2.13. *Assume either Hypothesis 2.2 or Hypothesis 2.4. Then we have the following:*

(i) For every $z \in H$ and $r \ge 0$ we have $e^{rA} z \in H$; moreover the semigroup $e^{tA}|_H$ is strongly continuous in H. In particular, for each $T > 0$ there exists $c_T > 0$ such that

$$\|e^{rA} z\|_H \le c_T \|z\|_H \qquad \forall z \in H, \quad \forall r \in [0,T].$$

We denote by A_0 the generator $e^{tA}|_H$ and set $e^{tA_0} := e^{tA}|_H$.

(ii) For every $\lambda \in \rho(A)$ we have $\lambda \in \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)|_H$.

(iii) The operator A_0 is given by

$$\begin{cases} \mathcal{D}(A_0) = \{x \in \mathcal{D}(A) \cap H \;:\; Ax \in H\} \\ A_0 x = Ax \quad \forall x \in \mathcal{D}(A_0). \end{cases} \tag{36}$$

Proof. (i) Fix any $z \in H$. When Hypothesis 2.2 holds, for $t > T_0$ we have $z \in \mathcal{R}(Q_\infty^{1/2}) = \mathcal{R}(Q_t^{1/2}) = \mathcal{R}(\mathcal{L}_{-t,0})$ (see (23)). When Hypothesis 2.4 holds, we have the same for all $t > 0$ (see [13, Proposition C.2-(iii)]). Hence there exists $u \in L^2(0,r;U)$ such that

$$z = \mathcal{L}_{-t,0}(u) = \int_{-t}^0 e^{-\sigma A} Bu(\sigma)\, d\sigma.$$

Thus, for every $r > 0$,

$$e^{rA} z = \int_{-t}^0 e^{(r-\sigma)A} Bu(\sigma)\, d\sigma = \int_{-t}^r e^{(r-\sigma)A} B\overline{u}(\sigma)\, d\sigma,$$

where

$$\overline{u}(s) = \begin{cases} u(s) & \text{if } s \in [-t,0] \\ 0 & \text{if } s \in [0,r]. \end{cases}$$

Setting $r - \sigma = -s$ and $v(s) = \overline{u}(s+r)$, it follows that

$$e^{rA} z = \int_{-t-r}^0 e^{-sA} B\overline{u}(r+s)\, ds = \mathcal{L}_{-t-r,0}(v) \in \mathcal{R}(\mathcal{L}_{-t-r,0}) = \mathcal{R}(Q_{t+r}^{1/2}) = \mathcal{R}(Q_\infty^{1/2}) = H.$$

Let us now prove that the restriction of e^{rA} to H has closed graph in H: if $z, w, \{z_n\} \subset H$ and if $z_n \to z$ in H, $e^{rA} z_n \to w$ in H, then, since H is continuously embedded into X,

$$z_n \to z \text{ in } X, \qquad e^{rA} z_n \to w \text{ in } X;$$

but $e^{rA} \in \mathcal{L}(X)$, so that $w = e^{rA} z$. Thus $e^{rA} z_n \to e^{rA} z$ in H, and it follows that $e^{rA} \in \mathcal{L}(H)$.

Now fix $x \in H$ and consider for $t > 0$ the quantity $e^{tA}x - x$. We have

$$\|e^{tA}x - x\|_H = \sup_{\|y\|_H=1} \langle e^{tA}x - x, y\rangle_H.$$

Now, for every $\varepsilon \in \,]0,1[\,$ there exists $y_\varepsilon \in H$ with $\|y_\varepsilon\|_H = 1$ such that

$$\|e^{tA}x - x\|_H < \varepsilon + \langle e^{tA}x - x, y_\varepsilon\rangle_H;$$

then, using Lemma 2.11 and choosing $z_\varepsilon \in \mathcal{R}(Q_\infty)$ such that $\|z_\varepsilon - y_\varepsilon\|_H < \varepsilon$, we obtain

$$\begin{aligned}\|e^{tA}x - x\|_H &< \varepsilon + \langle e^{tA}x - x, y_\varepsilon - z_\varepsilon\rangle_H + \langle e^{tA}x - x, z_\varepsilon\rangle_H \\ &\le \varepsilon + \|e^{tA}x - x\|_H \, \|y_\varepsilon - z_\varepsilon\|_H + \langle e^{tA}x - x, Q_\infty^\dagger z_\varepsilon\rangle_X \\ &\le \varepsilon + \|e^{tA}x - x\|_H \, \varepsilon + \|e^{tA}x - x\|_X \, \|Q_\infty^\dagger z_\varepsilon\|_X \,.\end{aligned}$$

Hence

$$(1-\varepsilon)\|e^{tA}x - x\|_H < \varepsilon + \|e^{tA}x - x\|_X \, \|Q_\infty^\dagger z_\varepsilon\|_X,$$

and letting $t \to 0^+$ we get

$$\limsup_{t\to 0^+} \|e^{tA}x - x\|_H \le \frac{\varepsilon}{1-\varepsilon} + 0.$$

The arbitrariness of ε leads to the conclusion.

(ii) Let $\lambda \in \rho(A)$. Then, by the resolvent formula (see [24, Theorem II.1.10]),

$$R(\lambda, A)x = \int_0^{+\infty} e^{-\lambda t} e^{tA}x \, \mathrm{d}t \qquad \forall x \in X.$$

The same holds, in particular, for $x \in H$; by [24, Theorem II.1.10-(i)], this implies that $\lambda \in \rho(A_0)$ and that the above integral is equal to $R(\lambda, A_0)$.

(iii) Let $z \in \mathcal{D}(A_0) \subseteq H$. By definition we have

$$\lim_{t\to 0^+} \frac{e^{tA_0}z - z}{t} = A_0 z \quad \text{ in } H.$$

Hence the above incremental ratio must also converge in the topology of X. This means that $z \in \mathcal{D}(A) \cap H$ and $Az = A_0 z \in H$. This proves that

$$\mathcal{D}(A_0) \subseteq \{x \in \mathcal{D}(A) \cap H \,:\, Ax \in H\}\,.$$

To prove the converse we first observe that, using (ii) and the definition of resolvent, we get, for $n \in \mathbb{N} \setminus \{0\}$ and $x \in H$,

$$nAR(n,A)x = nx - n^2 R(n,A)x = nx - n^2 R(n,A_0)x = nA_0 R(n,A_0)x.$$

We also recall that, by the properties of Yosida approximations of A_0 (see [25, Section 1.3]), we get

$$nR(n, A_0)x \to x \qquad \text{in } H \qquad \forall x \in H. \tag{37}$$

Now assume that $z \in \mathcal{D}(A) \cap H$ with $Az \in H$. To prove that $z \in \mathcal{D}(A_0)$ it is enough to show that $A_0 n R(n, A_0)z$ converges to some element y of H when $n \to +\infty$: in this case, since A_0 is a closed operator, such element is $A_0 z$. To do this we observe that, by the above remarks for the resolvents and by the assumptions on z, we have

$$nA_0 R(n, A_0)z = nAR(n,A)z = nR(n,A)Az = nR(n,A_0)Az.$$

The latter, by (37), converges in H to Az as $n \to +\infty$, since $Az \in H$. This shows that $z \in \mathcal{D}(A_0)$ and $A_0 z = Az$. $\square$

Lemma 2.14. *Assume Hypothesis 2.4. Then we have the following:*

(i) Q_∞ and $Q_\infty^{1/2}$ commute with $R(\lambda, A)$ for every $\lambda \in \rho(A)$, and with e^{tA} for every $t \geq 0$. Moreover, for every $x \in \mathcal{D}(A)$ we have $Q_\infty x \in \mathcal{D}(A)$, $Q_\infty^{1/2} x \in \mathcal{D}(A)$ and

$$AQ_\infty x = Q_\infty Ax, \qquad AQ_\infty^{1/2} x = Q_\infty^{1/2} Ax, \quad \forall x \in \mathcal{D}(A),$$

i.e. Q_∞ and $Q_\infty^{1/2}$ commute with A.

(ii) $Q_\infty^\dagger$ and $(Q_\infty^{1/2})^\dagger$ commute with $R(\lambda, A)$ for every $\lambda \in \rho(A)$, and with e^{tA} for every $t \geq 0$. This means that for every $x \in \mathcal{R}(Q_\infty)$ (respectively $x \in \mathcal{R}(Q_\infty^{1/2})$) we have $e^{tA}x \in \mathcal{R}(Q_\infty)$ (respectively $e^{tA}x \in \mathcal{R}(Q_\infty^{1/2})$), and

$$Q_\infty^\dagger e^{tA} x = e^{tA} Q_\infty^\dagger x \quad \textit{(respectively } (Q_\infty^{1/2})^\dagger e^{tA} x = e^{tA}(Q_\infty^{1/2})^\dagger x\textit{),}$$

and similarly for $R(\lambda, A)$.

(iii) We have

$$\|e^{tA_0}\|_{\mathcal{L}(H)} \leq M e^{-\omega t} \qquad \forall t \geq 0,$$

where M and ω are the constants in (4).

Proof. (i) Hypothesis 2.4 and [13, Lemma B.2] easily imply that Q_∞ commutes with A, $R(\lambda, A)$ and e^{tA}. By [26, Theorem VI.9] we immediately deduce that $Q_\infty^{1/2}$ commutes with $R(\lambda, A)$ and e^{tA}: so in particular it commutes with A^{-1}. Hence, for all $z \in H$

$$Q_\infty^{1/2} A^{-1} z = A^{-1} Q_\infty^{1/2} z.$$

Let $x \in \mathcal{D}(A)$ be such that $z = Ax$. Then the above implies

$$Q_\infty^{1/2} x = A^{-1} Q_\infty^{1/2} Ax.$$

This in turn gives $Q_\infty^{1/2} x \in \mathcal{D}(A)$ and the claim.

(ii) We just prove the result for $(Q_\infty^{1/2})^\dagger$ and e^{tA}, as the others are completely similar. Let $z \in H$: then, since $e^{tA}z = e^{tA_0}z \in H$, we have

$$Q_\infty^{1/2}(Q_\infty^{1/2})^\dagger e^{tA} z = e^{tA} z.$$

Moreover, since $Q_\infty^{1/2}$ and e^{tA} commute, and $z \in H$,

$$Q_\infty^{1/2} e^{tA}(Q_\infty^{1/2})^\dagger z = e^{tA} Q_\infty^{1/2}(Q_\infty^{1/2})^\dagger z = e^{tA} z.$$

It then follows

$$Q_\infty^{1/2}(Q_\infty^{1/2})^\dagger e^{tA} z = Q_\infty^{1/2} e^{tA}(Q_\infty^{1/2})^\dagger z.$$

Applying $(Q_\infty^{1/2})^\dagger$ on both sides we get the claim.

(iii) Let $x \in H$ and $t > 0$. We have, using the above statement (ii),

$$\begin{aligned}\|e^{tA_0}x\|_H &= \|e^{tA}x\|_H = \|Q_\infty^{1/2}(Q_\infty^{1/2})^\dagger e^{tA} x\|_H = \|(Q_\infty^{1/2})^\dagger e^{tA} x\|_X \\ &= \|e^{tA}(Q_\infty^{1/2})^\dagger x\|_X \leq M e^{-\omega t}\|(Q_\infty^{1/2})^\dagger x\|_X = M e^{-\omega t}\|x\|_H . \quad \square\end{aligned}$$

Lemma 2.15. *Assume either Hypothesis 2.2 or Hypothesis 2.4. Then we have the following:*

(i) $Q_\infty(H)$ is dense in H.

(ii) $Q_\infty(\mathcal{D}(A_0^*))$ is dense in H.

(iii) Let Hypothesis 2.4 hold. Then $Q_\infty(\mathcal{D}(A_0^*)) \subseteq \mathcal{D}(A_0)$; moreover A_0 is selfadjoint in H.

Proof. **(i)** Since $\ker Q_\infty^{1/2} = \ker Q_\infty$, we have $\overline{\mathcal{R}(Q_\infty^{1/2})} = \overline{\mathcal{R}(Q_\infty)}$. Fix $x \in H$ and set $z := Q_\infty^{-1/2}x \in \mathcal{R}(Q_\infty^{1/2})$. Then there exists $\{w_n\} \subset X$ such that, defining $z_n = Q_\infty w_n \in \mathcal{R}(Q_\infty)$, we have $z_n \to z$ in X. Set

$$x_n = Q_\infty^{1/2} z_n = Q_\infty^{1/2} Q_\infty w_n = Q_\infty Q_\infty^{1/2} w_n.$$

Clearly $x_n \in Q_\infty(H)$. Moreover

$$\|x_n - x\|_H = \|Q_\infty^{1/2} z_n - x\|_H = \|z_n - z\|_X \to 0 \quad \text{as } n \to +\infty,$$

which proves the claim.

(ii) Fix $x \in H$. By part (i) there exists $\{x_n\} \subset Q_\infty(H)$ such that $x_n \to x$ in H. We must have $x_n = Q_\infty z_n$, with $z_n \in H$. Since $\mathcal{D}(A_0^*)$ is dense in H, then, for every $n \in \mathbb{N} \setminus \{0\}$ there exists $w_n \in \mathcal{D}(A_0^*)$ such that $\|z_n - w_n\|_H < 1/n$. Consequently, setting $y_n = Q_\infty w_n$, we have, using Lemma 2.10(iv),

$$\|y_n - x\|_H \le \|Q_\infty(w_n - z_n)\|_H + \|x_n - x\|_H \le \|Q_\infty\|_{\mathcal{L}(H)} \frac{1}{n} + \|x_n - x\|_H \,.$$

This proves the claim.

(iii) Let A be selfadjoint and commuting with BB^*. Observe first that $\mathcal{D}(A_0^*) \subseteq \mathcal{D}(A^*) = \mathcal{D}(A)$. Indeed, when $x \in \mathcal{D}(A_0^*)$, the linear map $y \to \langle x, A_0 y \rangle_H$ is bounded in H. Using such boundedness and the fact that A and Q_∞ commute (see [13, Lemma B.2 or Proposition C.1-(v)])), we get, for every $y \in \mathcal{D}(A)$,

$$\langle x, Ay \rangle_X = \langle x, Q_\infty A y \rangle_H = \langle x, A Q_\infty y \rangle_H = \langle x, A_0 Q_\infty y \rangle_H = \langle A_0^* x, Q_\infty y \rangle_H \le C\|Q_\infty y\|_H \le C'\|y\|_X,$$

which implies $x \in \mathcal{D}(A^*) = \mathcal{D}(A)$.

