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Optimal Control with State Constraint and Non-concave Dynamics: A Model Arising in Economic Growth

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Abstract We consider an optimal control problem arising in the context of economic theory of growth, on the lines of the works by Skiba and Askenazy–Le Van. The framework of the model is intertemporal infinite horizon utility maximization. The dynamics involves a state variable representing total endowment of the social planner or average capital of the representative dynasty. From the mathematical viewpoint, the main features of the model are the following: (i) the dynamics is an increasing, unbounded and not globally concave function of the state; (ii) the state variable is subject to a static constraint; (iii) the admissible controls are merely locally integrable in the right half-line. Such assumptions seem to be weaker than those appearing in most of the existing literature. We give a direct proof of the existence of an optimal control for any initial capital $k_0 \geq 0$ and we carry on a qualitative study of the value function; moreover, using dynamic programming methods, we show that the value function is a continuous viscosity solution of the associated Hamilton–Jacobi–Bellman equation.

Keywords Optimal control · Utility maximization · Non-concave production function · Hamilton Jacobi Bellman equation · Viscosity solutions

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1 Introduction

Utility maximization problems constitute a fundamental part of modern economic growth models, since the works by Ramsey [10], Romer [11], Lucas [9], Barro and Sala-i-Martin [2].

These models aim to formalize the dynamics of an economy throughout the quantitative description of the consumers' behaviour. Consumers are seen as homogeneous entities, as far as their operative decisions are concerned; hence the time series of their consuming choices, or consumption path, is represented by a single function, and they as a collective are named *social planner*, or simply *agent*.

The agent's purpose is to maximize the utility as a function of the consumption path in a fixed time interval; this can be finite or more often (as far as economic growth literature is concerned) infinite.

In order to enlarge the applicability range, we introduce in our model three main features, which however imply additional technical difficulties.

First, the dynamics contains a convex–concave function representing production. It is well known that the presence of non-concavity in an optimization problem may complicate the study of the regularity properties of the value function.

Secondly, an additional state constraint is present—which may be called "static" since it does not involve the derivative of the state variable. This makes any admissibility proof much more involved than usual.

As a third relevant feature, we require that the admissible controls are not more than locally integrable in the positive half-line: this is the maximal class if one wants the control strategy to be a function and the state equation to have a solution. This is a weak regularity requirement which is of very little help; on the other side, it generates unexpected difficulties in various respects.

From the applications viewpoint, the target of the analysis is the study of the optimal trajectories: regularity, monotonicity, asymptotic behaviour properties and similar are expected to be investigated. These properties are still not characterized in recent literature, at least in the above described case.

Hence the program is quite complex and has to be dealt with in many phases. Here we undertake the work, providing an existence result and several necessary conditions related to the Hamilton–Jacobi–Bellman problem (HJB), remembering Skiba [12] and Askenazy–Le Van [1] and developing part of the studies carried on by Fiaschi and Gozzi [6].

We can summarize the main criticalities as follows:

1. Certain questions arise, that in other bounded-control models are not even present. For instance, the finiteness of the value function and the well-posedness of the Hamiltonian problem, consisting in the question whether the value function is a viscosity solution to the HJB equation. The notion of viscosity solution can be characterized both in terms of super- and sub-differentials and of test functions; in any case these auxiliary tools must match the necessary restrictions to the domain of the Hamiltonian function, at least for the solutions we are interested in verifying. We are able to prove certain regularity properties of the value function ensuring that this is the case.

2. As far as the existence of the optimum is concerned, we did not find a general result covering our case, either in the classical literature, e.g. Fleming and Rishel [7], Cesari [4], Zabczyk [14], Yong–Zhou [13], or in more recent publications, such as the book by Carlson et al. [3] and Zaslavski's books [15,16]. So we give a direct proof of the existence of an optimal control for every fixed initial state. To this purpose, it is a natural idea to make use of the traditional compactness results, in order to generate a convergent approximation procedure. However, in presence of unbounded controls and state constraints, the application of such compactness results is not straightforward. Something more about the technical tools needed to overcome these difficulties will be said at the end of the introduction.
3. Additional work to the usual proof of the fact that the value function solves HJB is needed: we use a result which is involved in the construction of the optimum, i.e. the fundamental Lemma 3.2.
4. The regularity property stated in Theorem 6.1(ii), which is necessary in order that the HJB problem is well-posed, not only requires optimal controls. It can be proven by a standard argument under the hypothesis that the admissible controls are locally bounded; in our case it is useful to come back again to the preliminary tools (Lemmas 3.2 and 3.3) in order to prove the result with merely integrable controls.

The contents are consequently arranged. First, the reader will find a section which intends to clear up the genesis of the model and the economic motivations for the assumptions.

Then, the preliminary results that are crucial for the development of the theory are proven.

Afterwards, some basic properties of the value function are proven, such as its behaviour near the origin and near $+\infty$. These results require careful manipulations of the data and some standard results about ordinary differential equations, but do not require the existence of optimal control functions.

In the subsequent section we prove the existence of an optimal control strategy for every initial state. Here we make wide use of the preliminary lemmas in association with a special diagonal procedure which we will speak about later.

After providing the existence theorem, we are able to prove other important regularity properties of the value function (such as the Lipschitz continuity in the closed intervals of $(0, +\infty)$), using optimal controls.

Eventually we give an application of the methods of dynamic programming to our model. As mentioned before, the proof of the admissibility of the value function as a viscosity solution of HJB is made more complicated by the use of the preliminary lemmas, but it allows to obtain the result independently of the regularity of the Hamiltonian function.

Many important results about infinite horizon optimal control were obtained during the last 25 years. Several of them can be found in [3] and more recently in [15,16]. Our problem does not seem to fit into any of these results: indeed, first of all our integrand is non-autonomous, due to the presence of the factor $e^{-\rho t}$, unlike the situation described in Chapter 4 of [3]. Next, as previously remarked, we prove the existence of an optimal control (a strongly optimal control, according to the terminology of

[3]) without any compactness assumption on the set of admissible controls, whereas in [3] some compactness hypotheses both on controls and on trajectories seem to be essential in proving the existence of a weak over-taking optimal trajectory. Moreover, in order to reach this goal, many control problems in that book are "deparametrized" and transformed in a Lagrange-like problem of the calculus of variations. Although this approach is interesting, the possibility of transforming the original problem into a Lagrange one, and then going back after having found the maximum, is subject to a list of assumptions which seem to us somewhat heavy.

We note that in [3] many other properties of the optimal trajectories are described, such as asymptotics, stability and turnpike properties, but this matter is not the purpose of our work. Similarly, [15, 16] focus on turnpike properties of both discrete-time and Lagrange-type control problems, reporting several interesting results which are far from the goal of our paper.

Finally, a few words are worth spending about some mathematical tools that may possibly be considered new. For the sake of generality we do not want to assume the control space to be compact: this way, the admissible consumption strategies are allowed to be unbounded on a bounded interval. In order to use a proper compactness result to prove the existence of the optimum, we look for a procedure generating a sequence of bounded controls from a given optimizing sequence; this procedure must guarantee that the optimizing property is preserved. Due to the presence of the state constraint, another issue steps in, since the bounding procedure may not guarantee the admissibility of the new controls.

This leads to the construction of certain new mathematical tools, which may be called "uniform localization" results.

Our localization result is Lemma 3.2, where we prove that the "localized" optimizing sequence is bounded in the L^∞ norm, in every compact set $[0, T]$, *independently* of the initial optimizing sequence. Moreover, the upper bound can be explicitly defined as a function N of T and of the initial state, which is proven to be increasing in both variables. This lemma can be considered an improvement of Lemma 5.3 in [8]—which is used there to prove *necessary* conditions for optimality.

The uniform boundedness and the monotonicity of N make possible to use Lemma 3.2 in the *existence* proof. Indeed, by monotonicity we can define a diagonalization procedure which is different from the traditional one (like Ascoli–Arzela's theorem), since it deals with *two* families of sequences, say $\mathcal{A}$ and $\mathcal{B}$, with the following property: the ith sequence in $\mathcal{B}$ is extracted from the ith sequence in $\mathcal{A}$, but the $(i+1)$th sequence in $\mathcal{A}$ is not extracted from the ith sequence in $\mathcal{B}$; it is instead generated by applying the localization Lemma to the former. Here comes the monotonicity of the bound respect to time, which allows to prove that an appropriate diagonal sequence is indeed weakly convergent in the space $L^1([0, T])$ for every $T > 0$.

As a last remark, we observe that the localization Lemma turns out to be useful in order to solve a different problem, namely the study of the value function. The question whether the value function solves HJB (in some sense) is standard in the literature, but apparently has not been answered yet for this particular type of models. We reach the goal by proving property (ii) in Theorem 6.1 (ensuring the well posedness of the problem), and Theorem 7.3, and both results make use of the localization Lemma.

2 The Model

2.1 Qualitative Description

We assume the existence of a representative dynasty in which all members share the same endowments and consume the same amount of a certain good. Our goal is to describe the dynamics of the capital accumulated by each member of the dynasty in an infinite-horizon period and to maximize its intertemporal utility (considered as a function of the quantity of good c that has been consumed). Clearly, consuming is seen as the agent's control strategy, and the set of consumption functions (over time) will be a superset of the set of the admissible control strategies.

First, we need a notion of instantaneous utility, depending on the consumptions, in order to define the inter-temporal utility functional. We will assume that instantaneous utility, which we denote by u, is a strictly increasing and strictly concave function of the consumptions, and that it is twice continuously differentiable. Moreover, we will assume the usual Inada's conditions, that is to say:

$$\lim_{c\to 0^+} u'(c) = +\infty, \quad \lim_{c\to +\infty} u'(c) = 0.$$

We will also use the following assumptions on u:

$$u(0) = 0, \quad \lim_{c\to +\infty} u(c) = +\infty.$$

With this material, we can define the inter-temporal utility functional, which, as usual, must include a (exponential) discount factor expressing time preference for consumption:

$$U(c(\cdot)) := \int_0^{+\infty} e^{-\hat{\rho}t} e^{nt} u(c(t))\, \mathrm{d}t \tag{1}$$

where $\hat{\rho} \in \mathbb{R}$ is the rate of time preference and $n \in \mathbb{R}$ is the growth rate of population. The number of members of the dynasty at time zero is normalized to 1.

2.2 Production Function and Constraints

We consider the production or output, denoted by F, as a function of the average capital of the representative dynasty, which we denote by k. First, we assume the usual hypothesis of monotonicity, regularity and unboundedness about the production, that is to say: F is strictly increasing and continuously differentiable from $\mathbb{R}$ to $\mathbb{R}$, and

$$F(0) = 0, \quad \lim_{k\to +\infty} F(k) = +\infty$$

where we may assume $F(x) < 0$ for every $x \in (-\infty, 0)$, since the assumption that F is defined in $(-\infty, 0)$ is merely technical, as we will see later; this way we distinguish the "admissible" values of the production function from the ones which are not.

