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Analisi matematica. — *On the abstract Cauchy problem in the case of constant domains.* Nota di PAOLO ACQUISTAPACE e BRUNELLO TERRENI, presentata (*) dal Corrisp. E. VESENTINI.

RIASSUNTO. — Si studiano esistenza, unicità e regolarità delle soluzioni strette, classiche e forti $u \in C([0,T],E)$ dell'equazione di evoluzione non autonoma $u'(t) - A(t)u(t) = f(t)$ con il dato iniziale $u(0) = x$, in uno spazio di Banach E. Gli operatori $A(t)$ sono generatori infinitesimali di semi-gruppi analitici ed hanno dominio indipendente da t e non necessariamente denso in E. Si danno condizioni necessarie e sufficienti per l'esistenza e la regolarità hölderiana della soluzione e della sua derivata.

0. INTRODUCTION

Let E be a Banach space, and $\{A(t)\}_{t\in[0,T]}$ a family of closed linear operators on E; suppose that for each $t \in [0,T]$ $A(t)$ generates an analytic semigroup and has a domain $D(A(t)) = D(A(0))$ which does not depend on t and is possibly not dense in E. We consider the following Cauhy problem:

$$(P) \qquad \begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [0,T] \\ u(0) = x \\ x \in E, f\colon [0,T] \to E & \text{prescribed.} \end{cases}$$

Many authors have studied this problem (see the references). The case of variable domains has been treated in [1]; thus, in this paper we will assume the same hypotheses of Tanabe [12]; more precisely, we suppose the following:

I) For each $t \in [0,T]$ $A(t)$ is a closed linear operator on the Banach space E, with domain $D(A(t)) = D(A(0))$ independent on t, generating the analytic semi-group $\{e^{\xi A(t)}\}_{\xi \geq 0}$: in particular

(i) there exists $\theta^0 \in]\pi/2, \pi]$ such that

$$\rho(A(t)) \supseteq \Sigma_{\theta^0} := \{z \in \mathbf{C} : z = \rho e^{i\theta}, \rho \in [0, \infty[, \theta \in]-\theta^0, \theta^0[\}, \forall t \in [0,T];$$

(ii) there exists $M > 0$ such that $\forall t \in [0,T]$

$$\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leq \frac{M}{|\lambda|} \; \forall \lambda \in \Sigma_{\theta^0} - \{0\}, \; \|A(t)^{-1}\|_{\mathscr{L}(E)} \leq M$$

(*) Nella seduta del 14 gennaio 1984.

II) There exist $\alpha \in]0, 1[$ and $K > 0$ such that

$$\| 1 - A(t) A(\tau)^{-1} \|_{\mathscr{L}(E)} \leq K \mid t - \tau \mid^{\alpha} \; \forall t , \tau \in [0, T] .$$

Let us specify now what we mean as a solution of Problem (P). Denote by $C([0, T], E)$ (resp. $C(]0, T], E)$) the space of continuous functions $[0, T] \to E$ (resp. $]0, T] \to E$), and set

$$D^0 = \{u \in C([0, T], E) : u(t) \in D(A(0)) \; \forall t \in [0, T] \quad \text{and} \quad u', A(\cdot) u(\cdot) \in C([0, T], E)\},$$

$$D = \{u \in C([0, T], E) : u(t) \in D(A(0)) \; \forall t \in]0, T] \quad \text{and} \quad u', A(\cdot) u(\cdot) \in C(]0, T], E)\}.$$

Now we define our solutions.

Definition 0.1. *$u : [0, T] \to E$ is a* strict solution *of Problem* (P) *if $u \in D$ and*

$$u'(t) - A(t) u(t) = f(t) \; \forall t \in [0, T], \qquad u(0) = x .$$

Definition 0.2. *$u : [0, T] \to E$ is a* classical solution *of Problem* (P) *if $u \in D^0$ and*

$$u'(t) - A(t) u(t) = f(t) \; \forall t \in]0, T], \qquad u(0) = x .$$

Definition 0.3. *$u : [0, T] \to E$ is a* strong solution *if $u \in C([0, T], E)$ and there exists $\{u_n\}_{n \in N} \subseteq D$ such that*

(i) $u_n \to u$ *in* $C([0, T], E)$,

(ii) $u_n' - A(\cdot) u_n(\cdot) =: f_n \to f$ *in* $C([0, T], E)$,

(iii) $u_n(0) =: x_n \to x$ *in* E.

Remark 0.4. By definition every strict solution is also a classical and a strong one whereas it is not obvious that a classical solution is also a strong one; this is true under hypotheses I and II. The situation in the case of variable domains is different (see Remark 6.5 of [1]).