Now, let $x \in Q_\infty(\mathcal{D}(A_0^*))$ (which is contained in $\mathcal{D}(A)$ by Lemma 2.11, since $\mathcal{D}(A_0^*) \subseteq \mathcal{D}(A^*)$) and let $z \in \mathcal{D}(A_0^*)$ be such that $x = Q_\infty z$. Using again the fact that A and Q_∞ commute, we get $Ax = AQ_\infty z = Q_\infty A z \in H$. Hence, by definition of A_0, we deduce that $x \in \mathcal{D}(A_0)$ and $A_0 x = Ax$.

Now we prove that A_0 is selfadjoint in H. Let $x \in \mathcal{D}(A_0)$ and $y \in Q_\infty(\mathcal{D}(A_0^*))$. Then for some $z \in \mathcal{D}(A_0^*)$ we have $y = Q_\infty z$ and $Q_\infty^\dagger y = z + z_0$, where $z_0 \in \ker Q_\infty$. Hence it must be $\langle Ax, z_0 \rangle_X = 0$, since $Ax = A_0 x \in H \subseteq [\ker Q_\infty]^\perp$. Using this fact, we get

$$\begin{aligned}\langle A_0 x, y \rangle_H &= \langle Ax, y \rangle_H = \langle Ax, Q_\infty^\dagger y \rangle_X = \langle Ax, z \rangle_X = \langle x, Az \rangle_X \\ &= \langle x, Q_\infty A z \rangle_H = \langle x, A Q_\infty z \rangle_H = \langle x, Ay \rangle_H = \langle x, A_0 y \rangle_H \,,\end{aligned}$$

where in the last step we used the inclusion $\mathcal{D}(A_0^*) \subseteq \mathcal{D}(A)$, the fact that Q_∞ and A commute, and the inclusion $Q_\infty(\mathcal{D}(A_0^*)) \subseteq \mathcal{D}(A_0)$. This implies that, for every $x \in \mathcal{D}(A_0)$, the linear map $y \to \langle x, A_0 y \rangle_H$ is defined on $Q_\infty(\mathcal{D}(A_0^*))$ (which is dense in H) and is bounded in H. This implies that $x \in \mathcal{D}(A_0^*)$ and $A_0^* x = A_0 x$. Hence A_0^* extends A_0. Since both A_0 and A_0^* generate a semigroup, we can choose $\lambda > 0$ such that $\lambda \in \rho(A_0) \cap \rho(A_0^*)$. For such λ we now prove that $R(\lambda, A_0^*) = R(\lambda, A_0)$, which immediately implies that $\mathcal{D}(A_0) = \mathcal{D}(A_0^*)$. Indeed for $z \in H$ we have

$$z = (\lambda - A_0) R(\lambda, A_0) z = (\lambda - A_0^*) R(\lambda, A_0) z,$$

where in the last equality we used that $\mathcal{D}(A_0) \subseteq \mathcal{D}(A_0^*)$ and that $A_0^* x = A_0 x$ for all $x \in \mathcal{D}(A_0)$. Applying $R(\lambda, A_0^*)$ to both sides we get the claim. □

3. A motivating example: from equilibrium to non-equilibrium states

In this section we describe, in a simple one-dimensional case, the optimal control problem outlined in the papers [3–8]. Such special case fits into the application studied e.g. in [6,8], in the case of the Landau–Ginzburg model.

We consider a controlled dynamical system whose state variable is described by a function $\rho :]-\infty, 0]$ (the choice of the letter ρ comes from the fact that in many physical models ρ is a density). The control variable is a function $F :]-\infty, 0] \times [0,1] \to \mathbb{R}$ which we assume to belong to $L^2\left(-\infty, 0; L^2(0,1)\right)$. The state equation is formally given by

$$\begin{cases} \frac{\partial \rho}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}(t,x) + \nabla F(t,x), & t \in\,]-\infty, 0[\,,\ x \in\,]0,1[\,, \\ \rho(-\infty, x) = \bar{\rho}(x), & x \in [0,1], \\ \rho(t,0) = \rho_-, \qquad \rho(t,1) = \rho_+, & t \in\,]-\infty, 0[\,, \\ \rho(0,x) = \rho_0(x), & x \in [0,1], \end{cases} \tag{38}$$

where $\rho_+, \rho_- \in (0,1)$, and $\bar{\rho}$ is an equilibrium state for the uncontrolled problem. Hence $\bar{\rho}$ is the unique solution of the following system

$$\begin{cases} v''(x) = 0, \\ v(0) = \rho_-, \\ v(1) = \rho_+; \end{cases}$$

so we have $\bar{\rho}(x) = (\rho_+ - \rho_-)x + \rho_-$.

For any datum $\rho_0 \in L^2(0,1)$ we consider any control driving (in Eq. (38)) the equilibrium state $\bar{\rho}$ (at time $t = -\infty$) to ρ_0 (at time $t = 0$). Then we consider the problem of minimizing, over the set of such controls, the energy functional

$$J^0_\infty(F) = \frac{1}{2}\int_{-\infty}^0 \|F(s)\|^2_{L^2(0,1)}\,\mathrm{d}s.$$

Given the above structure it is natural to consider the new control

$$\nu = \nabla F \in L^2\left(-\infty, 0; H^{-1}(0,1)\right)$$

and take both the state space X and the control space U equal to $H^{-1}(0,1)$: here $H^{-1}(0,1)$ is the dual space of $H^1_0(0,1)$.

We now rewrite (38) in our abstract setting as follows. First we denote by A the Laplace operator in the space $H^{-1}(0,1)$ with Dirichlet boundary conditions, i.e.

$$\mathcal{D}(A) = H^1_0(0,1), \qquad A\eta = \eta'' \quad \forall \eta \in H^1_0(0,1).$$

As A is dissipative, the fractional powers $(-A)^\alpha$ of $-A$ are well defined (see [9, Proposition 6.1, p. 113]). Hence, formally, the state equation (38) becomes

$$\begin{cases} \rho'(t) = A[\rho(t) - \bar{\rho}] + \nu(t), & t < 0, \\ \rho(-\infty) = \bar{\rho}. \end{cases} \tag{39}$$

Using a standard argument (see e.g. [27, Appendix C]), the state equation (38) can be rewritten in the space X and in the new variable $y(t) := \rho(t) - \bar{\rho}$ as

$$\begin{cases} y'(t) = Ay(t) + \nu(t), & t < 0, \\ y(-\infty) = 0. \end{cases} \tag{40}$$

The function

$$y(t; -\infty, 0, \nu) = \int_{-\infty}^t e^{(t-s)A}\nu(s)\,\mathrm{d}s, \qquad t \le 0, \tag{41}$$

corresponding to $\rho(t;\nu) = \bar{\rho} + \int_{-\infty}^t e^{(t-s)A}\nu(s)\,\mathrm{d}s$, is the unique solution of (40), adopting Definition 2.6 and applying Lemma 2.7.

The energy functional, in the new control variable ν, becomes

$$\bar{J}^0_\infty(\nu) = \frac{1}{2}\int_{-\infty}^0 \|(-A)^{-1/2}\nu(s)\|^2_{L^2(0,1)}\,\mathrm{d}s = \frac{1}{2}\int_{-\infty}^0 \|\nu(s)\|^2_{H^{-1}(0,1)}\,\mathrm{d}s.$$

The set of admissible controls here is exactly $\mathcal{U}_{[-\infty,0]}(0,y_0)$ (see Section 2.4), which is nonempty if and only if $y_0 \in H := R(Q_\infty^{1/2}) = \mathcal{D}((-A)^{1/2}) = L^2(0,1)$ (see e.g. [13, Section 5.2]). The value function V_∞ is defined as

$$V_\infty(y_0) := \inf_{\nu \in \mathcal{U}_{[-\infty,0]}(0,y_0)} \bar{J}^0_\infty(\nu). \tag{42}$$

Now, recalling that $X = U = H^{-1}(0,1)$ and setting $B = I_{H^{-1}(0,1)} \in \mathcal{L}(U,X)$, this problem belongs to the class of the minimum energy problems studied in this paper and, in particular, all Hypotheses 2.1, 2.2, 2.3, 2.4 hold true. Hence all the results of the subsequent sections apply: in particular Proposition 5.4, Theorems 5.8 and 6.5, Corollary 6.8. In the last Remark 6.10 we summarize what can be said using such results in this case.

4. The auxiliary problem

In this section we introduce an auxiliary problem which can be considered a "time reversed" version of the auxiliary problem considered in [10] (see also Remark 4.3 about this). This problem will be a key tool to prove our first main result, Theorem 5.8. Indeed, as we will see, any solution of our Algebraic Riccati Eq. (31) can be associated, under appropriate assumptions, with a constant solution of this auxiliary problem with itself as initial datum; a comparison argument will then lead to the main result.

In this section we consider, for $x \in X$, the following set of controls:

$$\overline{\mathcal{U}}_{[-t,0]}(x) = \{(z,u) \in H \times L^2(-t,0;U) : y(0) = x\}, \tag{43}$$

where $y(\cdot) := y(\cdot; -t, z, u)$ is the solution of the Cauchy problem (similar to (3) but with $[s,t]$ replaced by $[-t,0]$), i.e.

$$\begin{cases} y'(r) = Ay(r) + Bu(r), & r \in\ ]-t,0], \\ y(-t) = z. \end{cases} \tag{44}$$

Note that a control in $\overline{\mathcal{U}}_{[-t,0]}(x)$ is a pair: an initial point $z \in H$ and a control $u \in \mathcal{U}_{[-t,0]}(z,x)$, where (see (14))

$$\mathcal{U}_{[-t,0]}(z,x) = \{u \in L^2(-t,0;U) :\ y(0;-t,z,u) = x\}. \tag{45}$$

The following is true:

Proposition 4.1. *The set $\overline{\mathcal{U}}_{[-t,0]}(x)$ introduced in (43) is nonempty if and only if $x \in \bar{\mathbf{R}}_{[-t,0]}$. Moreover we have*

$$\bar{\mathbf{R}}_{[-t,0]} \subseteq H. \tag{46}$$

Furthermore, if Hypothesis 2.2 holds, we have equality in (46) for $t \geq T_0$. Finally, if Hypothesis 2.4 holds, we have equality in (46) for $t > 0$.

Proof. The first statement is an immediate consequence of the definition of reachable set in (17) and of Theorem 2.9(i). The second one follows from (22), Lemma 2.13(i), the fact that $\mathcal{R}(Q_t^{1/2}) \subseteq \mathcal{R}(Q_\infty^{1/2})$ (with equality for $t \geq T_0$, when Hypothesis 2.2 holds, and for $t > 0$, when Hypothesis 2.4 holds), and the equality, proved in Theorem 2.9(i)–(ii),

$$\mathcal{R}(\mathcal{L}_{-t,0}) = \mathbf{R}^0_{[-t,0]} = \mathcal{R}(Q_t^{1/2}), \qquad t \in [0,+\infty] \tag{47}$$

(here $\mathcal{L}_{-t,0}$ is the operator defined in (19)). $\square$

Given a bounded selfadjoint positive operator N on H we want to minimize, in the class $\overline{\mathcal{U}}_{[-t,0]}(x)$, the following functional with an initial cost:

$$J^N_{[-t,0]}(z,u) = \frac{1}{2}\langle Nz, z\rangle_H + \frac{1}{2}\int_{-t}^{0} \|u(s)\|_U^2 \,\mathrm{d}s. \tag{48}$$

The presence of the operator $N \in \mathcal{S}_+(H)$ forces us to fix the starting point z at time $-t$ in H, rather than in X. Define

$$V^N(t,x) = \inf_{(z,u)\in\overline{\mathcal{U}}_{[-t,0]}(x)} J^N_{[-t,0]}(z,u) = \inf_{z\in H}\left[\inf_{u\in\mathcal{U}_{[-t,0]}(z,x)} J^N_{[-t,0]}(z,u)\right], \quad t>0, \quad x\in X, \tag{49}$$

with the agreement that the infimum over the emptyset is $+\infty$, so that $V^N(t,x)$ is finite only when $x \in H$. Now we provide a relation between V^N and the value function V defined in (25).

Proposition 4.2. *We have*

$$V^N(t,x) = \inf_{z\in H}\left[V(t, x - e^{tA}z) + \frac{1}{2}\langle Nz, z\rangle_H\right], \quad t>0, \quad x\in X, \tag{50}$$

and, in particular,

$$V^N(t,x) \le V(t,x) \qquad \forall x\in X, \quad \forall t>0. \tag{51}$$

Proof. We use (16), (24) and (25) getting

$$\inf_{u\in\mathcal{U}_{[-t,0]}(z,x)} J^N_{[-t,0]}(z,u) = V_1(-t,0;z,x) + \frac{1}{2}\langle Nz,z\rangle_H = V(t,x-e^{tA}z) + \frac{1}{2}\langle Nz,z\rangle_H.$$

This equality immediately implies (50). Taking $z = 0$ we get (51). □

It is possible to associate to our auxiliary problem a Differential Riccati Equation (DRE). Our aim is to establish a comparison between the quadratic form associated to "stationary solutions" Q (see Definition 4.6) of such DRE and the value function V^N above when $N = Q$. This result will be a key tool to prove our main result (Theorem 5.8 in Section 5).

Observe that the above mentioned DRE will not be studied in this paper. Here we only explain, in Remark 4.3 just below, how such DRE arises, while we concentrate, in next Section 4.1, to give the precise definition of stationary solutions of it and to prove the announced comparison result.