Next, we make some specific requirements. As we want to deal with a non-monotonic marginal product of capital, we assume that, in $[0, +\infty)$, F is first strictly concave, then strictly convex and then again strictly concave up to $+\infty$. This means that in the first phase of capital accumulation, the production shows decreasing returns to scale, which become increasing from a certain level of *pro capite* capital $\underline{k}$. Then, when *pro capite* endowment exceed a threshold $\overline{k} > \underline{k}$, decreasing returns to scale characterize the production anew.

Moreover, we ask that the marginal product in $+\infty$ is strictly positive, so that we can deal with endogenous growth. Observe that this limit surely exists, as F' is (strictly) decreasing in a neighbourhood of $+\infty$. Of course the assumption is equivalent to the fact that the average product of capital tends to a strictly positive quantity for large values of the average stock of capital. Moreover, requiring that the marginal product has a strictly positive lower bound is necessary to ensure a positive long-run growth rate.

As far as the agent's behaviour is concerned, the following constraints must be satisfied, for every time $t \geq 0$:

$$
\begin{aligned}
&k(t) \geq 0, \;\; c(t) \geq 0 \\
&i(t) + c(t) \leq F(k(t)), \;\; \dot{k}(t) = i(t)
\end{aligned}
$$

where $i(t)$ is the per capita investment at time t. Observe that the first assumption is needed in order to make the agent's optimal strategy possibly different from the case of monotonic marginal product. In fact if condition $\forall t \geq 0{:}k(t) \geq 0$ were not present, then heuristically the convex range of production function would be not relevant to establish the long-run behaviour of economy, since every agent would have the possibility to get an amount of resources such that he can fully exploit the increasing return; therefore only the form of production function for large k would be relevant.

Another heuristic remark turns out to be crucial: the monotonicity of u respect to c implies that, if c is an optimal consumption path, then the production is completely allocated between investment and consumption, that is to say $i(t) + c(t) = F(k(t))$ for every $t \geq 0$. This remark, combined with the last of the above conditions implies that the dynamics of capital allocation, for an initial endowment $k_0 \geq 0$, is described by the following Cauchy's problem:

$$
\begin{cases}
\dot{k}(t) = F(k(t)) - c(t) & \text{for } t \geq 0 \\
k(0) = k_0
\end{cases}
\tag{2}
$$

Considering the first two constraints, the agent's target can be expressed the following way: given an initial endowment of capital $k_0 \geq 0$, maximize the functional in (1), when $c(\cdot)$ varies among measurable functions which are everywhere positive in $[0, +\infty)$ and such that the unique solution to problem (2) is also everywhere positive in $[0, +\infty)$; the latter requirement is usually called a *state constraint*.

A few reflections are still necessary in order to begin the analytic work. First, we will consider only the case when the time discount rate $\hat{\rho}$ and the population growth rate n satisfy

$$\hat{\rho} - n > 0,$$

which is the most interesting from the economic point of view. Second, we weaken the requirement that c is integrable and positive in $[0, +\infty)$ (in order that c is admissible) to the requirement that c is locally integrable and almost everywhere positive in $[0, +\infty)$.

Finally, we need another assumption about instantaneous utility u so that the functional in (1) is finite. To identify the best hypothesis, we temporarily restrict our attention to the particular but significant case in which u is a concave power function and F is linear; namely:

$$u(c) = c^{1-\sigma}, \quad c \geq 0$$
$$F(k) = Lk, \quad k \geq 0$$

for some $\sigma \in (0, 1)$ and $L > 0$ (of course in this case F does not satisfy all of the previous assumptions). Using Gronwall's Lemma, it is easy to verify that for any admissible control c (starting from an initial state k_0) and for every time $t \geq 0$, $\int_0^t c(s)\,\mathrm{d}s \leq k_0 e^{Lt}$. Hence, setting $\rho = \hat{\rho} - n$:

$$\begin{aligned} U(c(\cdot)) &= \lim_{T \to +\infty} \int_0^T e^{-\rho t} u(c(t))\mathrm{d}t \\ &= \lim_{T \to +\infty} e^{-\rho T} \int_0^T u(c(s))\mathrm{d}s + \lim_{T \to +\infty} \rho \int_0^T e^{-\rho t} \int_0^t u(c(s))\mathrm{d}s\mathrm{d}t. \end{aligned}$$

Hence using Jensen inequality, we reduce the problem of the convergence of $U(c(\cdot))$ to the problem of the convergence of

$$\int_1^{+\infty} t e^{-\rho t} e^{L(1-\sigma)t} \mathrm{d}t$$

which is equivalent to the condition $L(1-\sigma) < \rho$. Perturbing this clause by the addition of a positive quantity ϵ_0 we get $(L + \epsilon_0)(1 - \sigma) < \rho - \epsilon_0$ which is in its turn equivalent to the requirement that the function $e^{\epsilon_0 t} e^{-\rho t} \left(e^{(L+\epsilon_0)t}\right)^{1-\sigma} = e^{\epsilon_0 t} e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right)$ tends to 0 as $t \to +\infty$.

Turning back to the general case, we are suggested to assume precisely the same condition, taking care of defining the constant L as $\lim_{k \to +\infty} F'(k)$ (which has already been assumed to be strictly positive).

2.3 Quantitative Description

Hence the mathematical frame of the economic problem can be defined precisely as follows:

Definition 2.1 For every $k_0 \geq 0$ and for every $c \in L^1_{loc}([0, +\infty), \mathbb{R})$:

$k(\cdot; k_0, c)$ is the only solution to the Cauchy's problem

$$\begin{cases} k(0) = k_0 \\ \dot{k}(t) = F(k(t)) - c(t), \quad t \geq 0 \end{cases} \tag{3}$$

in the unknown k, where $F : \mathbb{R} \to \mathbb{R}$ has the following properties:

$$F \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}), \; F' > 0 \text{ in } \mathbb{R}, \; F(0) = 0, \; \lim_{x \to +\infty} F(x) = +\infty, \; \lim_{x \to +\infty} F'(x) > 0,$$
$$F \text{ is concave in } \left[0, \underline{k}\right] \cup \left[\overline{k}, +\infty\right) \text{ for some } 0 < \underline{k} < \overline{k} \text{ and } F \text{ is convex over } \left[\underline{k}, \overline{k}\right]$$

Moreover, we set $L := \lim_{x \to +\infty} F'(x)$.

Definition 2.2 Let $k_0 \geq 0$.
The set of *admissible consumption strategies* with initial capital k_0 is

$$\Lambda(k_0) := \left\{ c \in L^1_{loc}([0, +\infty), \mathbb{R}) / c \geq 0 \text{ almost everywhere}, \; k(\cdot; k_0, c) \geq 0 \right\}$$

The *intertemporal utility functional* $U(\cdot; k_0) : \Lambda(k_0) \to \mathbb{R}$ is

$$U(c; k_0) := \int_0^{+\infty} e^{-\rho t} u(c(t)) \, \mathrm{d}t \;\; \forall c \in \Lambda(k_0)$$

where $\rho > 0$, and the function $u : [0, +\infty) \to \mathbb{R}$, representing instantaneous utility, is strictly increasing and strictly concave and satisfies:

$$\begin{aligned} & u \in \mathcal{C}^2((0, +\infty), \mathbb{R}) \cap \mathcal{C}^0([0, +\infty), \mathbb{R}), \; u(0) = 0, \; \lim_{x \to +\infty} u(x) = +\infty \\ & \lim_{x \to 0^+} u'(x) = +\infty, \; \lim_{x \to +\infty} u'(x) = 0 \\ & \exists \epsilon_0 > 0 : \lim_{t \to +\infty} e^{\epsilon_0 t} e^{-\rho t} u\left(e^{(L + \epsilon_0) t}\right) = 0 \end{aligned} \tag{4}$$

The *value function* $V : [0, +\infty) \to \mathbb{R}$ is

$$V(k_0) := \sup_{c \in \Lambda(k_0)} U(c; k_0) \;\; \forall k_0 \geq 0$$

Remark 2.1 The last condition in (4) implies:

$$\int_0^{+\infty} e^{-\rho t} u\left(e^{(L + \epsilon_0) t}\right) \mathrm{d}t < +\infty, \; \int_0^{+\infty} t e^{-\rho t} u\left(e^{(L + \epsilon_0) t}\right) \mathrm{d}t < +\infty.$$

3 Preliminary Results

The main result of this section is the uniform localization Lemma 3.2.

Remark 3.1 Set

$$\overline{M} := \max_{[0,+\infty)} F' = \max\left\{F'(0), F'\left(\bar{k}\right), L\right\}.$$

Recalling that F is strictly increasing with $F(0) = 0$, we see that, for any $x, y \in [0, +\infty)$:

$$\begin{aligned} |F(x) - F(y)| &\leq \overline{M}\,|x - y| \\ F(x) &\leq \overline{M}x \end{aligned}$$

In particular F is Lipschitz-continuous.

This implies that the Cauchy's problem (3) admits a unique global solution (that is to say, defined on $[0, +\infty)$)—even if the dynamics is not continuous with respect to the time variable.

Indeed the mapping

$$\mathcal{F}(k)(t) := k_0 + \int_0^t F(k(s))\,\mathrm{d}s - \int_0^t c(s)\,\mathrm{d}s$$

is a contraction on the space $X := \left(\mathcal{C}^0\left(\left[0, \frac{1}{1+\overline{M}}\right]\right), \|\cdot\|_\infty\right)$, and so admits a unique fixed point $k(\cdot; k_0, c)$. Considering the mapping

$$\mathcal{F}(k)(t) := k\left(\frac{1}{1+\overline{M}}; k_0, c\right) + \int_{\frac{1}{1+\overline{M}}}^t F(k(s))\,\mathrm{d}s - \int_{\frac{1}{1+\overline{M}}}^t c(s)\,\mathrm{d}s$$

on the space $X' := \left(\mathcal{C}^0\left(\left[\frac{1}{1+\overline{M}}, \frac{2}{1+\overline{M}}\right]\right), \|\cdot\|_\infty\right)$, one can extend the function $k(\cdot; k_0, c)$ to the interval $\left[\frac{1}{1+\overline{M}}, \frac{2}{1+\overline{M}}\right]$, and so on.

In a few words, the existence and uniqueness of the solution depends on the fact that the dynamics in Eq. (3) is defined for *every* state and is *globally* Lipschitz-continuous.