In this paper we give existence, uniqueness and regularity results for strict, classical and strong solutions of (P); in addition we give a representation formula for the solutions, without passing through the construction of the fundamental solution; our method is mainly inspired by the techniques of Da Prato-Grisvard [5].

Our formula can be heuristically derived by the following argument: we look for a solution of this kind:

$$u(t) := e^{tA(0)} x + \int_0^t e^{(t-s)A(s)} g(s)\, ds, \qquad t \in [0, T], \tag{0.1}$$

where g is a suitable (integrable) function. Of course when $A(t) \equiv A$ this formula with $g = f$ gives the ordinary mild solution of (P); since $A(t)$ depends on t now, it is natural to expect that f has to be suitably modified. Taking the formal derivative of (0.1) we get

$$u'(t) = A(t)\, u(t) + (A(0) - A(t))\, e^{tA(0)} x + g(t) + \\ + \int_0^t (A(s) - A(t))\, e^{(t-s)A(s)}\, g(s)\, ds,$$

hence if we want (0.1) to be a solution of (P) we must choose g such that

$$g(t) + \int_0^t K(t, s)\, g(s)\, ds = f(t) - K(t, 0)\, x, \; t \in [0, T], \tag{0.2}$$
$$K(t, s) := (A(s) - A(t))\, e^{(t-s)A(s)}.$$

Denote by K the integral operator

$$K\varphi(t) := \int_0^t K(t, s)\, \varphi(s)\, ds; \tag{0.3}$$

then the representation formula for the solution of Problem (P) is formally given by

$$u(t) = e^{tA(0)} x + \int_0^t e^{(t-s)A(s)} (1 + K)^{-1} (f - K(\cdot, 0)\, x)(s)\, ds, \qquad t \in [0, T]. \tag{F}$$

We also study the maximal time regularity of the strict solution: in other words we prove that the solution u has Hölder continuous derivative in $[0, T]$ if f is Hölder continuous with the same exponent, provided the vectors x and $f(0)$ satisfy a suitable compatibility condition which is necessary and sufficient. Our condition generalizes the results of Poulsen [8], Sinestrari [9], [10], and Da Prato-Sinestrari [7].

Finally we give some space regularity results for the strong solution, i.e. we prove that the solution is continuous with values in the interpolation spaces between D (A (0)) and E, provided x belongs to the same space. Regarding classical solutions, we obtain an «a priori» estimate for $A(t)u(t)$ which is of interest in the study of the quasi-linear version of (P). More results, with all proofs, will appear in a forthcoming paper.

1. Preliminaries

If $\beta \in [0,1[$, denote by $C_\beta(]0,T],E)$ (resp. $C_\beta([0,T],E)$) the space of functions $u \in C(]0,T],E)$ such that $t \to t^\beta u(t) \in L^\infty(0,T,E)$ (resp. $C([0,T],E)$) (note that $C_0([0,T],E) = C([0,T],E)$). $C_\beta(]0,T],E)$ is obviously a Banach space with norm

$$\|u\|_{C_\beta(]0,T],E)} = \|t^\beta u(\cdot)\|_{L^\infty(0,T,E)},$$

having $C_\beta([0,T],E)$ as a closed subspace.

For each $\sigma \in]0,1[$ we define the Hölder space $C^\sigma([0,T],E) = \{f \in C([0,T],E) : \|f(t) - f(\tau)\|_E = 0\,(t-\tau)^\sigma$ as $t - \tau \to 0^+\}$ and its closed sub space $h^\sigma([0,T],E) = \{f \in C^\sigma([0,T],E) : \|f(t) - f(\tau)\|_E = o\,(t-\tau)^\sigma$ as $t - \tau \to 0^+\}$; we shall also use the spaces

$$C^\sigma(]0,T],E) := \bigcap_{\varepsilon \in]0,T[} C^\sigma([\varepsilon,T],E),$$

$$h^\sigma(]0,T],E) := \bigcap_{\varepsilon \in]0,T[} h^\sigma([\varepsilon,T],E).$$