Remark 4.3. If A generates not just a C_0-semigroup but a C_0-group, the auxiliary problem can be shown, under appropriate assumptions, to be equivalent, reversing the time, to a standard optimization problem with final cost. Indeed, given $x \in H$, consider the problem of minimizing, over all $v(\cdot) \in L^2(0,t;U)$, the functional

$$\widehat{J}^N_{[0,t]}(x,v) = \frac{1}{2}\langle Nw(t), w(t)\rangle_H + \frac{1}{2}\int_0^t \|v(s)\|_U^2\,\mathrm{d}s, \tag{52}$$

where $w(\cdot) := w(\cdot;0,x,v)$ is the mild solution of the Cauchy problem

$$w'(s) = -Aw(s) + Bv(s), \quad s\in[0,t[\,, \qquad w(0) = x. \tag{53}$$

Assume now that, for every $x \in H$, the mild solution $w(\cdot;0,x,v)$ belongs to H for every $t > 0$. Setting

$$\widehat{V}^N(t,x) = \inf_{v\in L^2(0,t;U)} \widehat{J}^N_{[0,t]}(x,v),$$

it can be seen that

$$\widehat{V}^N(t,x) = V^N(t,x).$$

To see this, fix $(t,x) \in [0,+\infty[\, \times H$ and recall that, for every $(z,u) \in \overline{\mathcal{U}}_{[-t,0]}(x)$, we have

$$e^{tA}z + \int_{-t}^{0} e^{-sA}Bu(s)ds = x \quad \Longleftrightarrow \quad z + \int_{-t}^{0} e^{(-t-s)A}Bu(s)ds = e^{-tA}x;$$

hence, changing variable in the integral,

$$z = e^{t(-A)}x + \int_0^t e^{(t-s)(-A)}B(-u(-s))ds.$$

This means that $\bar{\mathbf{R}}_{[-t,0]} = H$ (see (18)). Moreover, with any $(z,u) \in \overline{\mathcal{U}}_{[-t,0]}(x)$ we can associate a function $v \in L^2(0,t;U)$ such that $w(t) = z$, namely, $v(s) = -u(-s)$; consequently

$$J^N_{[-t,0]}(z,u) = \widehat{J}^N_{[0,t]}(x,v). \tag{54}$$

Conversely, given any $v \in L^2(0,t;U)$, set $z = w(t;0,x,v)$ and $u(s) = -v(-s)$: then, clearly, $(z,u) \in \overline{\mathcal{U}}_{[-t,0]}(x)$ and, again, (54) holds. In conclusion, there is a one-to-one correspondence between the control set of the two problems and, in particular, $\widehat{V}^N(t,x) = V^N(t,x)$.

The equation for the "time-reversed" problem (52)–(53) turns out to be the following:

$$\begin{cases} \dfrac{d}{ds}\langle P^N(s)x,y\rangle_H = & -\langle Ax, P^N(s)y\rangle_H - \langle P^N(s)x, Ay\rangle_H \\ & -\langle B^*Q_\infty^\dagger P^N(s)x, B^*Q_\infty^\dagger P^N(s)y\rangle_U\,, \qquad s \in\,]0,t], \\ P^N(0) = N. \end{cases} \tag{55}$$

To give sense to (55) we must take $x,y \in \mathcal{D}(A)\cap H$ with $Ax, Ay \in H$ and $P^N(t)x, P^N(t)y \in \mathcal{R}(Q_\infty)$. When $B^*Q_\infty^\dagger$ can be extended to a bounded operator $H \to U$ and A generates a group, then it is known that the value function $\widehat{V}^N$ is quadratic and $\widehat{V}^N(t,x) = \langle \widehat{P}^N(t)x,x\rangle_H$, where $\widehat{P}^N : [0,+\infty[\, \to \mathcal{S}_+(H)$ is the unique solution of (55). In our case this is not obvious, but it suggests anyway the right form of the Riccati equation for our auxiliary problem. ■

Remark 4.4. *As in the case $N=0$ treated in [13], in the above Riccati equations the sign of the linear part is opposite to the usual one. In fact the control problem (44)–(48) involves an "initial cost", instead of a final cost like in the standard problems (see e.g. [10]).* ■

Our aim now is to prove that for every stationary solution Q of the Riccati equation (55) (in a suitable class to be defined later) there exists an operator N, namely Q itself, such that

$$\frac{1}{2}\langle Nx, x\,\rangle_H \le V^N(t,x), \qquad \text{for sufficiently large } t.$$

Remark 4.5. It is possible to prove much more about the auxiliary problem, namely:

(i) that, for every $N \in \mathcal{S}_+(H)$ the value function V^N is continuous and is a quadratic form in H;
(ii) that, when N is coercive (i.e., for some $\nu > 0$, $\langle Nx, x\,\rangle_H \ge \nu\|x\|_H^2$ for all $x \in H$), the linear operator P^N associated with the value function solves the Riccati equation (55);
(iii) that the comparison result mentioned above translates in the inequality $P^N \ge Q$, in the preorder of selfadjoint positive operators, for every constant solution Q of the Riccati equation (55) in a suitable class.

This is the subject of a paper in progress.

4.1. A key comparison result

Given any initial datum $N \in \mathcal{S}_+(H)$, we want to compare the "stationary" solutions of the Riccati equation (55) with the value function V^N of the auxiliary problem. This fact will be used, in the next section, as a key tool to prove our first main result, Theorem 5.8. In order to do this we need first to give a precise meaning to the concept of stationary solution of (55).

Roughly speaking, a stationary solution $P \in \mathcal{S}_+(H)$ of the Riccati Eq. (55) should also be a solution of the following equation, which comes from the right-hand side of (55):

$$0 = -\langle Ax, Py\rangle_H - \langle Px, Ay\rangle_H - \langle B^* Q_\infty^\dagger Px, B^* Q_\infty^\dagger Py\rangle_U. \tag{56}$$

This equation is meaningful for every $x, y \in \mathcal{D}(A) \cap H$ with $Px, Py \in \mathcal{R}(Q_\infty)$ and $Ax, Ay \in H$. Since the last requirement appears too restrictive, we rewrite (56) by taking the first two inner products in X, getting:

$$0 = -\langle Ax, Q_\infty^\dagger Py\rangle_X - \langle Q_\infty^\dagger Px, Ay\rangle_X - \langle B^* Q_\infty^\dagger Px, B^* Q_\infty^\dagger Py\rangle_U. \tag{57}$$

This makes sense in a larger set of vectors x, y, namely for every $x, y \in \mathcal{D}(A) \cap H$ with $Px, Py \in \mathcal{R}(Q_\infty)$. It is important to note that (57) is precisely the ARE (31) if we formally set $R = Q_\infty^\dagger P$ (which is, in general, an unbounded operator).

We can now provide the precise definition of solution of (57).

Definition 4.6. Let $P \in \mathcal{S}_+(H)$ and define the operator Λ_P as follows:

$$\begin{cases} \mathcal{D}(\Lambda_P) = \{x \in H : \ Px \in \mathcal{R}(Q_\infty)\} \\ \Lambda_P x = Q_\infty^\dagger Px \qquad \forall x \in \mathcal{D}(\Lambda_P). \end{cases} \tag{58}$$

We say that P is a solution of (57) (or, alternatively, a stationary solution of (55)) if $\mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$ is dense in $[\ker Q_\infty]^\perp$ and

$$0 = -\langle Ax, \Lambda_P y\rangle_X - \langle \Lambda_P x, Ay\rangle_X - \langle B^* \Lambda_P x, B^* \Lambda_P y\rangle_U \qquad \forall x, y \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P). \tag{59}$$

Remark 4.7. In the above definition we added, beyond the fact that Eq. (57) is satisfied in the set where it makes sense (i.e. $\mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$), a density condition on such set. We made this choice in order to rule out solutions which would be not significant. For instance, consider the case when Hypothesis 2.4 holds and choose P to be an orthogonal projection on the line generated by a vector $v \notin \mathcal{R}(Q_\infty)$. Clearly, in this case $\mathcal{D}(\Lambda_P) = \{0\}$, hence $\mathcal{D}(A) \cap \mathcal{D}(\Lambda_P) = \{0\}$. Since the ARE (57) is obviously satisfied when $x = y = 0$, it follows that any such P would be a solution. ■

We now define a subclass $\mathcal{Q}$ of the class of all solutions of (57). First of all we recall that, by Lemma 2.13, $e^{tA}|_H$ is a strongly continuous semigroup in H, whose generator is denoted by A_0 (see (36)).

Definition 4.8. Let $P \in \mathcal{S}_+(H)$. We say that $P \in \mathcal{Q}$ if there exists $D \subseteq \mathcal{D}(\Lambda_P)$ such that D is dense in $\mathcal{D}(A) \cap H$ with respect to the norm $\|\cdot\|_H + \|A \cdot\|_X$;

Lemma 4.9. *The set $\mathcal{R}(Q_\infty) \cap \mathcal{D}(A)$ is dense in $\mathcal{D}(A) \cap H$, equipped with the norm $\|\cdot\|_H + \|A\cdot\|_X$. Hence, choosing $D = \mathcal{R}(Q_\infty) \cap \mathcal{D}(A)$, we have $I_H \in \mathcal{Q}$.*

Proof. Let $x \in H \cap \mathcal{D}(A)$ such that

$$\langle x, z\rangle_H + \langle Ax, Az\rangle_X = 0, \qquad \forall z \in \mathcal{R}(Q_\infty) \cap \mathcal{D}(A).$$

It is enough to prove that $x = 0$. Observe that, writing $z = Q_\infty y$,

$$\langle x, Q_\infty y\ \rangle_H + \langle Ax, AQ_\infty y\ \rangle_X = 0, \qquad \forall y \in \mathcal{D}(AQ_\infty).$$

Then

$$\langle Ax, AQ_\infty y\ \rangle_X = -\langle x, Q_\infty y\ \rangle_H = -\langle x, y\ \rangle_X \qquad \forall y \in \mathcal{D}(AQ_\infty).$$

This means that $Ax \in \mathcal{D}((AQ_\infty)^*)$ and $(AQ_\infty)^* Ax = -x$. Hence

$$\langle (AQ_\infty)^* Ax, Ax\ \rangle_X = -\langle x, Ax\ \rangle_X = \|(-A)^{1/2} x\|_X^2 \geq 0;$$

the fractional powers of $-A$ are well defined since, by Hypothesis 2.1(ii), $-A$ is a positive operator: see e.g. [28, Chapter 4]. On the other hand we know, from [13, Lemma 3.1-(ii)], that, for every $y \in \mathcal{D}((AQ_\infty)^*) \subseteq \mathcal{D}(AQ_\infty)$

$$2\langle (AQ_\infty)^* y, y\ \rangle_X = -\|B^* y\|_U^2,$$

so that

$$2\langle (AQ_\infty)^* Ax, Ax\ \rangle_X = -\|B^* Ax\|_U^2 \leq 0.$$

This implies that $\|(-A)^{1/2} x\|_X^2 = 0$; hence $Ax = 0$ and, since A is invertible, $x = 0$. □

Lemma 4.10. *Assume Hypotheses 2.2 and 2.3. Let $P \in \mathcal{S}_+(H)$ be a solution of (57), according to Definition 4.6, and suppose moreover that $P \in \mathcal{Q}$. Then the following estimate holds:*

$$\frac{1}{2}\langle Px, x\rangle_H \leq V^P(t - T_0, x) \qquad \forall x \in H, \ \ \forall t > T_0,$$

where V^P is the value function defined in (49) with $N = P$.

Proof. Step 1 We prove the estimate

$$\langle Px, x\rangle_H \leq \langle Py(T_0 - t), y(T_0 - t)\rangle_H + \int_{T_0 - t}^0 \|u(s)\|_U^2 \, \mathrm{d}s, \qquad t > T_0, \tag{60}$$

for every $(z, u) \in \overline{\mathcal{U}}_{[-t,0]}(x)$ with $x \in H$, where y is the state corresponding to (z, u), i.e.

$$y(s) = e^{(s+t)A} z + \int_{-t}^s e^{(s-\sigma)A}\, Bu(\sigma)\, d\sigma, \quad s \in [-t, 0]. \tag{61}$$

Such inequality would be easy to prove if we were able to compute $\frac{d}{ds}\langle Py(s), y(s)\rangle_H$ and prove that

$$\frac{d}{ds}\langle Py(s), y(s)\rangle_H \leq \|u(s)\|_U^2, \qquad s \in [-t, 0].$$

Unfortunately we even do not know if such a derivative exists. Hence we need to build a delicate approximation procedure as follows.