Remark 3.2 We recall that if k_1 and k_2 are two solutions of (3), then the function

$$h(t) := \begin{cases} \dfrac{F(k_1(t)) - F(k_2(t))}{k_1(t) - k_2(t)} & \text{if } k_1(t) \neq k_2(t) \\ F'(k_1(t)) & \text{if } k_1(t) = k_2(t) \end{cases}$$

is continuous in $[0, +\infty)$.

As a consequence, we have a well known comparison result, which in our case can be stated as follows:

Let $k_1, k_2 \geq 0$, $c_1, c_2 \in L^1_{loc}([0, +\infty), \mathbb{R})$, $T_0 \geq 0$ and $T_1 \in (T_0, +\infty]$ such that $c_1 \leq c_2$ almost everywhere in $[T_0, T_1]$. Then the following implications hold:

$$k(T_0; k_1, c_1) = k(T_0; k_2, c_2) \implies \forall t \in [T_0, T_1] : k(t; k_1, c_1) \geq k(t; k_2, c_2) \quad (5)$$
$$k(T_0; k_1, c_1) > k(T_0; k_2, c_2) \implies \forall t \in [T_0, T_1] : k(t; k_1, c_1) > k(t; k_2, c_2). \quad (6)$$

Lemma 3.1 *There exists a function $g : (0, +\infty) \to (0, +\infty)$ which is convex, strictly decreasing and such that*

$$g(x) \leq u'(x) \quad \forall x > 0.$$

Proof Let

$$\Sigma_{u'} := \left\{(x, y) \in (0, +\infty)^2 / y \geq u'(x)\right\}$$
$$K_{u'} := \bigcap \left\{K \in \mathcal{P}\left(\mathbb{R}^2\right) / K = \overline{K},\ K \text{ is convex},\ K \supseteq \Sigma_{u'}\right\}.$$

In particular $K_{u'}$ is a closed-convex superset of $\Sigma_{u'}$. Observe that, for any $x > 0$, the function $H_x(y) := (x, y)$ belongs to $\mathcal{C}^0\left(\mathbb{R}, \mathbb{R}^2\right)$, so any set of the form

$$\{y \geq 0 / (x, y) \in K_{u'}\} = H_x^{-1}(K_{u'}) \bigcap [0, +\infty)$$

is closed in $\mathbb{R}$, and consequently it has a minimum element. Now define

$$\forall x > 0 : g(x) := \min\{y \geq 0 / (x, y) \in K_{u'}\}.$$

(i) This definition implies that for every $(x, y) \in K_{u'}$, $g(x) \leq y$; hence

$$g(x) \leq u'(x) \quad \forall x > 0$$

because for any $x > 0$, $\left(x, u'(x)\right) \in \Sigma_{u'} \subseteq K_{u'}$.

(ii) Secondly, g is convex in $(0, +\infty)$. Let $x_0, x_1 > 0$ and $\lambda \in (0, 1)$. By definition of g, $(x_0, g(x_0)), (x_1, g(x_1)) \in K_{u'}$, which is a convex set. Hence

$$(1 - \lambda)(x_0, g(x_0)) + \lambda(x_1, g(x_1)) \in K_{u'}.$$

By the first property in (i), this implies

$$g((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda) g(x_0) + \lambda g(x_1).$$

(iii) Observe that the definition of g does not exclude that $g(x) = 0$ for some $x > 0$. Indeed we show that $g > 0$ in $(0, +\infty)$.
Fix $x > 0$, and consider the closed-convex approximation of $\Sigma_{u'}$

$$K_x := \left\{(t, y) \in [0, x] \times [0, +\infty) / y \geq \frac{u'(x)}{x}(x - t)\right\} \bigcup [x, +\infty) \times [0, +\infty).$$

By construction $K_{u'} \subseteq K_x$ which implies $(t, g(t)) \in K_x$ for any $t > 0$. In particular, for every $t \in (0, x)$:

$$g(t) \geq \frac{u'(x)}{x}(x - t) > 0$$

because $u' > 0$. Hence $g > 0$ in $(0, x)$. Since $x > 0$ is generic, we obtain $g > 0$ in $(0, +\infty)$.

(iv) Finally we show that g is strictly decreasing. Take $0 < x_0 < x_1$. By (ii) and by definition of convexity, for every $n \in \mathbb{N}$:

$$g(n(x_1 - x_0) + x_0) \geq n[g(x_1) - g(x_0)] + g(x_0).$$

Hence by the assumptions on u and by (i):

$$\begin{aligned} 0 = \lim_{n\to+\infty} u'(n(x_1 - x_0) + x_0) &\geq \limsup_{n\to+\infty} g(n(x_1 - x_0) + x_0) \\ &\geq \lim_{n\to+\infty} n[g(x_1) - g(x_0)] + g(x_0) \end{aligned}$$

which implies $g(x_1) < g(x_0)$, remembering that $g > 0$ by (iii).

□

Remark 3.3 The function h defined in Remark 3.2 satisfies

$$|h| \leq \overline{M}.$$

where $\overline{M}$ is defined as in Remark 3.1.

Remark 3.4 Let $k_0 \geq 0$ and $c \in \Lambda(k_0)$. Then, for every $t \geq 0$:

$$k(t; k_0, c) \leq k_0 e^{\overline{M}t}$$
$$\int_0^t c(s)\, \mathrm{d}s \leq k_0 e^{\overline{M}t}$$

Indeed, by Remark 3.1 and remembering that $c \geq 0$, we have, for every $t \geq 0$, $\dot{k}(t; k_0, c) \leq \overline{M}k(t; k_0, c)$—which implies by (5):

$$k(t; k_0, c) \leq k_0 e^{\overline{M}t} \quad \forall t \geq 0.$$

Now integrating both sides of the state equation, again by Remark 3.1 and by the fact that $k(\cdot; k_0, c) \geq 0$ we see that, for every $t \geq 0$:

$$\int_0^t c(s)\,\mathrm{d}s = k_0 - k(t;k_0,c) + \int_0^t F(k(s;k_0,c))\,\mathrm{d}s$$
$$\leq k_0 + \overline{M}\int_0^t k(s;k_0,c)\,\mathrm{d}s$$
$$\leq k_0 + \overline{M}k_0 \int_0^t e^{\overline{M}s}\mathrm{d}s = k_0 e^{\overline{M}t}.$$

Lemma 3.2 *There exists a function* $N : (0,+\infty)^2 \to (0,+\infty)$*, increasing in both variables, such that:*

for every $(k_0, T) \in (0,+\infty)^2$ *and every* $c \in \Lambda(k_0)$*, there exists a control function* $c^T \in \Lambda(k_0)$ *satisfying*

$$U\left(c^T;k_0\right) \geq U(c;k_0)$$
$$c^T = c \wedge N(k_0,T) \text{ almost everywhere in } [0,T]$$

In particular, c^T *is bounded above, in* $[0,T]$*, by a quantity which does not depend on the original control c, but only on T and on the initial status* k_0*.*

Proof Let g be the function defined in Lemma 3.1 and $\beta := \frac{\log(1+\overline{M})}{\overline{M}}$. Define, for every $(k_0,T) \in (0,+\infty)^2$:

$$\alpha(k_0,T) := \beta e^{-\rho(T+\beta)} g\left[k_0\left(\frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T}\right)\right]$$
$$N(k_0,T) := \inf\left\{\tilde{N} > 0 / \forall N \geq \tilde{N} : u'(N) < \alpha(k_0,T)\right\}$$
$$= \inf\left\{\tilde{N} > 0 / u'\left(\tilde{N}\right) < \alpha(k_0,T)\right\}.$$

In the first place, $N(k_0,T) \neq +\infty$, because $\alpha(k_0,T) > 0$ for every $k_0 > 0$, $T > 0$ and $\lim_{N\to+\infty} u'(N) = 0$.

In the second place, $u'((0,+\infty)) = (0,+\infty)$, which implies $N(k_0,T) > 0$: otherwise, since $\left(u'\right)^{-1}(\alpha(k_0,T)) > 0$, there would exist $N > 0$ such that

$$N < \left(u'\right)^{-1}(\alpha(k_0,T))$$
$$u'(N) < \alpha(k_0,T)$$

which is absurd because u' is decreasing; hence the quantity $u'(N(k_0,T))$ is well defined. Moreover by the continuity of u',

$$u'(N(k_0,T)) = \alpha(k_0,T). \tag{7}$$

The function $N(\cdot,\cdot)$ is also increasing in both variables, because $\alpha(\cdot,\cdot)$ is decreasing in both variables and u' is decreasing.

Indeed, for $k_0 \leq k_1$ and for a fixed $T > 0$, suppose that $N(k_1, T) < N(k_0, T)$. Then by definition of infimum we could choose $\tilde{N} \in [N(k_1, T), N(k_0, T))$ such that $u'\left(\tilde{N}\right) < \alpha(k_1, T)$, which implies

$$u'\left(\tilde{N}\right) < \alpha(k_0, T)$$

by the monotonicity of α. Since $\tilde{N} > 0$, this implies $N(k_0, T) \leq \tilde{N}$, a contradiction. With an analogous argument we prove that $N(\cdot, \cdot)$ is increasing in the second variable.

Now let $k_0, T > 0$ and $c \in \Lambda(k_0)$ as in the hypothesis. If $c \leq N(k_0, T)$ almost everywhere in $[0, T]$, then define $c^T := c$. If, on the contrary, $c > N(k_0, T)$ in a non-negligible subset of $[0, T]$, then define:

$$c^T(t) := \begin{cases} c(t) \wedge N(k_0, T) & \text{if } t \in [0, T] \\ c(t) + I_T & \text{if } t \in (T, T+\beta] \\ c(t) & \text{if } t > T + \beta \end{cases}$$

where $I_T := \int_0^T e^{-\rho t}(c(t) - c(t) \wedge N(k_0, T))\,\mathrm{d}t$. Observe that by Remark 3.4:

$$\begin{aligned} 0 < \ I_T &\leq \int_0^T (c(t) - c(t) \wedge N(k_0, T))\,\mathrm{d}t \\ &\leq \int_0^T c(t)\,\mathrm{d}t \\ &\leq k_0 e^{\overline{M}T} \end{aligned} \tag{8}$$

In order to prove the admissibility of such control function, we compare the orbit $k := k(\cdot; k_0, c)$ to the orbit $k^T := k\left(\cdot; k_0, c^T\right)$. In the first place, observe that by (5) and by definition of c^T:

$$k^T(t) \geq k(t) \quad \forall t \in [0, T] \tag{9}$$

Now by the state equation, we have:

$$\dot{k^T} - \dot{k} = F\left(k^T\right) - F(k) + c - c^T. \tag{10}$$

Set for every $t \geq 0$:

$$h(t) := \begin{cases} \frac{F\left(k^T(t)\right) - F(k(t))}{k^T(t) - k(t)} & \text{if } k^T(t) \neq k(t) \\ F'(k(t)) & \text{if } k^T(t) = k(t) \end{cases}$$