The interpolation spaces $D_{A(0)}(\theta,\infty)$ and $D_{A(0)}(\theta)$ between D (A (0)) and E ($\theta \in]0,1[$) can be defined in several equivalent ways (see Butzer–Berens [3], Da Prato-Grisvard [6]). In particular we have

$$\begin{aligned} D_{A(0)}(\theta,\infty) &= \{x \in E : \sup_{t>0} t^{-\theta} \|e^{tA(0)}x - x\|_E < \infty\} = \\ &= \{x \in E : \sup_{t>0} t^{1-\theta} \|A(0)\,e^{tA(0)}x\|_E < \infty\} = \\ &= \{x \in E : \sup_{\lambda \in \rho(A(0))} |\lambda|^\theta \|A(0)\,R(\lambda, A(0))\,x\|_E < \infty\}, \end{aligned}$$

and similarly

$$\begin{aligned} D_{A(0)}(\theta) &= \{x \in E : \lim_{t \to 0^+} t^{-\theta} \|e^{tA(0)}x - x\|_E = 0\} = \\ &= \{x \in E : \lim_{t \to 0^+} t^{1-\theta} \|A(0)\,e^{tA(0)}x\|_E = 0\} = \\ &= \{x \in E : \lim_{\substack{\lambda \in \rho(A(0)) \\ |\lambda| \to \infty}} |\lambda|^\theta \|A(0)\,R(\lambda, A(0))\,x\|_E\} = 0. \end{aligned}$$

$D_{A(0)}(\theta, \infty)$ is a Banach space with norm $\| x \|_\theta := \sup_{t>(0)} t^{-\theta} \| e^{tA(0)} x - x \|_E$; moreover the quantities $\sup_{t>0} t^{1-\theta} \| A(0) e^{tA(0)} x \|_E$ and $\sup_{\lambda \in \rho(A)(0))} | \lambda |^\theta \| A(0) R(\lambda, A(0)) x \|_E$ define two norms on $D_{A(0)}(\theta, \infty)$ which are equivalent to $\| x \|_\theta$.

Clearly $D(A(0)) \subseteq D_{A(0)}(\theta, \infty) \subseteq E$ with continuous inclusions (when $D(A(0))$ is equipped with the graph norm); in addition $D_{A(0)}(\theta)$—which is a closed subspace of $D_{A(0)}(\theta, \infty)$—coincides with the closure of $D(A(0))$ iu $D_{A(0)}(\theta, \infty)$ (see [3]).

The analytic semi-groups $\{ e^{\xi A(t)} \}_{\xi \geq 0}$, $t \in [0, T]$, have a standard representation by a Dunford integral: if $\theta \in]\pi/2, \pi]$, set $\gamma := \gamma^0 \cup \gamma^- \cup \gamma^+$, where

$$\gamma^0 := \{ \lambda \in \mathbf{C} : | \lambda | = 1, | \arg \lambda | \leq \theta \},$$

$$\gamma^\pm := \{ \lambda \in \mathbf{C} : \lambda = \rho e^{\pm i\theta}, \rho \geq 1 \};$$

then under hypothesis I we have

$$A(t)^n e^{\xi A(t)} x = \frac{1}{2\pi i} \int_\gamma \lambda^n e^{\xi\lambda} R(\lambda, A(t))\, d\lambda, \ \forall \xi > 0, \forall t \in [0, T], \forall n \in \mathbf{N}.$$

the integrals being absolutely convergent.

In addition, under hypotheses I, II it is clear that

$$\| K(t, s) \|_{\mathscr{L}(E)} = \| (1 - A(t) A(s)^{-1}) A(s) e^{(t-s) A(s)} \|_{\mathscr{L}(E)} \leq \frac{C}{(t-s)^{1-\alpha}},$$
$$0 \leq s < t \leq T.$$

Hence it is easy to show that the operator K defined in (0.3) is continuous on $L^p(0, T, E)$ and $C_\beta(]0, T], E)$, $p \in [1, \infty]$, $\beta \in [0, 1[$, and $(1 + K)^{-1}$ exists and is also continuous in such spaces. As a direct consequence, formula (F) defines an element u of $C_0(]0, T], E)$; moreover $u \in C([0, T], E)$ if and only if $x \in \overline{D(A(0))}$.

2. MAIN RESULTS

Here is a list of our main results, whose proofs will appear in a forthcoming paper.

THEOREM 2.1. (Necessary conditions). *Under hypotheses I, II let* $x \in E$ *and* $f \in C([0, T], E)$.

(i) *If u is a strict solution of (P), then* $x \in D(A(0))$ *and* $A(0)x + f(0) \in \overline{D(A(0))}$.

(ii) *If u is a classical (resp. a strong) solution of* (P), *then* $x \in \overline{D(A(0))}$.

THEOREM 2.2. (Uniqueness and continuous dependence on data). *Under hypotheses I, II let* $x \in E$ *and* $f \in C([0, T], E)$. *Then Problem* (P) *has at most one strict (resp. classical or strong) solution. Moreover if u is such a solution, then*

$$\| u(t) \|_E \leq C \left\{ \| x \|_E + \int_0^t \| f(s) \|_E \, ds \right\} \forall t \in [0, T].$$

THEOREM 2.3. (Existence of the strict solution). *Under hypotheses I, II let* $x \in D(A(0))$ *and* $f \in C^\sigma([0, T], E)$, $\sigma \in]0, \alpha]$ *(resp.* $h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$), *and suppose that* $A(0)x + f(0) \in \overline{D(A(0))}$. *Then the function u defined by* (F) *is the strict solution of* (P) *and belongs to* $C^{1,\sigma}(]0, T], E)$ *(resp.* $h^{1,\sigma}(]0, T], E)$).