Fix $t > T_0$ and $x \in H$; consider any $(z, u) \in \overline{\mathcal{U}}_{[-t,0]}(x)$. It is not restrictive to assume in (61) that $u(\sigma) \in \overline{\mathcal{R}(B^*)}$ for every $\sigma \in [-t, 0]$: indeed, writing, for every such σ,

$$u(\sigma) = u_1(\sigma) + u_2(\sigma), \quad u_1(\sigma) \in \overline{\mathcal{R}(B^*)}, \quad u_2(\sigma) \in \overline{\mathcal{R}(B^*)}^\perp = \ker B,$$

it is clear that $e^{(s-\sigma)A} Bu_2(\sigma) = 0$. Hence

$$y(s) = e^{(s+t)A} z + \int_{-t}^s e^{(s-\sigma)A}\, Bu_1(\sigma)\, d\sigma, \quad s \in [-t, 0].$$

Since, evidently, $J^P_{[-t,0]}(z,u) \geq J^P_{[-t,0]}(z,u_1)$, we can always choose u_1 in place of u. Next, select a sequence $\{(z_n,u_n)\} \subseteq [\mathcal{D}(A_0)] \times C^1_0([-t,0];U)$,[3] such that u_n is $\mathcal{R}(B^*)$-valued and $(z_n,u_n) \to (z,u)$ in $H \times L^2(-t,0;U)$. Thus we can set $u_n = B^*v_n$, where $v_n \in C^1_0([-t,0],X)$ and, denoting by y_n the corresponding state, we have $y_n \in C^1([-t,0];H) \cap C([-t,0];\mathcal{D}(A))$ (see e.g. [25, Chapter 4, Corollary 2.5]) and

$$y_n(s) = e^{(s+t)A} z_n + \int_{-t}^{s} e^{(s-\sigma)A} BB^* v_n(\sigma)\, d\sigma, \qquad s \in [-t,0].$$

Thanks to the properties of the set D of Definition 4.8, we can now choose, for every $n \in \mathbb{N}$, another approximating sequence $\{y_{nk}\}_{h\in N} \subset C^1([-t,0],H) \cap C([-t,0],\mathcal{D}(A))$, such that $y_{nk}(s) \in D$ for every $s \in [-t,0]$ and satisfying, as $k \to +\infty$,

$$y_{nk} \to y_n \text{ in } C^1([-t,0];H), \qquad Ay_{nk} \to Ay_n \text{ in } C([-t,0];X) \tag{62}$$

(see e.g. [25, Chapter 4, Theorem 2.7]). Set now $w_{nk} = y'_{nk} - Ay_{nk}$. By (62) we get, for every $n \in \mathbb{N}$,

$$w_{nk} \to y'_n - Ay_n = BB^* v_n \text{ in } C([-t,0];X) \qquad \text{as } k \to +\infty. \tag{63}$$

We now can differentiate the quantity $\langle Py_{nk}(s), y_{nk}(s)\rangle_H$ for $s \in [-t,0]$. Indeed, taking into account the above definition of w_{nk}, we obtain, for $s \in [-t,0]$ and $n,k \in \mathbb{N}$:

$$\begin{aligned}
&\frac{d}{ds}\langle Py_{nk}(s), y_{nk}(s)\rangle_H = \langle y'_{nk}(s), Py_{nk}(s)\rangle_H + \langle Py_{nk}(s), y'_{nk}(s)\rangle_H \\
&\quad = \langle y'_{nk}(s), \Lambda_P y_{nk}(s)\rangle_X + \langle \Lambda_P y_{nk}(s), y'_{nk}(s)\rangle_X \\
&\quad = \langle Ay_{nk}(s) + w_{nk}(s), \Lambda_P y_{nk}(s)\rangle_X + \langle \Lambda_P y_{nk}(s), Ay_{nk}(s) + w_{nk}(s)\rangle_X .
\end{aligned}$$

Since P solves the ARE (59) we get, for every $s \in [-t,0]$,

$$\begin{aligned}
&\frac{d}{ds}\langle Py_{nk}(s), y_{nk}(s)\rangle_H \\
&\quad = -\|B^*\Lambda_P y_{nk}(s)\|^2_U + \langle w_{nk}(s), \Lambda_P y_{nk}(s)\rangle_X + \langle \Lambda_P y_{nk}(s), w_{nk}(s)\rangle_X \\
&\quad = -\|B^*\Lambda_P y_{nk}(s)\|^2_U + \langle B^*v_n(s), B^*\Lambda_P y_{nk}(s)\rangle_U + \langle B^*\Lambda_P y_{nk}(s), B^*v_n(s)\rangle_U \\
&\qquad + \langle w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X + \langle \Lambda_P y_{nk}(s), w_{nk}(s) - BB^*v_n(s)\rangle_X \\
&\quad = -\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|^2_U + \|B^*v_n(s)\|^2_U \\
&\qquad + \langle w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X + \langle \Lambda_P y_{nk}(s), w_{nk}(s) - BB^*v_n(s)\rangle_X .
\end{aligned}$$

Before going on, we make some remarks on the terms

$$\begin{aligned}
&\langle w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X + \langle \Lambda_P y_{nk}(s), w_{nk}(s) - BB^*v_n(s)\rangle_X \\
&\qquad\qquad = 2\mathrm{Re}\, \langle w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X .
\end{aligned}$$

Denote by Π the orthogonal projection onto $\overline{\mathcal{R}(Q_\infty)} = \mathcal{R}(BB^*)$. As both $\Lambda_P y_{nk}(s)$ and $BB^*v_n(s)$ belong to $\overline{\mathcal{R}(Q_\infty)} = \mathcal{R}(BB^*)$, we may write

$$\langle w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X = \langle \Pi w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X .$$

Next, the pseudoinverse $(BB^*)^\dagger$ is well defined and, since $R(BB^*)$ is closed, it satisfies $(BB^*)^\dagger \Pi \in \mathcal{L}(X)$, due to the closed graph theorem. In addition, we have $BB^*(BB^*)^\dagger \Pi = \Pi$, so that

$$\begin{aligned}
\langle w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X &= \langle \Pi w_{nk}(s) - BB^*v_n(s), \Lambda_P y_{nk}(s)\rangle_X \\
&= \left\langle BB^* \left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right], \Lambda_P y_{nk}(s)\right\rangle_X .
\end{aligned}$$

[3] $C^1_0([-t,0];U)$ is the set of C^1 U-valued functions which take the value 0 at the extrema.

We go back now to the expression of the derivative of $\langle Py_{nk}(s), y_{nk}(s)\rangle_H$:

$$\begin{aligned}\frac{d}{ds}\langle Py_{nk}(s), y_{nk}(s)\rangle_H &= -\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|_U^2 + \|B^*v_n(s)\|_U^2 \\ &\quad + 2\mathrm{Re}\left\langle BB^*\left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right], \Lambda_P y_{nk}(s)\right\rangle_X \\ &= -\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|_U^2 + \|B^*v_n(s)\|_U^2 \\ &\quad + 2\mathrm{Re}\left\langle B^*\left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right], B^*\Lambda_P y_{nk}(s)\right\rangle_X .\end{aligned}$$

Hence we may write for every $\varepsilon > 0$,

$$\begin{aligned}\frac{d}{ds}\langle Py_{nk}(s), y_{nk}(s)\rangle_H &\le -\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|_U^2 + \|B^*v_n(s)\|_U^2 \\ &\quad + 2\left\|B^*\left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right]\right\|_X \|B^*\Lambda_P y_{nk}(s)\|_X \\ &\le -\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|_U^2 + \|B^*v_n(s)\|_U^2 \\ &\quad + \frac{1}{\varepsilon}\left\|B^*\left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right]\right\|_X^2 + \varepsilon\|B^*\Lambda_P y_{nk}(s)\|_X^2.\end{aligned} \tag{64}$$

Now observe that

$$\varepsilon\|B^*\Lambda_P y_{nk}(s)\|_X^2 \le 2\varepsilon\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|_U^2 + 2\varepsilon\|B^*v_n(s)\|_U^2.$$

Inserting this inequality into (64) we get

$$\begin{aligned}\frac{d}{ds}\langle Py_{nk}(s), y_{nk}(s)\rangle_H &\le -(1-2\varepsilon)\|B^*\Lambda_P y_{nk}(s) - B^*v_n(s)\|_U^2 \\ &\quad + (1+2\varepsilon)\|B^+v_n(s)\|_U^2 + \frac{1}{\varepsilon}\left\|B^*\left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right]\right\|_X^2 .\end{aligned} \tag{65}$$

Hence, for all positive $\varepsilon \le \frac{1}{2}$ we get

$$\frac{d}{ds}\langle Py_{nk}(s), y_{nk}(s)\rangle_H \le (1+2\varepsilon)\|B^*v_n(s)\|_U^2 + \frac{1}{\varepsilon}\left\|B^*\left[(BB^*)^\dagger \Pi w_{nk}(s) - v_n(s)\right]\right\|_X^2 . \tag{66}$$

Now we have as $k \to \infty$, for every $s \in [-t, 0]$,

$$\|y_{nk}(s) - y_n(s)\|_H \to 0, \quad \|y'_{nk}(s) - y'_n(s)\|_H \to 0, \quad \|w_{nk}(s) - BB^*v_n(s)\|_X \to 0;$$

as a consequence, using the fact that $(BB^*)^\dagger \Pi BB^* = (BB^*)^\dagger BB^* = \Pi$, we obtain as $k \to \infty$

$$(BB^*)^\dagger \Pi w_{nk}(s) \to (BB^*)^\dagger \Pi BB^* v_n(s) = \Pi v_n(s) \text{ in } X,$$

and also, since $\ker BB^* = \ker B^*$,

$$B^*(BB^*)^\dagger \Pi w_{nk}(s) \to B^*\Pi v_n(s) = B^*v_n(s) \text{ in } X.$$

Thus by (66) we get, for every $n \in \mathbb{N}\setminus\{0\}$, $s \in [-t, 0]$ and $0 < \varepsilon \le \frac{1}{2}$,

$$\frac{d}{ds}\langle Py_n(s), y_n(s)\rangle_H \le (1+2\varepsilon)\|B^*v_n(s)\|_U^2.$$

Finally, letting $\varepsilon \to 0$ and recalling that $u_n = B^*v_n$,

$$\frac{d}{ds}\langle Py_n(s), y_n(s)\rangle_H \le \|u_n(s)\|_U^2 \quad \forall n \in \mathbb{N}\setminus\{0\}, \quad \forall s \in [-t, 0].$$

We now integrate in the smaller interval $[T_0 - t, 0]$:

$$\langle Py_n(0), y_n(0)\rangle_H \le \langle Py_n(T_0 - t), y_n(T_0 - t)\rangle_H + \int_{T_0 - t}^0 \|u_n(s)\|_U^2 \, \mathrm{d}s.$$

Letting $n \to \infty$, since $y_n(s) \to y(s)$ for every $s \in [-t, 0]$, $y(0) = x$, and $u_n \to u$ in $L^2(-t, 0; U)$, we deduce for every $(z, u) \in \overline{\mathcal{U}}_{[-t,0]}(x)$

$$\langle Px, x\rangle_H \le \langle Py(T_0 - t), y(T_0 - t)\rangle_H + \int_{T_0-t}^{0} \|u(s)\|_U^2 \, \mathrm{d}s, \qquad t > T_0;$$

this is Eq. (60).

Step 2 We complete the proof of the lemma. Consider a sequence $(\hat{z}_n, \hat{u}_n) \in \overline{\mathcal{U}}_{[T_0-t,0]}(x)$, such that, as $n \to \infty$,

$$J^P_{[T_0-t,0]}(\hat{z}_n, \hat{u}_n) \to \inf_{(z,u)\in\overline{\mathcal{U}}_{[T_0-t,0]}(x)} J^P_{[T_0-t,0]}(z, u) = V^P(t - T_0, x). \tag{67}$$

Thus $\hat{z}_n \in H$, $\hat{u}_n \in L^2(T_0 - t, 0; U)$ and the corresponding state is

$$\hat{y}_n(s) = e^{(s+t-T_0)A}\hat{z}_n + \int_{T_0-t}^{s} e^{(s-\sigma)A}B\hat{u}_n(\sigma)\,\mathrm{d}\sigma, \quad s \in [T_0 - t, 0];$$

in particular $\hat{y}_n(0) = x$. Now choose $\hat{v}_n \in L^2(-t, T_0 - t; U)$ such that

$$\int_{-t}^{T_0-t} e^{(T_0-t-\sigma)A}B\hat{v}_n(\sigma)\,\mathrm{d}\sigma = \hat{z}_n; \tag{68}$$

this is possible since, due to Hypothesis 2.2, the range of the operator (defined in (19))

$$v \mapsto \mathcal{L}_{-t,T_0-t}(v) = \mathcal{L}_{-T_0,0}(v(\cdot + t - T_0))$$

is all of H (see [14, Theorem 2.3]). Then, setting

$$\overline{u}_n = \begin{cases} \hat{v}_n & \text{in } [-t, T_0 - t] \\ \hat{u}_n & \text{in } [T_0 - t, 0], \end{cases}$$

the state corresponding to $(0, \overline{u}_n)$ in $[-t, 0]$ is

$$\overline{y}_n(s) = \int_{-t}^{s} e^{(s-\sigma)A}B\overline{u}_n(\sigma)\,\mathrm{d}\sigma.$$

By (68) we have

$$\overline{y}_n(T_0 - t) = \int_{-t}^{T_0-t} e^{(T_0-t-\sigma)A}B\overline{u}_n(\sigma)\,\mathrm{d}\sigma = \hat{z}_n;$$

hence, by uniqueness,

$$\overline{y}_n(s) = e^{(s+t-T_0)A}\hat{z}_n + \int_{T_0-t}^{s} e^{(s-\sigma)A}B\hat{u}_n(\sigma)\,\mathrm{d}\sigma = \hat{y}_n(s) \qquad \forall s \in [T_0 - t, 0],$$

so that $\overline{y}_n(0) = \hat{y}_n(0) = x$. This shows that $(0, \overline{u}_n) \in \overline{\mathcal{U}}_{[-t,0]}(x)$, and consequently, by (60),

$$\langle Px, x\rangle_H \le \langle P\hat{z}_n, \hat{z}_n\rangle_H + \int_{T_0-t}^{0} \|\hat{u}_n(s)\|_U^2 \, \mathrm{d}s = 2J^P_{[T_0-t,0]}(\hat{z}_n, \hat{u}_n).$$

Finally, by (67), as $n \to \infty$ we get

$$\frac{1}{2}\langle Px, x\rangle_H \le V^P(t - T_0, x) \qquad \forall t > T_0, \quad \forall x \in H. \quad \square$$

5. Results on the minimum energy problem

5.1. Optimal strategies

We start proving the existence of optimal strategies.

Proposition 5.1. *The set $\mathcal{U}_{[-\infty,0]}(0,x)$ is nonempty if and only if $x \in H$. Moreover, for every $x \in H$ there exists a unique $\hat{u}_x \in \mathcal{U}_{[-\infty,0]}(0,x)$ such that*

$$V_\infty(x) = J_{[-\infty,0]}(\hat{u}_x).$$

Proof. The first statement follows from (17) as in Proposition 4.1. Now take $x \in H$ and observe that any minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ must be bounded in $L^2(-\infty,0;U)$; so, passing to a subsequence, we have $u_n \rightharpoonup \hat{u}_x$ in $L^2(-\infty,0;U)$. As the functional $J_{[-\infty,0]}$ is weakly lower semicontinuous, we get

$$V_\infty(x) \le J_{[-\infty,0]}(\hat{u}_x) \le \liminf_{n\to\infty} J_{[-\infty,0]}(u_n) = V_\infty(x),$$

i.e. $\hat{u}_x$ is optimal. Uniqueness is an easy consequence of the strict convexity of the functional $J_{[-\infty,0]}$. $\square$

Moreover we have the following result about the optimal pairs when $x \in \mathcal{R}(Q_\infty)$ (see [13, Proposition C.3 and Remark C.4]).