Hence by (10)

$$\dot{k^T}(t) - \dot{k}(t) = h(t)\left[k^T(t) - k(t)\right] + c(t) - c^T(t) \quad \forall t \geq 0.$$

By Remark 3.2, the function h is continuous in $[0,+\infty)$, so this is a typical linear equation with measurable coefficient of degree one, satisfied by $k^T - k$. Hence, multiplying both sides by the continuous function $t \to \exp\left(-\int_0^t h(s)\,\mathrm{d}s\right)$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[k^T(t) - k(t)\right] e^{-\int_0^t h(s)\mathrm{d}s}\right\} = \left[c(t) - c^T(t)\right] e^{-\int_0^t h(s)\mathrm{d}s} \quad \forall t \geq 0$$

which implies, integrating between 0 and any $t \geq 0$:

$$k^T(t) - k(t) = \int_0^t \left[c(s) - c^T(s)\right] e^{\int_s^t h(r)\mathrm{d}r} \tag{11}$$

Now observe that

$$h \leq \overline{M} \text{ in } [0,+\infty) \text{ and } h \geq 0 \text{ in } [0,T] \tag{12}$$

by (9) and the monotonicity of F. Set $t \in (T, T+\beta]$; then by (11) and (12):

$$\begin{aligned} k^T(t) - k(t) &= \int_0^T [c(s) - c(s) \wedge N(k_0,T)]\, e^{\int_s^t h(r)\mathrm{d}r} - I_T \cdot \int_T^t e^{\int_s^t h(r)\mathrm{d}r} \\ &\geq \int_0^T [c(s) - c(s) \wedge N(k_0,T)]\,\mathrm{d}s - I_T \cdot \int_T^t e^{\overline{M}(t-s)}\mathrm{d}s \\ &\geq \int_0^T e^{-\rho s}[c(s) - c(s) \wedge N(k_0,T)]\,\mathrm{d}s - I_T \cdot \int_T^{T+\beta} e^{\overline{M}(T+\beta-s)}\mathrm{d}s \\ &= I_T\left(1 - \frac{e^{\overline{M}\beta} - 1}{\overline{M}}\right) = 0 \end{aligned} \tag{13}$$

This also implies, by (5) and by definition of c^T,

$$k^T(t) \geq k(t) \quad \forall t \geq T + \beta$$

Such inequality, together with (9) and (13), gives us the general inequality

$$k^T(t) \geq k(t) \geq 0 \quad \forall t \geq 0.$$

This implies, associated with the obvious fact that $c^T \geq 0$ almost everywhere in $[0,+\infty)$, that $c^T \in \Lambda(k_0)$.

Now we prove the "optimality" property of c^T respect to c. By the concavity of u, and setting $N := N(k_0,T)$ for simplicity of notation, we have:

$$
\begin{aligned}
U(c;k_0) - U\left(c^T;k_0\right) &= \int_0^{+\infty} e^{-\rho t}\left[u(c(t)) - u\left(c^T(t)\right)\right]\mathrm{d}t \\
&= \int_{[0,T]\cap\{c\geq N\}} e^{-\rho t}\left[u(c(t)) - u(c(t)\wedge N)\right]\mathrm{d}t \\
&\quad + \int_T^{T+\beta} e^{-\rho t}\left[u(c(t)) - u(c(t)+I_T)\right]\mathrm{d}t \\
&\leq \int_{[0,T]\cap\{c\geq N\}} e^{-\rho t}u'(c(t)\wedge N)\left[c(t) - c(t)\wedge N\right]\mathrm{d}t \\
&\quad - I_T\int_T^{T+\beta} e^{-\rho t}u'(c(t)+I_T)\,\mathrm{d}t \\
&= u'(N)\int_0^T e^{-\rho t}\left[c(t) - c(t)\wedge N\right]\mathrm{d}t \\
&\quad - I_T\int_T^{T+\beta} e^{-\rho t}u'(c(t)+I_T)\,\mathrm{d}t \\
&= I_T\left[u'(N) - \int_T^{T+\beta} e^{-\rho t}u'(c(t)+I_T)\,\mathrm{d}t\right] \qquad (14)
\end{aligned}
$$

Now we exhibit a certain lower bound which is independent on the particular control function c. By Jensen inequality, by Lemma 3.1 and by (8), we have:

$$
\begin{aligned}
\int_T^{T+\beta} e^{-\rho t}u'(c(t)+I_T)\,\mathrm{d}t &\geq \int_T^{T+\beta} e^{-\rho t}g(c(t)+I_T)\,\mathrm{d}t \\
&\geq e^{-\rho(T+\beta)}\int_T^{T+\beta} g(c(t)+I_T)\,\mathrm{d}t \\
&\geq \beta e^{-\rho(T+\beta)}g\left(\frac{1}{\beta}\int_T^{T+\beta}\left[c(t)+I_T\right]\mathrm{d}t\right) \\
&\geq \beta e^{-\rho(T+\beta)}g\left(\frac{1}{\beta}\int_0^{T+\beta} c(t)\,\mathrm{d}t + I_T\right) \\
&\geq \beta e^{-\rho(T+\beta)}g\left[k_0\left(\frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T}\right)\right] \\
&= \alpha(k_0,T).
\end{aligned}
$$

Hence by (7) and (14):

$$
\begin{aligned}
U(c;k_0) - U\left(c^T;k_0\right) &\leq I_T\left[u'(N(k_0,T)) - \int_T^{T+\beta} e^{-\rho t}u'(c(t)+I_T)\,\mathrm{d}t\right] \\
&\leq I_T\left[u'(N(k_0,T)) - \alpha(k_0,T)\right] = 0.
\end{aligned}
$$

□

Lemma 3.3 *Let* $0 < k_0 < k_1$ *and* $c \in \Lambda(k_0)$. *Then there exists a control function* $\underline{c}^{k_1-k_0} \in \Lambda(k_1)$ *such that*

$$U\left(\underline{c}^{k_1-k_0}; k_1\right) - U(c; k_0) \geq u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t} \mathrm{d}t$$

where N *is the function defined in Lemma* 3.2.

Proof Fix k_0, k_1 and c as in the hypothesis and take $c^{k_1-k_0}$ as in Lemma 3.2 (where it is understood that $T = k_1 - k_0$).Then define:

$$\underline{c}^{k_1-k_0}(t) := \begin{cases} c^{k_1-k_0}(t) + 1 & \text{if } t \in [0, k_1 - k_0) \\ c^{k_1-k_0}(t) & \text{if } t \geq k_1 - k_0 \end{cases}$$

In the first place we prove that $\underline{c}^{k_1-k_0} \in \Lambda(k_1)$, showing that

$$\underline{k} := k\left(\cdot; k_1; \underline{c}^{k_1-k_0}\right) > k\left(\cdot; k_0, c^{k_1-k_0}\right) =: k \tag{15}$$

over $(0, +\infty)$. Suppose by contradiction that this is not true; take $\tau := \inf\left\{t > 0 / \underline{k}(t) \leq k(t)\right\}$. Then by the continuity of the orbits, $\underline{k}(\tau) \leq k(\tau)$, which implies $\tau > 0$. Considering the orbits as solutions to an integral equation we have:

$$k(\tau) = k_0 + \int_0^\tau F(k(t))\,\mathrm{d}t - \int_0^\tau c^{k_1-k_0}(t)\,\mathrm{d}t$$

$$\underline{k}(\tau) = k_1 + \int_0^\tau F\left(\underline{k}(t)\right)\mathrm{d}t - \int_0^\tau c^{k_1-k_0}(t)\,\mathrm{d}t - \min\{\tau, k_1 - k_0\}.$$

Hence

$$\begin{aligned} 0 \geq \underline{k}(\tau) - k(\tau) &= k_1 - k_0 + \int_0^\tau \left[F\left(\underline{k}(t)\right) - F(k(t))\right]\mathrm{d}t - \min\{\tau, k_1 - k_0\} \\ &\geq \int_0^\tau \left[F\left(\underline{k}(t)\right) - F(k(t))\right]\mathrm{d}t \end{aligned}$$

By the definition of τ and the strict monotonicity of F, this quantity must be strictly positive, which is absurd. Hence

$$\begin{aligned} &k\left(\cdot; k_1; \underline{c}^{k_1-k_0}\right) > k\left(\cdot; k_0, c^{k_1-k_0}\right) \geq 0 \text{ in } [0, +\infty) \\ &\underline{c}^{k_1-k_0} \geq c^{k_1-k_0} \geq 0 \text{ a.e. in } [0, +\infty) \end{aligned}$$

which implies $\underline{c}^{k_1-k_0} \in \Lambda(k_0)$.

In the second place, remembering the properties of $c^{k_1-k_0}$ given by Lemma 3.2, we have

$$\begin{aligned} U\left(\underline{c}^{k_1-k_0}; k_1\right) - U(c; k_0) &\geq U\left(\underline{c}^{k_1-k_0}; k_1\right) - U\left(c^{k_1-k_0}; k_0\right) \\ &= \int_0^{k_1-k_0} e^{-\rho t}\left[u\left(c^{k_1-k_0}(t)+1\right) - u\left(c^{k_1-k_0}(t)\right)\right]\mathrm{d}t \\ &\geq \int_0^{k_1-k_0} e^{-\rho t} u'\left(c^{k_1-k_0}(t)+1\right)\mathrm{d}t \\ &\geq u'\left(N(k_0, k_1-k_0)+1\right)\int_0^{k_1-k_0} e^{-\rho t}\mathrm{d}t \end{aligned}$$

which concludes the proof. □

Remark 3.5 In the previous Lemma, the property (15) can also be proved with the "comparison technique", like we did for the admissibility of c^T in Lemma 3.2.

More generally, it can be proved that

$$k(\cdot; k_1, c_H) > k(\cdot; k_0, c)$$

where $k_1 > k_0 \geq 0$, $c \in L^1_{loc}([0, +\infty), \mathbb{R})$, $H > 0$ and

$$c_H(t) := \begin{cases} c(t) + H & \text{if } t \in [0, \delta_H) \\ c(t) & \text{if } t \geq \delta_H \end{cases}$$

and $\delta_H > 0$ satisfies $\delta_H \cdot H \leq k_1 - k_0$.