THEOREM 2.4. (Maximal time regularity of the strict solution). *Under hypotheses I, II let* $x \in D(A(0))$ *and* $f \in C^\sigma([0, T], E)$, $\sigma \in]0, \alpha]$ *(resp.* $h^\sigma([0, T], E)$, $\sigma \in]0, \alpha[$), *and suppose that u is a strict solution of* (P). *Then* $u \in C^{1,\sigma}([0, T], E)$ *(resp.* $h^{1,\sigma}([0, T], E)$) *if and only if* $A(0)x + f(0) \in D_{A(0)}(\sigma, \infty)$ *(resp.* $D_{A(0)}(\sigma)$).

THEOREM 2.5. (Existence of the classical solution). *Under hypotheses I, II let* $x \in \overline{D(A(0))}$ *and* $f \in C([0, T], E) \cap C^\sigma(]0, T], E)$, $\sigma \in]0, 1[$ *(resp.* $C([0, T], E) \cap h^\sigma(]0, T], E)$). *Then the function u defined by* (F) *is the classical solution of* (P) *and belongs to* $C^{1,\sigma}(]0, T], E)$ *(resp.* $h^{1,\sigma}(]0, T], E)$). *In addition if* $x \in D_{A(0)}(\beta, \infty)$ *and* $f \in C^\sigma([0, T], E)$, $\beta, \sigma \in]0, 1[$, *then*

$$\| A(t) u(t) \|_E \leq C \left\{ \frac{1}{t^{1-\beta}} \| x \|_\beta + \| f \|_{C^\sigma([0, T], E)} \right\} \forall t \in]0, T].$$

THEOREM 2.6. (Existence of the strong solution). *Under hypotheses I, II let* $x \in \overline{D(A(0))}$ *and* $f \in C([0, T], E)$. *Then the function u defined by* (F) *is the strong solution of* (P) *and belongs to* $C^\sigma(]0, T], E) \; \forall \sigma \in]0, 1[$. *In addition* $u \in C^\sigma([0, T], E)$ *(resp.* $h^\sigma([0, T], E)$), $\sigma \in]0, 1[$, *if and only if* $x \in D_{A(0)}(\sigma, \infty)$ *(resp.* $D_{A(0)}(\sigma)$).

THEOREM 2.7. (Space regularity of the strong solution). *Under hypotheses I, II let* $x \in \overline{D(A(0))}$, $f \in C([0, T], E)$ *and let u be a strong solution of* (P). *Then* $u \in C^{1-\theta}(]0, T], D_{A(0)}(\theta)) \; \forall \theta \in]0, 1[$, *and moreover*

(i) $u \in C_0(]0, T], D_{A(0)}(\theta))$ *(resp.* $C([0, T], D_{A(0)}(\theta))$), $\theta \in]0, 1[$, *if and only if* $x \in D_{A(0)}(\theta, \infty)$ *(resp.* $D_{A(0)}(\theta)$);

(ii) *if* $x \in D_{A(0)}(\theta, \infty)$ *(resp.* $D_{A(0)}(\theta)$), $\theta \in]0, 1[$, *then* $u \in C^{\theta-\beta}([0, T], D_{A(0)}(\beta))$ *(resp.* $h^{\theta-\beta}([0, T], D_{A(0)}(\beta))$) $\forall \beta \in]0, \theta[$.

Remark 2.7 Existence and uniqueness of the strict solution is still guaranteed if, instead of Hölder continuity, we suppose that $f \in C([0, T], E)$ and its oscillation $\omega(.)$ satisfies

$$\int_0^T \frac{\omega(s)}{s} \, ds < \infty . \tag{2.1}$$

This generalizes a result of Crandall-Pazy [4]. On the other hand the only assumption $f \in C([0, T], E)$ is not sufficient for existence, even if $x = 0$ and $A(t) \equiv A$ (see Baillon [2], Travis [14], Da Prato-Grisvard [7]).

Similarly, we have existence and uniqueness of the classical solution if $f \in C_\theta(]0, T], E)$ for some $\theta \in [0, 1[$ and in addition the oscillations $w_\varepsilon(\cdot)$ of $f|_{[\varepsilon, T]}$ satisfy (2.1).

Finally we note that the preceding results apply to many concrete situations; two examples will be described in our forthcoming paper.

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