Proposition 5.2. *Let $x \in \mathcal{R}(Q_\infty)$. Let $(\hat{y}_x, \hat{u}_x)$ be the optimal pair for our problem with target x. Then we have*

$$\hat{u}_x(r) = B^* e^{-rA^*} Q_\infty^\dagger x, \quad r \in\ ]-\infty,0]. \tag{69}$$

Moreover the corresponding optimal state $\hat{y}_x$ satisfies

$$\hat{y}_x(r) = Q_\infty e^{-rA^*} Q_\infty^\dagger x, \quad r \in\ ]-\infty,0]; \tag{70}$$

and the optimal pair satisfies the feedback formula

$$\hat{u}_x(r) = B^* Q_\infty^\dagger \hat{y}_x(r), \quad r \in\ ]-\infty,0]. \tag{71}$$

Remark 5.3. We observe that in Proposition 5.2 $\hat{y}_x$ is, formally, a solution of the backward closed loop equation

$$y'(r) = (A + BB^* Q_\infty^\dagger) y(r), \quad r \in\ ]-\infty,0[\,, \quad y(0) = x. \tag{72}$$

Since Q_∞ solves the Lyapunov equation

$$AQ + QA^* + BB^* = 0$$

(see [13, Proposition 3.3]), Eq. (72) rewrites as

$$y'(r) = -Q_\infty A^* Q_\infty^\dagger y(r), \quad r \in\ ]-\infty,0[\,. \tag{73}$$

Finally, if A^* commutes with Q_∞ (e.g. when A is selfadjoint and invertible, and A and BB^* commute), then (73) becomes

$$y'(r) = -A^* y(r), \quad r \in\ ]-\infty,0[\,. \tag{74}$$

This means that, in such case, the optimal trajectory arriving at x is given by

$$y(r) = e^{-rA^*} x, \quad r \in\ ]-\infty,0].$$

5.2. Connection with the finite horizon case

We now prove the connection between V_∞ and the value function V of the corresponding finite horizon problem studied in [13].

Proposition 5.4. *Let Hypothesis 2.2 or Hypothesis 2.4 hold. For every $x \in H$ we have*

$$V_\infty(x) = \lim_{t\to+\infty} V(t,x) = \inf_{t>0} V(t,x).$$

Moreover

$$V_\infty(x) = \frac{1}{2}\|x\|_H^2 = \frac{1}{2}\|(Q_\infty^{1/2})^\dagger x\|_X^2$$

and, for $x \in \mathcal{R}(Q_\infty)$, $V_\infty(x) = \frac{1}{2}\langle Q_\infty^\dagger x, x\ \rangle_X$.

Proof. **Step 1.** First of all, by [13, Proposition 4.8-(i)], the function $V(\cdot,x)$ is decreasing for every $x \in H$; hence, for every such x

$$\exists \lim_{t\to+\infty} V(t,x) = \inf_{t>0} V(t,x).$$

We now prove that $V_\infty(x) \le \inf_{t>0} V(t,x)$. With an abuse of notation we can write

$$\mathcal{U}_{[-t,0]}(0,x) \subseteq \mathcal{U}_{[-\infty,0]}(0,x) \qquad \forall t>0:$$

indeed, given a control bringing 0 to x in the interval $[-t,0]$, we can extend it to a control bringing 0 to x in the interval $[-\infty,0]$ just taking the null control on $]-\infty,-t]$. So, if the set $\mathcal{U}_{[-t,0]}(0,x)$ is not empty, *a fortiori* the set $\mathcal{U}_{[-\infty,0]}(0,x)$ will be not empty. This fact, together with the monotonicity of $V(\cdot,x)$, implies that

$$V_\infty(x) \le \inf_{t>0} V(t,x). \tag{75}$$

Note that this inequality is true without assuming Hypotheses 2.2 or 2.4.

Step 2. We prove now the reverse inequality and the last statement under Hypothesis 2.2. Fix any $\varepsilon > 0$ and consider the optimal state $\hat{u}_x \in \mathcal{U}_{[-\infty,0]}(0,x)$ corresponding to x, such that $J_{[-\infty,0]}(\hat{u}_x) = V_\infty(x)$. By (9) we get

$$x = \int_{-\infty}^0 e^{-\tau A} B\hat{u}_x(\tau)\,\mathrm{d}\tau = e^{tA}\hat{y}_x(-t) + \int_{-t}^0 e^{-\tau A} B\hat{u}_x(\tau)\,\mathrm{d}\tau \qquad \forall t>0;$$

hence we have $\hat{u}_x|_{[-t,0]} \in \mathcal{U}_{[-t,0]}(\hat{y}_x(-t),x)$, which in turn implies that

$$V(t, x - e^{tA}\hat{y}_x(-t)) \le \frac{1}{2}\int_{-t}^0 \|\hat{u}_x(s)\|_U^2\,\mathrm{d}s \le J_{[-\infty,0]}(\hat{u}_x) = V_\infty(x). \tag{76}$$

Now we claim that for every $\delta \in\ ]0,1[$ we may choose $t_\delta > T_0$ such that

$$\|e^{tA}\hat{y}_x(-t)\|_H \le \delta \qquad \forall t > t_\delta: \tag{77}$$

indeed, by Hypothesis 2.2 and Lemma 2.10(v) we have for $t > T_0$

$$\begin{aligned}\|e^{tA}\hat{y}_x(-t)\|_H &= \|(Q_\infty^{1/2})^\dagger e^{tA}\hat{y}_x(-t)\|_X \le \|(Q_\infty^{1/2})^\dagger e^{T_0 A}\|_{\mathcal{L}(X)}\|e^{(t-T_0)A}\hat{y}_x(-t)\|_X \\ &\le \|(Q_\infty^{1/2})^\dagger e^{T_0 A}\|_{\mathcal{L}(X)} M e^{-\omega(t-T_0)}\|\hat{y}_x(-t)\|_X\,.\end{aligned}$$

Since, as a straightforward consequence of Lemma 2.7, $\hat{y}_x(-t)$ is uniformly bounded in X for $t > 0$, the claim is proved.

Going ahead with the proof, we recall that, by [13, Proposition 4.8-(iii)-(b)], we have uniform continuity of V on $[T_0,+\infty] \times B_H(0,R)$ for every $R > 0$, where $B_H(0,R)$ is the ball of center 0 and radius R in H. So, setting $R=\|x\|_H+1$, and denoting by ρ_R the continuity modulus of V on $[T_0,+\infty]\times B_H(0,R)$, by (77) we have for $t>t_\delta$

$$V\left(t,x-e^{tA}\hat{y}_x(-t)\right)>V(t,x)-\rho_R(\delta).$$

The above, together with (76), implies that

$$V(t,x)-\rho_R(\delta)\le V_\infty(x)\qquad \forall t>t_\delta.$$

Choose now δ such that $\rho_R(\delta)<\varepsilon$: then for $t>t_\delta$ we get $V(t,x)<V_\infty(x)+\varepsilon$, so that

$$\inf_{t>0} V(t,x)<V_\infty(x)+\varepsilon;$$

by the arbitrariness of ε, (75) becomes an equality, as desired.

Finally the last statement follows from [13, Proposition 4.8-(iii)-(d)].

Step 3. Assume now Hypothesis 2.4. In order to prove the reverse of inequality (75), we repeat the argument of Step 2, with the only difference in estimating $\|e^{tA}\hat{y}_x(-t)\|_H$: since $\hat{y}_x(-t)\in H$, we have now by Hypothesis 2.4 and Lemma 2.14(iii)

$$\|e^{tA}\hat{y}_x(-t)\|_H=\|e^{tA_0}\mathcal{L}_{-\infty,-t}\hat{u}_x\|_H\le M\,e^{-\omega t}\|\mathcal{L}_{-\infty,-t}\hat{u}_x\|_H.$$

On the other hand, setting $\hat{u}_{x,t}(s)=\hat{u}_x(s-t)$, we have $\hat{u}_{x,t}\in L^2(-\infty,-t;U)$ and

$$\|\mathcal{L}_{-\infty,-t}\hat{u}_x\|_H=\|\mathcal{L}_{-\infty,0}\hat{u}_{x,t}\|_H,$$

so that, by Lemma 2.10(v),

$$\begin{aligned}\|e^{tA}\hat{y}_x(-t)\|_H&\le M\,e^{-\omega t}\|\mathcal{L}_{-\infty,0}\hat{u}_{x,t}\|_H=M\,e^{-\omega t}\|(Q_\infty^{1/2})^\dagger\mathcal{L}_{-\infty,0}\hat{u}_{x,t}\|_X\\&\le M\,e^{-\omega t}\|(Q_\infty^{1/2})^\dagger\mathcal{L}_{-\infty,0}\|_{\mathcal{L}(L^2(-\infty,0;U),X)}\|\hat{u}_{x,t}\|_{L^2(-\infty,0;U)}.\end{aligned}$$

Since

$$\|\hat{u}_{x,t}\|_{L^2(-\infty,0;U)}=\|\hat{u}_x\|_{L^2(-\infty,-t;U)}\le\|\hat{u}_x\|_{L^2(-\infty,0;U)}=\sqrt{2V_\infty(x)},$$

we obtain

$$\|e^{tA}\hat{y}_x(-t)\|_H\le\sqrt{2V_\infty(x)}M\,e^{-\omega t}\|(Q_\infty^{1/2})^\dagger\mathcal{L}_{-\infty,0}\|_{\mathcal{L}(L^2(-\infty,0;U),X)}.$$

Hence, again, for every $\delta\in\,]0,1[$ we may choose $t_\delta>0$ such that (77) holds. Proceeding as in Step 2, we conclude that, as before, (75) becomes an equality.

Step 4. We now prove the final statement under Hypothesis 2.4. Arguing as in [13, proof of Proposition 4.8-(iii)], for $t>0$ and $x\in\mathcal{R}(Q_\infty)=\mathcal{R}(Q_t)$ we have

$$\begin{aligned}|2V(t,x)-\|x\|_H^2|&=|\langle Q_\infty Q_t^\dagger x-x,x\rangle_H\\&=|\langle(Q_\infty-Q_t)Q_t^\dagger x,x\rangle_H|=|\langle(Q_\infty-Q_t)Q_t^\dagger x,Q_\infty^\dagger x\rangle_X|.\end{aligned}$$

Since, for suitable $c>0$,

$$\|(Q_\infty-Q_t)z\|_X=\left\|\int_t^\infty e^{sA}BB^*e^{sA^*}z\,\mathrm{d}s\right\|_X\le c\,e^{-2\omega t}\|z\|_X\quad\forall z\in X,$$

we obtain

$$|2V(t,x)-\|x\|_H^2|\le c\,e^{-2\omega t}\|Q_t^\dagger x\|_X\|Q_\infty^\dagger x\|_X.$$

Now we observe that, using [13, Proposition C.1-(iv)], it holds, for every $t>0$,

$$Q_\infty x=Q_tx+e^{2tA}Q_\infty x\qquad\forall x\in X.$$

This is equivalent to

$$(1-e^{2tA})Q_\infty x = Q_t x \qquad \forall x \in X$$

which implies, since $\mathcal{R}(Q_\infty) = \mathcal{R}(Q_t)$, that $(1-e^{2tA})$ maps $\mathcal{R}(Q_\infty)$ into itself. Hence, for large $t>0$,

$$Q_\infty x = (1-e^{2tA})^{-1} Q_t x \qquad \forall x \in X. \tag{78}$$

We claim now that passing to pseudoinverses we have

$$Q_\infty^\dagger z = Q_t^\dagger (1-e^{2tA}) z, \qquad \forall z \in \mathcal{R}(Q_\infty). \tag{79}$$

Indeed, fix $z \in \mathcal{R}(Q_\infty)$ and set

$$v = Q_\infty^\dagger z, \qquad w = Q_t^\dagger (1-e^{2tA}) z.$$

Applying Q_∞ we get, since $(1-e^{2tA})z \in \mathcal{R}(Q_\infty) = \mathcal{R}(Q_t)$, and using (78),

$$Q_\infty v = Q_\infty Q_\infty^\dagger z = z,$$
$$Q_\infty w = (1-e^{2tA})^{-1} Q_t w = (1-e^{2tA})^{-1} Q_t Q_t^\dagger (1-e^{2tA}) z = (1-e^{2tA})^{-1}(1-e^{2tA}) z = z.$$

Hence $Q_\infty w = Q_\infty v$: since Q_∞ is injective on $\mathcal{R}(Q_\infty)$ (where both v, w live), we obtain $w = v$, thus proving our claim.

Since $\|(1-e^{2tA})^{-1}\|_{\mathcal{L}(X)} \to 1$ as $t \to +\infty$, then, by (79) we deduce, for large $t>0$ and for some constant K,

$$|2V(t,x) - \|x\|_H^2| \le c\, e^{-2\omega t} \|Q_t^\dagger x\|_X \|Q_\infty^\dagger x\|_X = \|(1-e^{2tA})^{-1} Q_\infty^\dagger x\|_X \|Q_\infty^\dagger x\|_X \le K\, e^{-2\omega t} \|Q_\infty^\dagger x\|_X^2,$$

which proves that

$$\lim_{t\to\infty} V(t,x) = \frac{1}{2}\|x\|_H^2.$$

The proof is complete. □

5.3. Algebraic Riccati Equation

We now deal with the Algebraic Riccati Equation associated with our infinite horizon problem, i.e. (31). As usual we expect that the operator representing the value function is a solution of it. Moreover, as the solution cannot be unique (the zero operator is always a solution), we only expect the above solution to be maximal in some suitable sense. This is our main goal. Before starting we note that, by Proposition 5.4, V_∞ is a quadratic form represented, in H, by the identity operator $I_H \in \mathcal{L}(H)$ and, in X, by the possibly unbounded operator $Q_\infty^\dagger$.