Indeed, set $k_H := k(\cdot; k_1, c_H)$ and $k := k(\cdot; k_0, c)$ and suppose by contradiction that $-\infty < \inf\{t > 0 / k_H(t) \leq k(t)\} =: \tau$. Then for a suitable, positive continuous function $h : [0, +\infty) \to \mathbb{R}$, the following equality holds:

$$k_H(\tau) - k(\tau) = e^{\int_0^\tau h(r)\mathrm{d}r}\left[k_1 - k_0 + \int_0^\tau (c(s) - c_H(s))\, e^{-\int_0^s h(r)\mathrm{d}r}\right].$$

Moreover $\tau \leq \delta_H$, because on the contrary we would have $k_H > k$ in $[0, \delta_H]$; then remembering (6) and the definition of c_H we would conclude that $k_H > k$ everywhere in $[0, +\infty)$, which contradicts $\tau > -\infty$. Moreover $k_H(\tau) = k(\tau)$ by the continuity of k_h and k and by definition of infimum. Then the above equality implies

$$0 = k_1 - k_0 - H\int_0^\tau e^{-\int_0^s h(s)ds} > k_1 - k_0 - \tau H \geq k_1 - k_0 - \delta_H H \geq 0.$$

At the same time $k_H(\tau) \leq k(\tau)$ by the continuity of k_h and k and by definition of infimum (in fact the equality holds, again by continuity); hence we have reached the desired contradiction.

Next we establish a simple characterisation of the admissible constant controls, which will prove itself useful afterwards.

Proposition 3.1 *Let* $k_0, c \geq 0$. *Then*

(i) $k(\cdot; k_0, F(k_0)) \equiv k_0$

(ii) *the function constantly equal to c is admissible at* k_0 *(which we write* $c \in \Lambda(k_0)$*) if, and only if*

$$c \in [0, F(k_0)] .$$

In particular the null function is admissible at any initial state $k_0 \geq 0$.

Proof (i) By the uniqueness of the orbit.

(ii) ($\Longleftarrow$) In the first place, observe that $F(k_0) \in \Lambda(k_0)$, by (i). In the second place, assume $c \in [0, F(k_0))$ and set $k := k(\cdot; k_0, c)$. Hence

$$\dot{k}(0) = F(k_0) - c > 0$$

which means, by the continuity of $\dot{k}$, that we can find $\delta > 0$ such that k is strictly increasing in $[0, \delta]$. In particular $\dot{k}(\delta) = F(k(\delta)) - c > F(k_0) - c$ because F is strictly increasing too. By the fact that $\dot{k}(\delta) > 0$ we see that there exists $\hat{\delta} > \delta$ such that k is strictly increasing in $\left[0, \hat{\delta}\right]$—and so on. Hence k is strictly increasing in $[0, +\infty)$ and in particular $k \geq 0$. This shows that $c \in \Lambda(k_0)$.

($\Longrightarrow$) Suppose that $c > F(k_0)$ and set again $k := k(\cdot; k_0, c)$. Then

$$\dot{k}(0) = F(k_0) - c < 0$$

so that we can find $\delta > 0$ such that k is strictly decreasing in $[0, \delta]$, and $\dot{k}(\delta) = F(k(\delta)) - c < F(k_0) - c < 0$. Hence one can arbitrarily extend the neighbourhood of 0 in which $\dot{k}$ is strictly less than the strictly negative constant $F(k_0) - c$, which implies that

$$\lim_{t \to +\infty} k(t) = -\infty.$$

Hence k cannot be everywhere-positive and $c \notin \Lambda(k_0)$.

□

Corollary 3.1 *The set sequence* $(\Lambda(k))_{k \geq 0}$ *is strictly increasing, that is:*

$$\Lambda(k_0) \subsetneq \Lambda(k_1)$$

for every $0 \leq k_0 < k_1$.

Proof For every $c \in \Lambda(k_0)$, $k(\cdot; k_0, c) \leq k(\cdot; k_1, c)$ by (5), which implies the second orbit being positive, and so $c \in \Lambda(k_1)$.

On the other hand, by Proposition 3.1 and by the strict monotonicity of F, the constant control $\hat{c} \equiv F\left(\hat{k}\right)$ belongs to $\Lambda(k_1) \setminus \Lambda(k_0)$ for any $\hat{k} \in (k_0, k_1]$. □

4 Basic Qualitative Properties of the Value Function

4.1 Finiteness of the Value Function

Now we deal with the first problem one has to solve in order to develop the theory: the finiteness of the value function. The asymptotic properties of F' make F sub-linear: this allows us to prove certain uniform estimates (Lemma 4.1) leading to the desired result. These estimates will also reveal themselves useful both in the construction of the optimal control (as they assure the dominated convergence in a crucial step of the approximation) and in the characterization of the latter appearing in the Sect. 7.

Remark 4.1 Set $M_0, \hat{M} \geq 0$ such that:

$$\forall x \geq M_0 : F(x) \leq (L+\epsilon_0)\, x$$
$$\hat{M} := \max_{[0,M_0]} F.$$

(which is possible because $\lim_{x\to+\infty} \frac{F(x)}{x} = L$). Hence, for every $x \geq 0$:

$$F(x) \leq (L+\epsilon_0)\, x + \hat{M}$$

Remark 4.2 Since u is a concave function satisfying $u(0) = 0$, u is sub-additive in $[0,+\infty)$ and satisfies:

$$\forall x > 0 : \forall K > 1 : u(Kx) \leq Ku(x)$$

Lemma 4.1 *Let $k_0 \geq 0$. There exists a number $M(k_0) > 1$ and a continuous, strictly positive function $\psi_{k_0} : (0,+\infty) \to \mathbb{R}$ such that, for any $c \in \Lambda(k_0)$:*

$$\text{(i)} \quad \forall t \geq 0 : \int_0^t c(s)\, ds \leq tM(k_0)\left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{L+\epsilon_0}$$
$$\text{(ii)} \quad \forall t > 0 : e^{-\rho t}\int_0^t u(c(s))\, ds \leq \psi_{k_0}(t)$$
$$\text{(iii)} \quad U(c;k_0) = \rho\int_0^{+\infty} e^{-\rho t}\int_0^t u(c(s))\, ds dt.$$

Both $M(k_0)$ and ψ_{k_0} depend only on k_0 and the problem's data (in particular they don't depend on c). Moreover ψ_{k_0} satisfies

$$\lim_{t\to+\infty} \psi_{k_0}(t) = 0, \quad \int_0^{+\infty} \psi_{k_0}(t)\, dt < +\infty$$

Proof (i) Set $\kappa := k(\cdot; k_0, c)$ and $M(k_0) := 1 + \max\left\{(L+\epsilon_0)\, k_0, \hat{M}\right\}$, where $\hat{M}$ is the quantity defined in Remark 4.1. Observe that, by Remark 4.1, for every $x \geq 0$:

$$F(x) \leq (L+\epsilon_0)\, x + M(k_0).$$

Fix $t \geq 0$; by the state equation, we have for any $s \in [0, t]$

$$\kappa(s) \leq k_0 + sM(k_0) + (L + \epsilon_0) \int_0^s \kappa(\tau)\, d\tau$$

which implies by Gronwall's inequality:

$$\kappa(s) \leq [k_0 + sM(k_0)]\, e^{(L+\epsilon_0)s} \quad \forall s \in [0, t],$$

as $s \to k_0 + sM(k_0)$ is increasing. So

$$\begin{aligned}
\int_0^t (L + \epsilon_0)\,\kappa(s)\, ds &\leq k_0 (L + \epsilon_0) \int_0^t e^{(L+\epsilon_0)s} ds \\
&\quad + M(k_0)(L + \epsilon_0) \int_0^t s \cdot e^{(L+\epsilon_0)s} ds \\
&= k_0 e^{(L+\epsilon_0)t} - k_0 + tM(k_0)\, e^{(L+\epsilon_0)t} \\
&\quad - \frac{M(k_0)}{(L + \epsilon_0)} e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L + \epsilon_0)} \\
&= tM(k_0)\, e^{(L+\epsilon_0)t} + \left[k_0 - \frac{M(k_0)}{(L + \epsilon_0)}\right] e^{(L+\epsilon_0)t} \\
&\quad + \frac{M(k_0)}{(L + \epsilon_0)} - k_0 \\
&\leq tM(k_0)\, e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L + \epsilon_0)} - k_0
\end{aligned}$$

Hence, again by the state equation, for every $t \geq 0$:

$$\begin{aligned}
\int_0^t c(s)\, ds &= k_0 - \kappa(t) + \int_0^t F(\kappa(s))\, ds \\
&\leq k_0 + tM(k_0) + \int_0^t (L + \epsilon_0)\,\kappa(s)\, ds \;\leq\; tM(k_0)\left[1 + e^{(L+\epsilon_0)t}\right] \\
&\quad + \frac{M(k_0)}{(L + \epsilon_0)}.
\end{aligned}$$

which proves the first assertion.

(ii) In the second place, it follows by Jensen inequality, the monotonicity of u and Remark 4.2, that for every $t \geq 0$:

$$\begin{aligned}
0 \leq e^{-\rho t} \int_0^t u(c(s))\, ds \;&\leq\; te^{-\rho t} u\left(\frac{\int_0^t c(s)\, ds}{t}\right) \leq te^{-\rho t} u \\
&\quad \times \left(M(k_0)\left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{t(L + \epsilon_0)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \quad te^{-\rho t}\left\{u\left(M\left(k_0\right)\right)+M\left(k_0\right)u\left(e^{(L+\epsilon_0)t}\right)\right.\\
&\quad\left.+u\left(\frac{M\left(k_0\right)}{t\left(L+\epsilon_0\right)}\right)\right\}\\
&=:\psi_{k_0}\left(t\right).
\end{aligned}
$$

This proves the inequality in (ii); from the last assumption on u in (4) we deduce that

$$
\lim_{t\to+\infty}\psi_{k_0}\left(t\right)=0.
$$

Hence $\lim_{T\to\infty}e^{-\rho T}\quad\int_0^T u\left(c\left(s\right)\right)\mathrm{d}s=0$ and this implies the identity in (iii) by a simple integration by parts.
It remains to be proven that $\psi_{k_0}\in L^1\left([0,+\infty)\right)$. We have:

$$
\begin{aligned}
\int_0^{+\infty}\psi_{k_0}\left(t\right)\mathrm{d}t &= \int_0^{+\infty} te^{-\rho t}\left\{u\left(M\left(k_0\right)\right)+M\left(k_0\right)u\left(e^{(L+\epsilon_0)t}\right)\right.\\
&\quad\left.+u\left(\frac{M\left(k_0\right)}{t\left(L+\epsilon_0\right)}\right)\right\}\mathrm{d}t\\
&\le u\left(M\left(k_0\right)\right)\int_0^{+\infty} te^{-\rho t}\mathrm{d}t+M\left(k_0\right)\int_0^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t\\
&\quad+u\left(\frac{M\left(k_0\right)}{L+\epsilon_0}\right)\left\{\int_0^1 e^{-\rho t}\mathrm{d}t+\int_1^{+\infty} te^{-\rho t}\mathrm{d}t\right\}.
\end{aligned}
$$

This estimate follows again by the monotonicity of u and the concavity properties stated in Remark 4.2. By Remark 2.1 the upper bound is finite.