To prove our maximality result, it seems better, to avoid unboundedness issues, to work with the representation of V_∞ in H. Hence in analogy with what is done in Section 4.1 (see (57)) we consider Eq. (31) where the unknown R is formally set equal to $Q_\infty^\dagger P$ and P is the new unknown.

$$0 = -\langle Ax, Q_\infty^\dagger P y\rangle_X - \langle Q_\infty^\dagger P x, Ay\rangle_X - \langle B^* Q_\infty^\dagger P x, B^* Q_\infty^\dagger P y\rangle_U. \tag{80}$$

Note that such expression makes sense only when $Px, Py \in \mathcal{R}(Q_\infty)$ and $x, y \in \mathcal{D}(A) \cap H$.

Remark 5.5. In the finite-dimensional case, when the operator Q_∞ is invertible, it is proved that the operator $R = Q_\infty^{-1}$ solves (31), using the fact that its inverse $W = Q_\infty$ is the unique solution of the Lyapunov equation

$$AW + WA^* = -BB^* \tag{81}$$

among all definite positive bounded operators $X \to X$. This is reported by Scherpen [12, Theorem 2.2], who quotes Moore [11] for the proof (see also, among others, [29, Chapters 5 and 7], [30] and [31] for related results). In fact, as we will see, this procedure works in our infinite dimensional case, too, but with more difficulties.

Clearly the issue of maximality/minimality of solutions of ARE have been studied also in the infinite dimensional case, but in cases different from ours, see e.g. the books [21, Section 9.2] [22,23]. ■

We now provide the definition of solution of both forms of our ARE, i.e. (80) and (31), which include, along the same line of Definition 4.6, a density condition motivated by Remark 4.7.

Definition 5.6.

(i) An operator $P \in \mathcal{S}_+(H)$ is a solution of the ARE (80) if the set $\mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$ (see (58)) is dense in H and Eq. (80) is satisfied for all $x, y \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$.

(ii) A positive, selfadjoint, possibly unbounded operator $R : \mathcal{D}(R) \subset X \to X$ is a solution of the ARE (31) if the set $\mathcal{D}(A) \cap \mathcal{D}(R)$ is dense in $[\ker Q_\infty]^\perp$ (in the topology inherited by X) and Eq. (31) is satisfied for all $x, y \in \mathcal{D}(A) \cap \mathcal{D}(R)$.

Proposition 5.7. *The following statements hold.*

(i) If $P \in \mathcal{S}_+(H)$ is a solution to (80), then $R = Q_\infty^\dagger P$ is a solution to (31) and it satisfies, in addition, $Q_\infty^{1/2} R Q_\infty^{1/2} \in \mathcal{L}(X)$ and $\mathcal{D}(A) \cap \mathcal{D}(R)$ dense in H.

(ii) If R is a solution to (31) and it satisfies, in addition, $Q_\infty^{1/2} R Q_\infty^{1/2} \in \mathcal{L}(X)$[4] and $\mathcal{D}(A) \cap \mathcal{D}(R)$ dense in H, then $P = Q_\infty R \in \mathcal{S}_+(H)$ is a solution to (80).

Proof. **(i)** Assume that $P \in \mathcal{S}_+(H)$ solves (80). Setting $R = Q_\infty^\dagger P$, we easily see that R is selfadjoint and positive and that its domain is exactly $\mathcal{D}(\Lambda_P)$, which is dense in H by Definition 5.6(i). Hence it is also dense in $[\ker Q_\infty]^\perp$. Then, again by Definition 5.6(i), the set $\mathcal{D}(A) \cap \mathcal{D}(R) = \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$ is dense in H. The fact that such R satisfies (31) for every $x, y \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$ follows by simple substitution. Finally, for every $x \in X$ we have

$$\|Q_\infty^{1/2} R Q_\infty^{1/2} x\|_X = \|Q_\infty^{-1/2} P Q_\infty^{1/2} x\|_X = \|P Q_\infty^{1/2} x\|_H \le \|P\|_{\mathcal{L}(H)} \|Q_\infty^{1/2} x\|_H = \|P\|_{\mathcal{L}(H)} \|x\|_X.$$

(ii). Let $R : \mathcal{D}(R) \to X$ be a solution of (31), having the properties that $\mathcal{D}(A) \cap \mathcal{D}(R)$ is dense in H and $Q_\infty^{1/2} R Q_\infty^{1/2} \in \mathcal{L}(X)$. We set $P = Q_\infty R$: then we easily see that P is selfadjoint and positive. Hence $P \in \mathcal{S}_+(H)$ since, for every $x \in H$,

$$\begin{aligned} \|Px\|_H &= \|Q_\infty R x\|_H = \|Q_\infty^{1/2} [Q_\infty^{1/2} R Q_\infty^{1/2}] (Q_\infty^{1/2})^\dagger x\|_H = \|[Q_\infty^{1/2} R Q_\infty^{1/2}] (Q_\infty^{1/2})^\dagger x\|_X \\ &\le \|Q_\infty^{1/2} R Q_\infty^{1/2}\|_{\mathcal{L}(X)} \|(Q_\infty^{1/2})^\dagger x\|_X = \|Q_\infty^{1/2} R Q_\infty^{1/2}\|_{\mathcal{L}(X)} \|x\|_H \,. \end{aligned}$$

Moreover, we see immediately that $\mathcal{D}(\Lambda_P) = H \cap \mathcal{D}(R)$. In addition, (31) transforms into (80), and it holds for every $x, y \in \mathcal{D}(A) \cap \mathcal{D}(R)$, i.e. it holds for every $x, y \in \mathcal{D}(A) \cap H \cap \mathcal{D}(R) = \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$, as required by Definition 5.6. □

Concerning the two AREs (80) and (31) we have the following result:

Theorem 5.8. *Let Hypothesis 2.2 hold.*

[4] Here we mean that the set of $x \in X$ such that $Q_\infty^{1/2} x \in D(R)$ is dense in X and that the operator $Q_\infty^{1/2} R Q_\infty^{1/2}$ can be extended to a bounded operator in X.

(i) The operator $R = Q_\infty^\dagger$ is a solution of the Riccati equation (31) in the sense of Definition 5.6(ii).

(ii) The operator $P = I_H$ is a solution of the Riccati equation (80) in the sense of Definition 5.6(i).

(iii) Let also Hypothesis 2.3 hold. Then the operator I_H is the maximal solution of (80) in the following sense: if $\hat{P}$ is another solution of (80) in the sense of Definition 5.6(i), belonging to the class $\mathcal{Q}$ introduced in Definition 4.8, then

$$\frac{1}{2}\langle \hat{P}x, x\rangle_H \leq \frac{1}{2}\|x\|_H^2 = V_\infty(x) \qquad \forall x \in H.$$

(iv) Let also Hypothesis 2.3 hold. Then the operator $Q_\infty^\dagger$ is the maximal solution of (31) in the following sense: if $\hat{R}$ is another solution of (31) in the sense of Definition 5.6(ii), such that $Q_\infty^{1/2}\hat{R}Q_\infty^{1/2} \in \mathcal{L}(X)$, $\mathcal{D}(A) \cap \mathcal{D}(R)$ is dense in H, and $Q_\infty \hat{R} \in \mathcal{Q}$ (see Definition 4.8), then

$$\frac{1}{2}\langle \hat{R}x, x\rangle_X \leq \frac{1}{2}\langle \hat{Q}_\infty^\dagger x, x\rangle_X = V_\infty(x) \qquad \forall x \in \mathcal{R}(Q_\infty).$$

Proof. (i) By [13, Proposition 3.3], Q_∞ solves the Lyapunov equation, i.e. we have for every $\xi \in \mathcal{D}(A^*)$

$$AQ_\infty\xi + Q_\infty A^*\xi + BB^*\xi = 0.$$

This implies that, for every $\xi \in \mathcal{D}(A^*)$ and $\eta \in X$,

$$\langle AQ_\infty\xi, \eta\rangle_X + \langle Q_\infty A^*\xi, \eta\rangle_X + \langle B^*\xi, B^*\eta\rangle_U = 0.$$

When $\eta \in \mathcal{D}(AQ_\infty)$ the second term above rewrites as $\langle \xi, AQ_\infty\eta\rangle_X$. Consequently, when $\eta \in \mathcal{D}(AQ_\infty)$, the functional $\xi \to \langle AQ_\infty\xi, \eta\ \rangle_X$, well defined since $\xi \in \mathcal{D}(A^*)$, can be extended to a bounded linear operator on X since, by the above equation, it is equal to $-\langle \xi, AQ_\infty\eta\ \rangle_X - \langle B^*\xi, B^*\eta\ \rangle_U$. Hence, choosing in particular $\xi \in \mathcal{D}(AQ_\infty)$, we get

$$\langle AQ_\infty\xi, \eta\rangle_X + \langle \xi, AQ_\infty\eta\rangle_X + \langle B^*\xi, B^*\eta\rangle_U = 0, \qquad \xi, \eta \in \mathcal{D}(AQ_\infty). \tag{82}$$

Now set $x = Q_\infty\xi$ and $y = Q_\infty\eta$. Then $x, y \in \mathcal{D}(A)$ and the above rewrites as

$$\langle Ax, \eta\rangle_X + \langle \xi, Ay\rangle_X + \langle B^*\xi, B^*\eta\rangle_U = 0. \tag{83}$$

Observe that $\xi = Q_\infty^\dagger x + \xi_0$ and $\eta = Q_\infty^\dagger y + \eta_0$ for suitable $\xi_0, \eta_0 \in \ker Q_\infty \subseteq \ker B^*$. Hence, using the fact that Q_∞ solves the Lyapunov equation in the form (82), we have, for $\xi \in \mathcal{D}(AQ_\infty)$,

$$\langle Ax, \eta_0\rangle_X = \langle AQ_\infty\xi, \eta_0\rangle_X = -\langle \xi, AQ_\infty\eta_0\rangle_X - \langle B^*\xi, B^*\eta_0\rangle_U = 0$$

and, similarly, for $\eta \in \mathcal{D}(AQ_\infty)$, $\langle \xi_0, Ay\rangle_X = 0$. We then get, substituting into (83) and observing that $B^*\xi_0 = B^*\eta_0 = 0$,

$$\langle Ax, Q_\infty^\dagger y\rangle_X + \langle Q_\infty^\dagger x, Ay\rangle_X + \langle B^*Q_\infty^\dagger x, B^*Q_\infty^\dagger y\rangle_U = 0, \quad x, y \in Q_\infty(\mathcal{D}(AQ_\infty)). \tag{84}$$

The above is exactly equation (31) for $R = Q_\infty^\dagger$. To end the proof of (i), it is enough to observe that $Q_\infty(\mathcal{D}(AQ_\infty))$ is dense in $[\ker Q_\infty]^\perp$ (using Remark 2.12 and the fact that it contains $Q_\infty(\mathcal{D}(A^*))$), and moreover that

$$Q_\infty(\mathcal{D}(AQ_\infty)) = \mathcal{D}(A) \cap \mathcal{R}(Q_\infty) = \mathcal{D}(A) \cap \mathcal{D}(Q_\infty^\dagger).$$

Indeed if $x \in Q_\infty(\mathcal{D}(AQ_\infty))$ then it must be $x = Q_\infty\xi$ with $\xi \in (\mathcal{D}(AQ_\infty))$, so that $AQ_\infty\xi$ is well defined and clearly coincides with Ax, proving that $x \in \mathcal{D}(A)$. Obviously it must also be $x \in \mathcal{R}(Q_\infty)$. The converse is similar.

(ii) It is enough to observe that (84) coincides with (80) with $P = I_H$, and that $\mathcal{D}(\Lambda_{I_H}) = \mathcal{R}(Q_\infty)$.

(iii) Let $\hat{P}$ be a solution of (80) belonging to the class $\mathcal{Q}$ introduced in Definition 4.8. It is immediate to see that $\hat{P}$ is a stationary solution of (55) in the sense of Definition 4.6. Now we apply Lemma 4.10 and (51), getting

$$\frac{1}{2}\langle \hat{P}x, x\rangle_H \le V^{\hat{P}}(t,x) \le V(t,x) \quad x \in H, \quad t > T_0.$$

Taking the limit as $t \to +\infty$, the result follows by Proposition 5.4.

(iv) Let $\hat{R}$ be a solution of (31) with the required properties. Then, from Proposition 5.7, we have $\hat{P} := Q_\infty \hat{R} \in \mathcal{S}_+(H)$, $\hat{P}$ is a solution of (80) and $\hat{P} \in \mathcal{Q}$. By part (iii) we then obtain, for $x \in \mathcal{R}(Q_\infty)$,

$$\langle \hat{R}x, x\,\rangle_X = \langle Q_\infty^\dagger \hat{P}x, x\,\rangle_X = \langle \hat{P}x, x\,\rangle_H \le \langle x, x\,\rangle_H = \langle Q_\infty^\dagger x, x\,\rangle_X.$$

The claim follows. □

Remark 5.9. The statements of Theorem 5.8 still hold if we consider the slightly more general problem where the energy functional has the integrand $\langle Cu, u\rangle_U$ instead of $\langle u, u\rangle_U$, where $C \in \mathcal{S}_+(U)$ is coercive and hence invertible. Indeed it is enough to define the new control variable $v = C^{1/2}u$ and, consequently, to replace the control operator B in the state equation by $BC^{-1/2}$. ■

Remark 5.10. Theorem 5.8 can be applied to a variety of cases (e.g. delay equations treated in [13, Subsection 5.1] or wave equations). Here, according to our motivating example arising in physics (see Section 3), we develop a deeper analysis in the case where the operator A is selfadjoint and commutes with BB^*; in particular, when both are diagonal. This will be done in the next section. ■

6. The selfadjoint commuting case

We consider the case where A is selfadjoint and invertible, and commutes with BB^*. We do not use here Theorem 5.8: so we need neither Hypothesis 2.2, nor Hypothesis 2.3. We just assume Hypothesis 2.4.

From [13, Proposition C.1-(v)] we know that, for every $x \in X$,

$$Q_\infty x = -\frac{1}{2}A^{-1}BB^*x$$

and

$$2Ax = -BB^*Q_\infty^\dagger x \qquad \forall x \in \mathcal{R}(Q_\infty) \subseteq \mathcal{D}(A). \tag{85}$$

Multiplying at the right the last equality by $(BB^*)^\dagger$ and using that (see [13, Proposition C.1-(iii)]) $[\ker BB^*]^\perp = [\ker Q_\infty]^\perp$, we get

$$2(BB^*)^\dagger Ax = -(BB^*)^\dagger BB^*Q_\infty^\dagger x = Q_\infty^\dagger x \qquad \forall x \in \mathcal{R}(Q_\infty) \subseteq \mathcal{D}(A).$$

In this section we deal with the ARE (80) in H (with unknown $P \in \mathcal{L}(H)$) under Hypothesis 2.4. Such equation, using (85), becomes

$$0 = -\langle Ax, Q_\infty^\dagger Py\,\rangle_X - \langle Q_\infty^\dagger Px, Ay\,\rangle_X + 2\langle APx, Q_\infty^\dagger Py\,\rangle_X. \tag{86}$$

This makes sense, as for (80), when $x, y \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$ (see Definition 4.6). We now want to rewrite this equation using the inner products in H. Observe first that in $\mathcal{R}(Q_\infty)$ we have $Q_\infty^\dagger = (Q_\infty^{1/2})^\dagger(Q_\infty^{1/2})^\dagger$. Then, if Ax, Ay and APx belong to H, we rewrite (86) as

$$0 = -\langle Ax, Py\,\rangle_H - \langle Px, Ay\,\rangle_H + 2\langle APx, Py\,\rangle_H. \tag{87}$$

Now, recalling the definition of A_0 (see Lemma 2.13(iii)), Eq. (87) can be equivalently rewritten as

$$0 = -\langle A_0 x, Py\rangle_H - \langle Px, A_0 y\rangle_H + 2\langle A_0 Px, Py\rangle_H, \tag{88}$$

provided that x, y, Px, Py belong to $\mathcal{D}(A_0)$.

We now clarify the relationship between (86) and (88). First we set

$$D^P := \{x \in \mathcal{D}(A_0) : \; Px \in \mathcal{D}(A_0)\}\,. \tag{89}$$

Next, we provide the following definition of solution for (88) (compare with Definition 5.6):

Definition 6.1. An operator $P \in \mathcal{S}_+(H)$ is a solution of the ARE (88) if the set D^P is dense in H and Eq. (88) is satisfied for every $x, y \in D^P$.

Finally, we observe that every solution of (86) is also a solution of (88): indeed, if $P \in \mathcal{S}_+(H)$, then, by definition, we have $D^P \subseteq \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$. Hence, if $P \in \mathcal{S}_+(H)$ solves Eq. (86), then, choosing in particular $x, y \in D^P$ we can turn (86) into (88).

The reverse procedure is also possible: we postpone the proof at the end of the section (Proposition 6.6), since some more informations on solutions P of (88) are needed.

We now give a preparatory result about the properties of such solutions.

Proposition 6.2. *Assume Hypothesis 2.4. Then any solution P of (88) satisfies*

$$\langle A_0 x, A_0 Pz\ \rangle_H = \langle A_0 Px, A_0 z\ \rangle_H \qquad \forall x, z \in D^P. \tag{90}$$

Proof. Let P be a solution of (88). We observe that for all $x, y \in D^P$ we have, since A_0 is selfadjoint in H (see Lemma 2.15(iii)),

$$\langle A_0 Px, y\ \rangle_H + \langle PA_0 x, y\ \rangle_H = 2\langle PA_0 Px, y\ \rangle_H\,. \tag{91}$$

By density, this equation holds for every $x \in D^P$ and $y \in H$. Symmetrically we have also

$$\langle x, PA_0 y\ \rangle_H + \langle x, A_0 Py\ \rangle_H = 2\langle x, PA_0 Py\ \rangle_H \tag{92}$$

for every $x \in H$ and $y \in D^P$. We choose in (91) $y = PA_0 z - A_0 Pz$, with $z \in D^P$, and we obtain:

$$\begin{aligned}\langle A_0 Px, PA_0 z\rangle_H - \langle A_0 Px, A_0 Pz\rangle_H \;&+\; \langle PA_0 x, PA_0 z\rangle_H - \langle PA_0 x, A_0 Pz\rangle_H \\ &= 2\langle PA_0 Px, PA_0 z\rangle_H - 2\langle PA_0 Px, A_0 Pz\rangle_H\,.\end{aligned}$$

We isolate on the left the symmetric terms:

$$\begin{aligned}2\langle PA_0 Px, A_0 Pz\rangle_H \;&-\; \langle A_0 Px, A_0 Pz\rangle_H + \langle PA_0 x, PA_0 z\rangle_H \\ &= -\langle A_0 Px, PA_0 z\rangle_H + \langle PA_0 x, A_0 Pz\rangle_H + 2\langle PA_0 Px, PA_0 z\rangle_H\,.\end{aligned}$$

Next, we apply (91) to the last term on the right:

$$\begin{aligned}2\langle PA_0 Px, A_0 Pz\rangle_H \;&-\; \langle A_0 Px, A_0 Pz\rangle_H + \langle PA_0 x, PA_0 z\rangle_H \\ &= -\langle A_0 Px, PA_0 z\rangle_H + \langle PA_0 x, A_0 Pz\rangle_H + \langle A_0 Px, PA_0 z\rangle_H + \langle PA_0 x, PA_0 z\rangle_H\,,\end{aligned}$$

which simplifies to

$$2\langle PA_0 Px, A_0 Pz\rangle_H - \langle A_0 Px, A_0 Pz\rangle_H = \langle PA_0 x, A_0 Pz\rangle_H.$$

Applying (92) to the term on the right, rewritten as $\langle A_0x, PA_0Px\rangle_H$, we obtain for every $x, z \in D^P$

$$2\langle PA_0Px, A_0Pz\rangle_H - \langle A_0Px, A_0Pz\rangle_H - \frac{1}{2}\langle PA_0x, A_0z\rangle_H = \frac{1}{2}\langle A_0x, A_0Pz\rangle_H \,. \tag{93}$$

We now restart from (92), and choose $x = PA_0z - A_0Pz$, with $z \in D^P$: acting on the left variable of the inner product, and proceeding exactly in the same way as before, we get for every $z, y \in D^P$

$$2\langle A_0Pz, PA_0Py\rangle_H - \langle A_0Pz, A_0Py\rangle_H - \frac{1}{2}\langle PA_0z, A_0y\rangle_H = \frac{1}{2}\langle A_0Pz, A_0y\rangle_H \,. \tag{94}$$

Comparing Eqs. (93) and (94), both written with variables x, y, we immediately obtain

$$\frac{1}{2}\langle A_0x, A_0Py\rangle_H = \frac{1}{2}\langle A_0Px, A_0y\rangle_H \,, \quad x, y \in D^P,$$

which is (90). $\square$

We can now prove:

Theorem 6.3. *Assume Hypothesis 2.4. Then any solution P of (88) commutes with A_0, i.e. $Px \in \mathcal{D}(A_0)$ for every $x \in \mathcal{D}(A_0)$ and*

$$A_0Px = PA_0x \qquad \forall x \in \mathcal{D}(A_0).$$

In particular $D^P = \mathcal{D}(A_0)$.

Proof. We start from (90) with $w = A_0x$ and $y = A_0z$, i.e.

$$\langle w, A_0PA_0^{-1}y\,\rangle_H = \langle A_0PA_0^{-1}w, y\,\rangle_H \qquad \forall w, y \in A_0(D^P). \tag{95}$$

Notice that $A_0(D^P)$ is the natural domain of the operator $A_0PA_0^{-1}$; which might be (*a priori*) not dense in H. Let us denote by Z the closure of $\mathcal{D}(A_0PA_0^{-1})$ in H; so we have

$$Z := \overline{A_0(D^P)} = \overline{\mathcal{D}(A_0PA_0^{-1})}.$$

Obviously Z is a Hilbert space with the inner product of H. Eq. (95) then tells us that $A_0(D^P) \subseteq \mathcal{D}((A_0PA_0^{-1})^*)$ and

$$(A_0PA_0^{-1})^*w = A_0PA_0^{-1}w \qquad \forall w \in A_0(D^P) = \mathcal{D}(A_0PA_0^{-1}). \tag{96}$$

On the other hand, if $x \in \mathcal{D}(A_0)$ and $y \in \mathcal{D}(A_0PA_0^{-1})$ we may write

$$\langle x, A_0PA_0^{-1}y\,\rangle_H = \langle A_0^{-1}PA_0x, y\,\rangle_H \,;$$

consequently

$$\mathcal{D}(A_0) \subseteq \mathcal{D}((A_0PA_0^{-1})^*) \tag{97}$$

and

$$(A_0PA_0^{-1})^*x = A_0^{-1}PA_0x \qquad \forall x \in \mathcal{D}(A_0). \tag{98}$$

We now claim that $A_0PA_0^{-1}$ is selfadjoint in the space H, i.e.

$$\mathcal{D}((A_0PA_0^{-1})^*) = \mathcal{D}(A_0PA_0^{-1}) = A_0(D^P) \tag{99}$$

is dense in H and (96) holds.

Indeed, assume that $z \in \mathcal{D}((A_0PA_0^{-1})^*)$: then there is $c > 0$ such that

$$|\langle A_0PA_0^{-1}x, z\rangle_H| \le c\|x\|_H \qquad \forall x \in \mathcal{D}(A_0PA_0^{-1}).$$

In particular, by (96),

$$\langle x, (A_0PA_0^{-1})^*z\rangle_H = \langle A_0PA_0^{-1}x, z\rangle_H = \langle (A_0PA_0^{-1})^*x, z\rangle_H \quad \forall x \in \mathcal{D}(A_0PA_0^{-1}).$$

This shows that $z \in \mathcal{D}(A_0PA_0^{-1})$ and $A_0PA_0^{-1}z = (A_0PA_0^{-1})^*z$. Hence

$$\mathcal{D}((A_0PA_0^{-1})^*) \subseteq \mathcal{D}(A_0PA_0^{-1}) \quad \text{and} \quad A_0PA_0^{-1} = (A_0PA_0^{-1})^* \text{ on } \mathcal{D}((A_0PA_0^{-1})^*).$$

Conversely, we know from (96) that

$$\mathcal{D}(A_0PA_0^{-1}) = A_0(D^P) \subseteq \mathcal{D}((A_0PA_0^{-1})^*) \quad \text{and} \quad (A_0PA_0^{-1})^* = A_0PA_0^{-1} \text{ on } \mathcal{D}(A_0PA_0^{-1}).$$

In particular, by (97), Z coincides with H, i.e. both domains in (99) are dense in H. This proves our claim.

Take now $x \in \mathcal{D}(A_0)$. As, by (97), $\mathcal{D}(A_0) \subseteq \mathcal{D}(A_0PA_0^{-1})$, we have

$$PA_0^{-1}x \in \mathcal{D}(A_0) \quad \forall x \in \mathcal{D}(A_0), \quad \text{i.e.} \quad D^P = \mathcal{D}(A_0). \tag{100}$$

(see (89)). Moreover, by (98) and by the above claim we deduce

$$A_0^{-1}PA_0x = A_0PA_0^{-1}x \qquad \forall x \in \mathcal{D}(A_0).$$

Applying A_0^{-1} we have $A_0^{-2}PA_0x = PA_0^{-1}x$ for every $x \in \mathcal{D}(A_0)$, or, equivalently,

$$A_0^{-2}PA_0^2z = z \qquad \forall z \in \mathcal{D}(A_0^2), \qquad \text{i.e.} \qquad A_0^{-2}Pw = PA_0^{-2}w \qquad \forall w \in H.$$

This means that the bounded operators A_0^{-2} and P commute. Now, since A_0^{-1} is a non-negative operator such that $(A_0^{-1})^2 = A_0^{-2}$, by a well known result (see [26, Theorem VI.9]), A_0^{-1} must commute with every bounded operator B which commutes with A_0^{-2}, for instance $B = P$. So

$$A_0^{-1}Pw = PA_0^{-1}w \qquad \forall w \in H, \qquad \text{i.e.} \qquad Pz = A_0^{-1}PA_0z \qquad \forall z \in \mathcal{D}(A_0);$$

this implies that $P(\mathcal{D}(A_0)) \subseteq \mathcal{D}(A_0)$ and $A_0Pz = PA_0z$ for every $z \in \mathcal{D}(A_0)$. Thus P commutes with A_0, as required. Moreover $P(\mathcal{D}(A_0)) \subseteq \mathcal{D}(A_0)$ implies $\mathcal{D}(A_0) \subseteq D^P$. The reverse inclusion immediately follows from the definition of D^P. $\square$

Remark 6.4. *By Theorem 6.3 we easily deduce that any solution P of (88) commutes with the resolvents $R(\lambda, A_0)$ for every $\lambda \in \rho(A_0)$, i.e.*

$$PR(\lambda, A_0)x = R(\lambda, A_0)Px \qquad \forall x \in H.$$

Indeed, for all $x \in D(A_0)$ we have $PA_0x = A_0Px$; so, for all $\lambda \in \rho(A_0)$ we have $P(\lambda - A_0)x = (\lambda - A_0)Px$. Choosing $x = R(\lambda, A_0)z$ we get

$$P(\lambda - A_0)R(\lambda, A_0)z = (\lambda - A_0)PR(\lambda, A_0)z \quad \Longleftrightarrow \quad Pz = (\lambda - A_0)PR(\lambda, A_0)z.$$

We then conclude multiplying both sides by $R(\lambda, A_0)$.

We are now able to characterize all solutions of the ARE (88).

Theorem 6.5. *Assume Hypothesis 2.4 and let $P \in \mathcal{S}_+(H)$. Then P is a solution of (88) if and only if P is an orthogonal projection in H and it commutes with A_0. In particular the identity I_H is the maximal solution among all solutions of (88).*

Proof. Let P be a solution of (88): by Theorem 6.3 we have $Px \in \mathcal{D}(A_0)$ for every $x \in \mathcal{D}(A_0)$ and $A_0Px = PA_0x$. Hence the ARE (91), equivalent to (88), becomes

$$0 = -2\langle PA_0x, y\rangle_H + 2\langle PA_0Px, y\rangle_H, \quad x \in D^P,\ y \in H.$$

Since y is arbitrary, using (100) we get $2PA_0x = 2PA_0Px$ for every $x \in \mathcal{D}(A_0)$, and successively, for all $x \in D(A_0)$, $PA_0x - PA_0Px = 0$, $PA_0(I_H - P)x = 0$, $A_0P(I_H - P)x = 0$, $P(I_H - P)x = 0$, $Px = P^2x$; finally, by density, $P = P^2$.