□

So we have established the starting point of the theory.

Corollary 4.1 *The value function* $V:[0,+\infty)\to\mathbb{R}$ *is well defined; that is, for every* $k_0\ge 0$, $V\left(k_0\right)<+\infty$.

Proof By Lemma 4.1 we have:

$$
V\left(k_0\right)=\sup_{c\in\Lambda(k_0)}U\left(c;k_0\right)\le\rho\int_0^{+\infty}\psi_{k_0}\left(t\right)\mathrm{d}t<+\infty.
$$

□

4.2 Asymptotic Behaviour of the Value Function

These properties are called “basic” because they don’t rely on optimal controls. Nevertheless, their proof is not straightforward.

Theorem 4.1 *The value function* $V : [0, +\infty) \to \mathbb{R}$ *satisfies:*

$$\text{(i)} \quad \lim_{k \to +\infty} V(k) = +\infty$$
$$\text{(ii)} \quad \lim_{k \to +\infty} \frac{V(k)}{k} = 0$$
$$\text{(iii)} \quad \lim_{k \to 0} V(k) = V(0) = 0$$

Proof (i) For every $k_0 \geq 0$ the constant control $F(k_0)$ is admissible at k_0 by Proposition 3.1; hence

$$V(k_0) \geq U(F(k_0); k_0) = \frac{u(F(k_0))}{\rho} \to +\infty$$

as $k_0 \to +\infty$, by the assumptions on u and F.

(ii) Set $\hat{M} > 0$ as in Remark 4.1 and $k_0 > 0$ such that:

$$k_0 > \frac{1}{L + \epsilon_0} \hat{M} \tag{16}$$

Hence, for every $x > 0$:

$$F(x) \leq (L + \epsilon_0)(x + k_0) \tag{17}$$

By reasons that will be clear later, suppose also that:

$$k_0 > \frac{1}{L + \epsilon_0} \tag{18}$$

Observe that the proof of Lemma 4.1, (i) only requires $M(k_0) \geq \hat{M}, k_0(L + \epsilon_0)$; hence (16) and (17) imply that this property holds for $M(k_0) = k_0(L + \epsilon_0)$—which means that:

$$\forall t \geq 0 : \int_0^t c(s)\,\mathrm{d}s \leq k_0 + t k_0 (L + \epsilon_0)\left[1 + e^{(L+\epsilon_0)t}\right]. \tag{19}$$

In particular

$$\forall t \geq 1 : \frac{\int_0^t c(s)\,\mathrm{d}s}{t} \leq k_0 + k_0(L + \epsilon_0) + k_0(L + \epsilon_0)\, e^{(L+\epsilon_0)t}. \tag{20}$$

Now set

$$J_c(\alpha, \beta) := \int_\alpha^\beta t e^{-\rho t} u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right) \mathrm{d}t \tag{21}$$

and fix $N > 0$.

We provide three different estimates, over $J_c(0,1)$, $J_c(1,N)$ and $J_c(N,+\infty)$, using Remark 4.2.
First, we have by (19):

$$
\begin{aligned}
J_c(0,1) &\leq \int_0^1 te^{-\rho t}\frac{1}{t}u\left(\int_0^1 c(s)\,\mathrm{d}s\right)\mathrm{d}t\\
&\leq u\left[k_0\left(1+(L+\epsilon_0)\left(1+e^{(L+\epsilon_0)}\right)\right)\right]\frac{1-e^{-\rho}}{\rho}\\
&\leq u(k_0)\frac{1-e^{-\rho}}{\rho}\left[1+(L+\epsilon_0)\left(1+e^{(L+\epsilon_0)}\right)\right].
\end{aligned}
$$

Moreover, by (20):

$$
\begin{aligned}
J_c(1,N) &\leq \int_1^N te^{-\rho t}u\left(k_0+k_0(L+\epsilon_0)+k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}\right)\mathrm{d}t\\
&\leq u(k_0+k_0(L+\epsilon_0))\int_1^N te^{-\rho t}\mathrm{d}t\\
&\quad+u(k_0(L+\epsilon_0))\int_1^N te^{-\rho t}e^{(L+\epsilon_0)t}\mathrm{d}t\\
&\leq u[k_0(1+L+\epsilon_0)]\left(1+e^{(L+\epsilon_0)N}\right)\int_1^N te^{-\rho t}\mathrm{d}t
\end{aligned}
$$

Finally, remembering that $k_0(L+\epsilon_0)>1$ by (18),

$$
\begin{aligned}
J_c(N,+\infty) &\leq \int_N^{+\infty} te^{-\rho t}u\left(k_0+k_0(L+\epsilon_0)+k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}\right)\mathrm{d}t\\
&\leq u(k_0+k_0(L+\epsilon_0))\int_N^{+\infty} te^{-\rho t}\mathrm{d}t+k_0(L+\epsilon_0)\\
&\quad\times\int_N^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t
\end{aligned}
$$

Now we show that

$$
\lim_{k\to+\infty}\frac{V(k)}{k}=0.
$$

Fix $\eta>0$; by Remark 2.1, we can choose $N_\eta>0$ such that

$$
(L+\epsilon_0)\int_{N_\eta}^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t<\eta.
$$

Hence for k_0 satisfying:

$$k_0 > \max\left\{\frac{1}{L+\epsilon_0}\hat{M}, \frac{1}{L+\epsilon_0}\right\}$$

and for *every* $c \in \Lambda(k_0)$, the above estimates imply:

$$\begin{aligned} U(c; k_0) &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s))\, \mathrm{d}s \mathrm{d}t \\ &\le \rho J_c(0,1) + \rho J_c\left(1, N_\eta\right) + \rho J_c\left(N_\eta, +\infty\right) \\ &\le u(k_0)\left(1 - e^{-\rho}\right)\left[1 + (L+\epsilon_0)\left(e^{(L+\epsilon_0)} + 1\right)\right] \\ &\quad + u(k_0)(1 + L + \epsilon_0)\left(1 + e^{(L+\epsilon_0)N_\eta}\right)\int_1^{N_\eta} t e^{-\rho t}\mathrm{d}t \\ &\quad + u(k_0)(1 + L + \epsilon_0)\int_{N_\eta}^{+\infty} t e^{-\rho t}\mathrm{d}t + k_0\eta \end{aligned} \tag{22}$$

following Remark 4.2, Lemma 4.1, (iii), (21) and Jensen inequality. Now observe that:

$$\lim_{k_0 \to +\infty} \frac{u(k_0)}{k_0} = \lim_{k_0 \to +\infty} u'(k_0) = 0.$$

Hence for k_0 sufficiently large (say $k_0 > k^*$):

$$\begin{aligned} \frac{u(k_0)}{k_0} < \eta \Bigg\{ &\left(1 - e^{-\rho}\right)\left[1 + (L+\epsilon_0)\left(e^{(L+\epsilon_0)} + 1\right)\right] \\ &+ (1 + L + \epsilon_0)\left(1 + e^{(L+\epsilon_0)N_\eta}\right)\int_1^{N_\eta} t e^{-\rho t}\mathrm{d}t + (1 + L + \epsilon_0) \\ &\times \int_{N_\eta}^{+\infty} t e^{-\rho t}\mathrm{d}t \Bigg\}^{-1} \end{aligned}$$

Observe that this is possible because the expression into the brackets does not depend on k_0. In fact, like N_η, it depends only on η and on the problem's data L, ϵ_0, ρ—and so does k^*.
By (22), this implies for *every* $c \in \Lambda(k_0)$:

$$U(c; k_0) \le 2k_0\eta$$

which gives, taking the sup over $\Lambda(k_0)$:

$$V(k_0) \le 2k_0\eta.$$

Hence the assertion is proven, because the previous inequality holds for every

$$k_0 > \max\left\{\frac{1}{L+\epsilon_0}\hat{M}, \frac{1}{L+\epsilon_0}, k^*\right\},$$

and the last quantity is a threshold depending only on η and on the problem's data.

(iii) In the first place, we prove that

$$V(0) = 0.$$

Let $c \in \Lambda(0)$; by definition, $c \geq 0$ so that

$$\forall t \geq 0 : \dot{k}(t; 0, c) \leq F(k(t; 0, c)).$$

Observe that F is precisely the function which defines the dynamics of $k(\cdot; 0, 0)$, hence by (5):

$$\forall t \geq 0 : k(t; 0, c) \leq k(t; 0, 0) = 0$$

where the last equality holds by Lemma 3.1, (i).
Hence $k(\cdot; 0, c) \equiv 0$ which together with $F(0) = 0$ implies $c \equiv 0$. So $\Lambda(0) = \{0\}$, which implies

$$V(0) = U(0; 0) = \int_0^{+\infty} e^{-\rho t} u(0)\, \mathrm{d}t = 0$$

Now we show that

$$\lim_{k \to 0} V(k) = 0.$$

In this case we have to study the behaviour of $V(k_0)$ when $k_0 \to 0$, so we use the sublinearity of $F(x)$ for $x \to +\infty$ and the concavity of F near 0.
As a first step, we construct a linear function which is always above F with these two tools. Indeed we show that there is $m > 0$ such that the function

$$G(x) := \begin{cases} mx & \text{if } x \in [0, \bar{k}] \\ (L+\epsilon_0)(x-\bar{k}) + m\bar{k} & \text{if } x \geq \bar{k} \end{cases}$$

satisfies

$$\forall x \geq 0 : F(x) \leq G(x). \tag{23}$$

If $F'(\bar{k}) \leq L+\epsilon_0$ then it is enough to choose $m > \max\left\{F'(0), F'(\bar{k}), \frac{F(\bar{k})}{\bar{k}}\right\}$.

If $F'\left(\bar{k}\right) > L + \epsilon_0$ then take $\bar{x} > \bar{k}$ such that $F' \leq L + \epsilon_0$ in $(\bar{x}, +\infty)$; a first-order development in $\bar{x}$ with Lagrange remainder shows that

$$\forall x > \bar{k} : \; F(x) < F(\bar{x}) + (L + \epsilon_0)\left(x - \bar{k}\right) + \max_{\left[\bar{k}, \bar{x}\right]} F.$$

Hence it is enough to choose $m > \max\left\{F'(0), F'\left(\bar{k}\right), \frac{F(\bar{x})+M}{\bar{k}}\right\}$ (where $M = \max_{\left[\bar{k}, \bar{x}\right]} F$) in order that condition (23) is satisfied.
Observe that condition $m > F'\left(\bar{k}\right)$ is still necessary to ensure that $mx > F(x)$ for $x \in \left[\underline{k}, \bar{k}\right]$ (Lagrange's theorem proves that it is sufficient).
Suppose also, for reasons that will be clear later, that

$$m > 1. \tag{24}$$

Now take $k_0 > 0$, $c \in \Lambda(k_0)$ and consider the function $h : [0, +\infty) \to \mathbb{R}$ which is the unique solution to the Cauchy's problem

$$\begin{cases} h(0) = k_0 \\ \dot{h}(t) = G(h(t)) \quad t \geq 0 \end{cases}$$

Hence, by (23) and (5), $k := k(\cdot; k_0, c) \leq h$. So, setting

$$\bar{t} := \frac{1}{m} \log\left(\frac{\bar{k}}{k_0}\right) \text{ and } \hat{k} := \bar{k}(m - L - \epsilon_0)$$

we get, for every $t \in \left[0, \bar{t}\right]$:

$$h(t) = k_0 e^{mt}$$

and, for every $t \geq \bar{t}$:

$$\begin{aligned} h(t) &= e^{(L+\epsilon_0)t} \int_{\bar{t}}^{t} e^{-(L+\epsilon_0)s} \hat{k} \mathrm{d}s + \bar{k} e^{-(L+\epsilon_0)\bar{t}} \\ &= \frac{\hat{k} e^{-(L+\epsilon_0)\bar{t}}}{L + \epsilon_0} e^{(L+\epsilon_0)t} + \bar{k} e^{-(L+\epsilon_0)\bar{t}} - \frac{\hat{k}}{L + \epsilon_0} \\ &=: \omega_0(k_0) e^{(L+\epsilon_0)t} + \omega_1(k_0) - \frac{\hat{k}}{L + \epsilon_0} \end{aligned}$$

where by definition of $\bar{t}$ the functions ω_i satisfy:

$$\omega_0(k_0) = \frac{\hat{k}}{L+\epsilon_0} e^{-(L+\epsilon_0)\bar{t}} = \frac{\hat{k}}{L+\epsilon_0}\left(\frac{k_0}{\bar{k}}\right)^{\frac{L+\epsilon_0}{m}}$$

$$\omega_1(k_0) = \bar{k} e^{-(L+\epsilon_0)\bar{t}} = \bar{k}\left(\frac{k_0}{\bar{k}}\right)^{\frac{L+\epsilon_0}{m}}.$$

Using the state equation, we deduce by the above computations of h two estimates for the integrals of c.
For every $t \in [0, \bar{t}]$ (remembering that h is increasing so that $\forall s \leq t : h(s) \leq \bar{k}$):

$$\int_0^t c(s)\,\mathrm{d}s \leq k_0 + \int_0^t F(k(s))\,\mathrm{d}s \leq k_0 + \int_0^t G(h(s))\,\mathrm{d}s = k_0 + \int_0^t m k_0 e^{ms}\,\mathrm{d}s = k_0 e^{mt}. \tag{25}$$

Instead, for every $t > \bar{t}$:

$$\begin{aligned}
\int_0^t c(s)\,\mathrm{d}s &\leq k_0 + \int_0^{\bar{t}} G(h(s))\,\mathrm{d}s + \int_{\bar{t}}^t G(h(s))\,\mathrm{d}s \\
&\leq k_0 e^{m\bar{t}} + \int_{\bar{t}}^t \left\{(L+\epsilon_0) h(s) + \hat{k}\right\} \mathrm{d}s \\
&\leq \bar{k} + (t-\bar{t})\,\hat{k} + (L+\epsilon_0) \\
&\quad \times \int_{\bar{t}}^t \left\{\omega_0(k_0) e^{(L+\epsilon_0)s} + \omega_1(k_0) - \frac{\hat{k}}{L+\epsilon_0}\right\} \mathrm{d}s \\
&\leq \bar{k} + \omega_0(k_0)\left[e^{(L+\epsilon_0)t} - e^{(L+\epsilon_0)\bar{t}}\right] + (L+\epsilon_0)(t-\bar{t})\,\omega_1(k_0) \\
&\leq \bar{k} + \omega_0(k_0) e^{(L+\epsilon_0)t} - \frac{\hat{k}}{L+\epsilon_0} + (L+\epsilon_0)(t-\bar{t})\,\omega_1(k_0)
\end{aligned} \tag{26}$$

where we have used $h(s) \geq \bar{k}$ for $s \in (\bar{t}, t)$ and the fact that $k_0 e^{m\bar{t}} = \bar{k}$.
Now observe that

$$\lim_{k_0 \to 0} \omega_0(k_0) = \lim_{k_0 \to 0} \omega_1(k_0) = 0$$

$$\lim_{k_0 \to 0} \bar{t} = \lim_{k_0 \to 0} \frac{1}{m} \log\left(\frac{\bar{k}}{k_0}\right) = +\infty. \tag{27}$$

Hence if k_0 is small enough (say $k_0 < k^*$), we may assume $\bar{t} > 1$ and $\omega_i(k_0) \leq 1$ for $i = 0, 1$, so that (26) implies, for every $t > \bar{t}$:

$$\frac{\int_0^t c(s)\,\mathrm{d}s}{t} \le \bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0)\frac{(t-\bar{t})}{t} \le \bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0) \tag{28}$$

Hence, by Lemma 4.1, (iii), by Remark 4.2, and by (25), (28), the following inequality holds for every $k_0 < k^*$ and every $c \in \Lambda(k_0)$:

$$\begin{aligned}
0 \le U(c; k_0) = &\rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s))\,\mathrm{d}s\mathrm{d}t \\
&\le \rho \int_0^{+\infty} t e^{-\rho t} u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right)\mathrm{d}t \\
&\le \rho \int_0^1 e^{-\rho t} u\left(\int_0^t c(s)\,\mathrm{d}s\right)\mathrm{d}t + \rho \int_1^{\bar{t}} t e^{-\rho t} u\left(\frac{k_0 e^{mt}}{t}\right)\mathrm{d}t \\
&\quad + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(\bar{k} + e^{(L+\epsilon_0)t} + (L+\epsilon_0)\right)\mathrm{d}t \\
&\le \rho \int_0^1 e^{-\rho t} u\left(k_0 e^{mt}\right)\mathrm{d}t + \rho u\left(\frac{k_0 e^{m\bar{t}}}{\bar{t}}\right)\int_1^{\bar{t}} t e^{-\rho t}\mathrm{d}t \\
&\quad + \rho u\left(\bar{k} + (L+\epsilon_0)\right)\int_{\bar{t}}^{+\infty} t e^{-\rho t}\mathrm{d}t \\
&\quad + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\le \rho u\left(k_0 e^{m}\right)\int_0^1 e^{-\rho t}\mathrm{d}t + \rho u\left(\frac{\bar{k}}{\bar{t}}\right)\frac{e^{-\rho}(1+\rho)}{\rho^2} \\
&\quad + \rho u\left(\bar{k} + (L+\epsilon_0)\right)\int_{\bar{t}}^{+\infty} t e^{-\rho t}\mathrm{d}t \\
&\quad + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t
\end{aligned}$$

where we used also the fact that the function $t \to \frac{e^{mt}}{t}$ is increasing for $t > 1$, by condition (24).

It follows from (27) and the fact that $\lim_{x\to 0} u(x) = 0$, together with Remark 2.1, that the above quantity tends to 0 as $k_0 \to 0$; moreover, that quantity does not depend on c.

Hence, noticing that k^* depends only on the data and m, we see that for any $\epsilon > 0$ there exists $\delta \in (0, k^*]$ such that for every $k_0 \in (0, \delta)$ and for *every* $c \in \Lambda(k_0)$:

$$U(c; k_0) \le \epsilon,$$

which implies, taking the sup over $\Lambda(k_0)$, that $V(k_0) \le \epsilon$ - and the assertion follows.

□

5 Existence of the Optimal Control

In this section we deal with a fundamental topic of any optimization problem: the existence of an optimal control. For any fixed $k_0 \geq 0$, we look for a control $c^* \in \Lambda(k_0)$ satisfying

$$U\left(c^*; k_0\right) = \sup_{c \in \Lambda(k_0)} U(c; k_0) = V(k_0).$$

We preliminary observe that the peculiar features of our problem, particularly the absence of any boundedness conditions on the admissible controls, force us to make use of this result in proving certain regularity and monotonicity properties of the value function which usually do not require such a settlement—and which we postpone for this reason.

First observe that by Theorem 4.1, (iii) if we set $c_0 :\equiv 0$, then $U(c_0, 0) = 0 = V(0)$; hence c_0 is optimal at 0.

Let $k_0 > 0$; this will be the initial state which we will refer to during the whole section—hence the meaning of this symbol will not change in this context.

We split the construction in various steps; first we make a simple but important

Remark 5.1 Suppose that $(f_n)_{n \in \mathbb{N}}, f$ belong to $L^1_{loc}([0, +\infty), \mathbb{R})$, and are such that for *every* $N \in \mathbb{N}$, $f_n \rightharpoonup f$ in $L^1([0, N], \mathbb{R})$. If $T > 0$, $T \in \mathbb{R}$, then it follows from the definition of weak convergence that, for $g \in L^\infty([0, T], \mathbb{R})$:

$$\int_0^T g(s) f_n(s)\, ds = \int_0^{[T]+1} \chi_{[0,T]} g(s) f_n(s)\, ds \to \int_0^{[T]+1} \chi_{[0,T]} g(s) f(s)\, ds$$
$$= \int_0^T g(s) f(s)\, ds.$$

Hence $f_n \rightharpoonup f$ in $L^1([0, T], \mathbb{R})$, for every $T > 0$, $T \in \mathbb{R}$.

Step 1 The first step is to find a maximizing sequence of controls which are admissible at k_0 and a function $\gamma \in L^1_{loc}([0, +\infty), \mathbb{R})$, such that the sequence weakly converges to γ in $L^1([0, T], \mathbb{R})$, for every $T > 0$.

By definition of supremum, we can find a maximizing sequence; that is to say, there exist a sequence $(c_n)_{n \in \mathbb{N}} \subseteq \Lambda(k_0)$ of admissible controls satisfying:

$$\lim_{n \to +\infty} U(c_n; k_0) = V(k_0).$$

In order to apply the tools we set up at the beginning of the chapter, we need the following result.