Assume, conversely, that P is an orthogonal projection in H and it commutes with A_0. Then

$$PA_0Pz = P^2A_0z = PA_0z = A_0Pz \quad \forall z \in \mathcal{D}(A_0),$$

and consequently P solves (91). Finally, since I_H solves (88), the last statement is immediate. □

We conclude this section proving the equivalence of the two forms (86) and (88) of the ARE.

Proposition 6.6. *Every solution of (86) is also a solution of (88) and vice versa.*

Proof. We have already seen that every solution of (86) is also a solution of (88).

Consider now a solution P of (88). First of all, if $x, y \in D^P = \mathcal{D}(A_0)$, Eq. (88) transforms into (86), so that (86) holds true for $x, y \in D^P$.

We claim that D^P is dense in $\mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$ (see (89)) with respect to the norm $\|\cdot\|_H + \|A\cdot\|_X + \|AP\cdot\|_X$. Indeed, for $z \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$, recalling Lemma 2.13, we take the Yosida approximations of A (see [25, Section 1.3]), and define, for $n \in \mathbb{N}$,

$$z_n = nR(n, A)z = nR(n, A)|_H\, z = nR(n, A_0)z.$$

Then $z_n \in \mathcal{D}(A_0) = D^P$ and, as $n \to \infty$,

$$\begin{gathered} z_n \to z \quad \text{in } H, \\ A_0z_n = nA_0R(n, A_0)z = nAR(n, A)z \to Az \quad \text{in } X, \\ A_0Pz_n = nA_0PR(n, A_0)z = nAR(n, A)Pz \to APz \quad \text{in } X; \end{gathered}$$

this proves our claim.

Let now $x, y \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P)$; we select again $x_n = nR(n, A_0)x$, $y_n = nR(n, A_0)y$: we have $x_n, y_n \in \mathcal{D}(A_0)$ and, as $n \to \infty$,

$$\begin{gathered} x_n \to x \text{ in } H, \quad Ax_n \to Ax \text{ in } X, \quad APx_n \to APx \text{ in } X, \\ y_n \to y \text{ in } H, \quad Ay_n \to Ay \text{ in } X, \quad APy_n \to APy \text{ in } X. \end{gathered}$$

In addition we have by Remark 6.4, Lemma 2.10(ii)

$$Q_\infty^\dagger Px_n = Q_\infty^\dagger PnR(n, A_0)x = Q_\infty^\dagger nR(n, A_0)Px = Q_\infty^\dagger nR(n, A)Px = nR(n, A)Q_\infty^\dagger Px,$$

so that, as $n \to \infty$,

$$Q_\infty^\dagger Px_n \to Q_\infty^\dagger Px \text{ in } X,$$

and similarly $Q_\infty^\dagger Py_n \to Q_\infty^\dagger Py$ in X as $n \to \infty$. For x_n and y_n, (86) holds:

$$0 = -\langle Ax_n, Q_\infty^\dagger y_n\rangle_X - \langle Q_\infty^\dagger Px_n, Ay_n\rangle_X + 2\langle APx_n, Q_\infty^\dagger Py_n\rangle_X.$$

By what established above, we can pass to the limit as $n \to \infty$ in all terms, obtaining

$$0 = -\langle Ax, Q_\infty^\dagger Py \rangle_X - \langle Q_\infty^\dagger Px, Ay \rangle_X + 2\langle APx, Q_\infty^\dagger Py \rangle_X \qquad \forall x, y \in \mathcal{D}(A) \cap \mathcal{D}(\Lambda_P),$$

i.e. P solves (86). □

Remark 6.7. It is easy to verify that for every solution P of (88) the space $D^P = \mathcal{D}(A_0)$ is dense in $\mathcal{D}(A) \cap H$ with respect to the norm $\|\cdot\|_H + \|A\cdot\|_X$: it suffices to repeat the argument above, i.e. to consider, for fixed $x \in \mathcal{D}(A) \cap H$, the approximation $x_n = nR(n, A_0)x$, observing that $x_n \to x$ in H and $A_0 x_n = Ax_n \to Ax$ in X. Thus, P belongs to the class $\mathcal{Q}$ introduced in Definition 4.8.

Corollary 6.8. *Assume that A_0 is a diagonal operator with respect to an orthonormal complete system $\{e_n\}$ in H with sequence of eigenvalues $\{\lambda_n\} \subset \,]-\infty, 0[\,$. Let P be a solution of the ARE (88). Then*

(i) every eigenspace of A_0 is invariant for P;
(ii) if all eigenvalues are simple, then P is diagonal with respect to the system $\{e_n\}$;
(iii) if there exists an eigenspace M whose dimension is $m \geq 2$, then the restriction of P to M may not be diagonal: in particular, if $m = 2$ a non-diagonal P on M must have the following explicit form:

$$\begin{pmatrix} a & \pm\sqrt{a(1-a)} \\ \pm\sqrt{a(1-a)} & 1-a \end{pmatrix} \qquad \textit{for some } \ a \in \,]0,1[\,. \tag{101}$$

Proof. To prove (i) it is enough to show that, for every eigenvalue λ of A_0 and x eigenvector of A_0 associated with λ, we have $\lambda Px = A_0 Px$. This is immediate since A_0 and P commute.

Concerning (ii) we observe that, for every $n \in \mathbb{N}$ we have $A_0 e_n = \lambda_n e_n$, so that $\lambda_n Pe_n = A_0 Pe_n$. Since λ_n is simple, it is $Pe_n = ke_n$ for some $k \in \mathbb{R}$. Since P is a projection, it must be $k = 0$ or $k = 1$.

Now, to prove (iii), let us assume that M is an eigenspace of dimension 2 associated to an eigenvalue $\lambda < 0$ and generated by the orthonormal eigenvectors e and f. We know that M is invariant with respect to P and that P is symmetric. This implies that P on M (with basis (e, f)) can be identified with a 2×2 symmetric matrix T of the form

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \text{for some } \ a, b, c \in \mathbb{R}. \tag{102}$$

Moreover T is a projection. If T has rank 2 then it must be the identity matrix. If not, it must be of the form $Tx = (x_1 v_1 + x_2 v_2)v$ for some vector v of M of norm 1. Hence

$$T = \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}$$

Setting $v_1^2 = a$ we get the claim. □

Remark 6.9. Let A be a diagonal operator with respect to an orthonormal complete system $\{e_n\}$ in H, with sequence of eigenvalues $\{\lambda_n\} \subset \,]-\infty, 0[\,$, where all λ_n are simple. Then BB^* must be diagonal, too. Indeed we have, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{+\infty} \langle BB^* e_n, e_k \rangle_H \, e_k = BB^* e_n = \frac{1}{\lambda_n} BB^* A e_n = \frac{1}{\lambda_n} ABB^* e_n = \frac{1}{\lambda_n} \sum_{k=0}^{+\infty} \lambda_k \langle BB^* e_n, e_k \rangle_H \, e_k,$$

which implies

$$\langle BB^* e_n, e_k \rangle_H \left(1 - \frac{\lambda_k}{\lambda_n}\right) = 0 \qquad \forall k, n \in \mathbb{N}.$$

Since all eigenvalues are distinct, it must be $BB^*e_n = b_n e_n$ for all $n \in \mathbb{N}$ for a suitable sequence $\{b_n\} \in \ell^\infty$. This implies that Q_∞ and $Q_\infty^{1/2}$ are diagonal with respect to $\{e_n\}$, too. Following [13, Subsection 5.2] we may also consider the case when BB^* is unbounded and characterize the space H, for specific choices of BB^*, in terms of the domain of a suitable power of $(-A)$. The example developed in Section 3 fits into this framework. ■

Remark 6.10. We can now apply our abstract theory to the example of Section 3. First of all, by Proposition 5.4, we know that the value function (42) is given by

$$V_\infty(y_0) = \frac{1}{2}\|y_0\|^2_{L^2(0,1)}.$$

Moreover, by Theorem 5.8, we obtain that:

- by point (i), the operator $Q_\infty^\dagger = 2A$ solves the ARE (31) where we replace B^* by $I_{H^{-1}(0,1)}$;
- by point (ii), the identity in $L^2(0,1)$, $I_{L^2(0,1)}$, solves the ARE (80) where we replace B and B^* by $I_{H^{-1}(0,1)}$;
- by point (iii) $I_{L^2(0,1)}$ is the maximal solution of the ARE (80) among those belonging to the class $\mathcal{Q}$ introduced in Definition 4.8.

Finally, since Hypothesis 2.4 holds, we can apply Theorem 6.5. Then, noting that A_0 is the Laplace operator with Dirichlet boundary conditions in the space $H = L^2(0,1)$, whose domain is $H^2(0,1) \cap H^1_0(0,1)$, we obtain that:

- the identity $I_{L^2(0,1)}$, is a solution of the two (equivalent) AREs (86) and (88);
- the set of all solutions of (86) and (88) consists of all orthogonal projections P which commute with A_0, i.e. all projections whose image is generated by a subset of the eigenvectors of A_0;
- $I_{L^2(0,1)}$ is the maximal solution among all solutions of (86) and (88). ■

Acknowledgments

We warmly thank the three referees: their careful scrutiny and their useful suggestions helped us to considerably improve the paper. We also thank Prof. G. Jona-Lasinio for suggesting us the problem. Finally we thank Proff. L. Bertini, G. Fabbri, S. Faggian, E. Priola, A. Swiech, R. Vinter, J. Zabczyk for useful discussions on the topic.

References

[1] G. Da Prato, A.J. Pritchard, J. Zabczyk, On minimum energy problems, SIAM J. Control Optim. 29 (1) (1991) 209–221.
[2] J. Feng, T. Kurtz, Large Deviations for Stochastic Processes. Mathematical Surveys and Monographs, AMS, 2006.
[3] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Fluctuations in stationary nonequilibrium states of irreversible processes, Phys. Rev. Lett. 87 (2001) 040601.
[4] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Macroscopic fluctuation theory for stationary non equilibrium state, J. Stat. Phys. 107 (2002) 635–675.
[5] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Large deviations for the boundary driven simple exclusion process, Math. Phys. Anal. Geom. 6 (2003) 231–267.
[6] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Minimum dissipation principle in stationary non-equilibrium states, J. Stat. Phys. 116 (2004) 831–841.
[7] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, C. Landim, Action functional and quasi-potential for the Burgers equation in a bounded interval, Comm. Pure. Appl. Math. 64 (2011) 649–696.
[8] L. Bertini, D. Gabrielli, J.L. Lebowitz, Large deviations for a stochastic model of heat flow, J. Stat. Phys. 121 (2005) 843–885.
[9] A. Bensoussan, G. Da Prato, M.C. Delfour, S.K. Mitter, Representation and Control of Infinite Dimensional System, second ed., Birkhäuser, Boston, 2007.
[10] E. Priola, J. Zabczyk, Null controllability with vanishing energy, SIAM J. Control Optim. 42 (6) (2003) 1013–1032.

[11] B.C. Moore, Principal component analysis in linear systems: controllability, observability, and model reduction, IEEE Trans. Automat. Control 26 (1) (1981) 17–32.
[12] J.M.A. Scherpen, Balancing for nonlinear systems, Systems Control Lett. 21 (2) (1993) 143–153.
[13] P. Acquistapace, F. Gozzi, Minimum energy for linear systems with finite horizon: a non-standard Riccati equation, Math. Control Signals Systems 29 (2017) 19.
[14] J. Zabczyk, Mathematical Control Theory: An Introduction, Birkhäuser Verlag, Boston, 1995.
[15] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimension, Springer, 1992.
[16] O. Carja, The minimal time function in infinite dimensions, SIAM J. Contr. Optimiz. 31 (5) (1993) 1103–1114.
[17] Z. Emirsajlow, A feedback for an infinite-dimensional linear-quadratic control problem with a fixed terminal state, IMA J. Math. Control Inform. 6 (1) (1989) 97–117.
[18] Z. Emirsajlow, S.D. Townley, Uncertain systems and minimum energy control, J. Appl. Math. Comput. Sci. 5 (3) (1995) 533–545.
[19] F. Gozzi, P. Loreti, Regularity of the minimum time function and minimum energy problems: The linear case, SIAM J. Control Optim. 37 (4) (1999) 1195–1221.
[20] R. Curtain, A.J. Pritchard, Infinite Dimensional Linear Systems Theory, Springer, 1978.
[21] R. Curtain, H. Zwart, Introduction To Infinite-Dimensiuonal Systems Theory, Springer, 2020.
[22] I. Lasiecka, R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories, Part 1, Cambridge University Press, 2000.
[23] I. Lasiecka, R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Part 2, Cambridge University Press, 2000.
[24] K.-J. Engel, R. Nagel, One Parameter Semigroups for Linear Evolution Equations, Springer, 1999.
[25] A. Pazy, Semigroups of Linear Operators and Applications To Partial Differential Equations, Springer Verlag, New-York, 1983.
[26] M. Reed, B. Simon, Functional Analysis, Academic Press, London, 1980.
[27] G. Fabbri, F. Gozzi, A. Swiech, Stochastic Optimal Control in Infinite Dimension, Springer, 2017.
[28] Lunardi A., Interpolation theory, in: Appunti, Scuola Normale Superiore Di Pisa (Nuova Serie), second ed., Edizioni della Normale, Pisa, 2009.
[29] P. Lancaster, L. Rodman, Algebraic Riccati Equations, Oxford Science Publications, Clarendon Press, Oxford, 1995.
[30] J.C. Willems, Least squares stationary optimal control and the algebraic Riccati equation, IEEE Trans. Automat. Control AC-16 (1971) 621–634.
[31] T. Reis, M. Voigt, Linear-quadratic optimal control of differential–algebraic systems: the infinite time horizon problem with zero terminal state, SIAM J. Control Optim. 7 (3) (2019) S. 1567–1596, http://dx.doi.org/10.1137/18M1189609.