Lemma 5.1 *Let $T \in \mathbb{N}$ and $(f_n)_{n \in \mathbb{N}} \subseteq L^1_{loc}([0, +\infty), \mathbb{R})$, $M(T) > 0$ such that*

$$\forall n \in \mathbb{N} : \|f_n\|_{\infty, [0,T]} \leq M(T).$$

Then there exist a subsequence $\left(\overline{f}_n\right)_{n\in\mathbb{N}}$ *of* $(f_n)_{n\in\mathbb{N}}$ *and a function* $f \in L^1([0,T],\mathbb{R})$ *such that*

$$\overline{f}_n \rightharpoonup f \textit{ in } L^1([0,T],\mathbb{R}).$$

Proof For every $0 \le t_0 < t_1 \le T$:

$$\int_{t_0}^{t_1} |f_n(s)|\,\mathrm{d}s \le \|f_n\|_{\infty,[0,T]} \cdot (t_1 - t_0) \le M(T) \cdot (t_1 - t_0).$$

Hence, by the fact that the family $\{(t_0, t_1) \in \mathcal{P}([0,T]) / t_0, t_1 \in [0,T]\}$ generates the Borel σ-algebra in $[0,T]$, and by the regularity property of the Lebesgue measure, it is easy to verify that the latter relation holds for every measurable set $E \subseteq [0,T]$; that is to say

$$\int_E |f_n(s)|\,\mathrm{d}s \le M(T) \cdot \mu(E).$$

This implies easily that the densities $\{d_n / n \in \mathbb{N}\}$ given by $d_n(E) := \int_E f_n(s)\,\mathrm{d}s$ are absolutely equicontinuous. So the thesis follows from the Dunford–Pettis criterion. Observe that the third condition required by such theorem, that is to say, for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subseteq [0,T]$ such that

$$\forall n \in \mathbb{N} : \int_{[0,T]\setminus K_\epsilon} f_n(s)\,\mathrm{d}s \le \epsilon$$

is obviously satisfied. □

Now we apply Lemma 3.2 to $(c_n)_{n\in\mathbb{N}}$ in order to find a new sequence $\left(c_n^1\right)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ such that, for every $n \in \mathbb{N}$:

$$U\left(c_n^1; k_0\right) \ge U(c_n; k_0)$$
$$c_n^1 = c_n \wedge N(k_0, 1) \text{ a.e. in } [0,1].$$

In particular $\left(c_n^1\right)_{n\in\mathbb{N}} \subseteq L^1_{loc}([0,+\infty),\mathbb{R})$ and $\left\|c_n^1\right\|_{\infty,[0,1]} \le N(k_0, 1)$ for every $n \in \mathbb{N}$. Hence by Lemma 5.1, there exists a sequence $\left(\overline{c}_n^1\right)_{n\in\mathbb{N}}$ extracted from $\left(c_n^1\right)_{n\in\mathbb{N}}$ and a function $c^1 \in L^1([0,1],\mathbb{R})$ such that

$$\overline{c}_n^1 \rightharpoonup c^1 \text{ in } L^1([0,1],\mathbb{R}).$$

Now define, for every $n \in \mathbb{N}$:

$$c_n^2 := \left(\overline{c}_n^1\right)^2$$

where $\left(\overline{c}_n^1\right)^2$ is understood with the notation of Lemma 3.2.

Hence for every $n \in \mathbb{N}$:

$$\begin{aligned} &U\left(c_n^2; k_0\right) \geq U\left(\overline{c}_n^1; k_0\right) \\ &c_n^2 = \overline{c}_n^1 \wedge N\left(k_0, 2\right) \text{ a.e. in } [0, 2]. \end{aligned}$$

Again by Lemma 5.1, we can exhibit a subsequence $\left(\overline{c}_n^2\right)_{n\in\mathbb{N}}$ of $\left(c_n^2\right)_{n\in\mathbb{N}}$ and a function $c^2 \in L^1\left([0,2], \mathbb{R}\right)$ such that

$$\overline{c}_n^2 \rightharpoonup c^2 \text{ in } L^1\left([0,2], \mathbb{R}\right).$$

Following this pattern we are able to give a recursive definition of a family $\left\{\left(\left(c_n^T\right)_{n\in\mathbb{N}}, \left(\overline{c}_n^T\right)_{n\in\mathbb{N}}, c^T\right) / T \in \mathbb{N}\right\}$ and a family of sequences of indices $\left\{\sigma_T\left(\cdot\right) : \mathbb{N} \to \mathbb{N} / T \in \mathbb{N}\right\}$ satisfying, for every $T, n \in \mathbb{N}$:

$$\begin{aligned} &\sigma_T\left(\cdot\right) \text{ is strictly increasing and } \sigma_T\left(n\right) \geq n \\ &c_n^T \in \Lambda\left(k_0\right), \; \overline{c}_n^T = c_{\sigma_T(n)}^T \\ &U\left(c_n^{T+1}; k_0\right) \geq U\left(\overline{c}_n^T; k_0\right) \\ &c_n^{T+1} = \overline{c}_n^T \wedge N\left(k_0, T+1\right) \text{ a.e. in } [0, T+1] \\ &\overline{c}_n^T \rightharpoonup c^T \text{ in } L^1\left([0,T], \mathbb{R}\right) \end{aligned} \tag{29}$$

Now fix $T \in \mathbb{N}$. The above relations clearly imply that for every $n \in \mathbb{N}$ there exist sets $U_n^T, V_n^T \subseteq [0, T]$ such that $\mu\left([0,T] \setminus U_n^T\right) = \mu\left([0,T] \setminus V_n^T\right) = 0$ and

$$\begin{aligned} &\overline{c}_n^{T+1} = c_{\sigma_{T+1}(n)}^{T+1} = \overline{c}_{\sigma_{T+1}(n)}^T \wedge N\left(k_0, T+1\right) \text{ in } U_n^T \\ &\overline{c}_{\sigma_{T+1}(n)}^T = c_{\sigma_T \circ \sigma_{T+1}(n)}^T \leq N\left(k_0, T\right) \text{ in } V_n^T \end{aligned}$$

By the monotonicity of the function $N\left(\cdot, \cdot\right)$ in the second variable (Lemma 3.2) we obtain

$$\overline{c}_n^{T+1} = \overline{c}_{\sigma_{T+1}(n)}^T \quad \text{in} W_n^T := U_n^T \cap V_n^T. \tag{30}$$

Hence $\left(\overline{c}_n^{T+1}\right)_n$ coincides, as a sequence, with $\left(\overline{c}_{\sigma_{T+1}(n)}^T\right)_n$ in $\bigcap_n W_n^T$—that is to say almost everywhere in $[0, T]$. By the properties of σ_{T+1} in (29), the latter is a subsequence of $\left(\overline{c}_n^T\right)_n$. By the essential uniqueness of the weak limit in $L^1\left([0,T]\right)$ we have:

$$c^{T+1} = c^T \text{ almost everywhere in } [0, T]. \tag{31}$$

It remains to be constructed a maximizing sequence $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ and a function $\gamma \in L^1_{loc}([0,+\infty),\mathbb{R})$ such that

$$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0.$$

Definition 5.1 (i) $\gamma : [0,+\infty) \to \mathbb{R}$ is the function

$$\gamma(t) := c^{[t]+1}(t) \quad \forall t \geq 0$$

(ii) $\forall n \in \mathbb{N}$: $\gamma_n := \overline{c}^n_n$.

Now, if we consider (for any fixed $T \in \mathbb{N}$), the restriction to $[0,T]$ of the sequence $\gamma_T, \gamma_{T+1}, \gamma_{T+2}, \ldots$ we see that there exists a subset of $[0,T]$, with negligible complementary, in which such sequence coincides with a subsequence of $\left(\overline{c}^T_n\right)_n$. Indeed, by computations similar to those carried out after Remark 30 we find that:

$$\begin{aligned}
&\gamma_T = \overline{c}^T_T \\
&\gamma_{T+1} = \overline{c}^T_{\sigma_{T+1}(T+1)} \quad \text{in } W^T_{T+1} \\
&\gamma_{T+2} = \overline{c}^T_{\sigma_{T+1}\circ\sigma_{T+2}(T+2)} \quad \text{in } W^{T+1}_{T+2} \cap W^T_{\sigma_{T+2}(T+2)} \cap [0,T] \\
&\ldots
\end{aligned}$$

Any of these sets almost coincides with $[0,T]$ (and so does the intersection); moreover by the properties of the σ_n's we have $T < \sigma_{T+1}(T+1) < \sigma_{T+1}\circ\sigma_{T+2}(T+2)$.

Now we can state the following

Proposition 5.1 *Let $(\gamma_n)_{n\in\mathbb{N}}$, γ as in Definition 5.1. Then we have $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$, $\gamma \in L^1_{loc}([0,+\infty),\mathbb{R})$ and*

$$\lim_{n\to+\infty} U(\gamma_n; k_0) = V(k_0).$$

Moreover, for every $T \in \mathbb{N}$, $(\gamma_n)_{n\geq T}$, as a sequence, coincides almost everywhere in $[0,T]$ with a subsequence of $\left(\overline{c}^T_n\right)_{n\in\mathbb{N}}$. Consequently

$$\begin{aligned}
&\|\gamma_n\|_{\infty,[0,T]} \leq N(k_0,T) \quad \forall T, n \in \mathbb{N},\ n \geq T, \\
&\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0, T \in \mathbb{R}.
\end{aligned}$$

Proof By Definition 5.1 and by the second condition in (29), $\gamma_n = c^n_{\sigma_n(n)} \in \Lambda(k_0)$.

Moreover, for every $T \in \mathbb{N}$, $\gamma = c^T$ almost everywhere in $[0,T]$; hence $\gamma \in L^1([0,T],\mathbb{R})$, which implies $\gamma \in L^1_{loc}([0,+\infty),\mathbb{R})$ because T is generic.

Now fix $n \in \mathbb{N}$. The above equality for γ_n cannot be developed in $[0,+\infty)$, but the second and third condition in (29) imply that the following chain of inequalities for the functional holds:

$$
\begin{aligned}
U(\gamma_n; k_0) &\geq U\left(\overline{c}^{n-1}_{\sigma_n(n)}; k_0\right) = U\left(c^{n-1}_{\sigma_{n-1}\circ\sigma_n(n)}; k_0\right) \\
&\geq U\left(\overline{c}^{n-2}_{\sigma_{n-1}\circ\sigma_n(n)}; k_0\right) \geq \cdots \geq U\left(\overline{c}^{1}_{\sigma_2\circ\cdots\circ\sigma_n(n)}; k_0\right) \\
&= U\left(c^{1}_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_n(n)}; k_0\right) \geq U\left(c_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_n(n)}; k_0\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|U(\gamma_n; k_0) - V(k_0)| &= V(k_0) - U(\gamma_n; k_0) \\
&\leq V(k_0) - U\left(c_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_n(n)}; k_0\right) \\
&= \left|U\left(c_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_n(n)}; k_0\right) - V(k_0)\right|;
\end{aligned}
$$

since $\sigma_1 \circ \cdots \circ \sigma_n(n) \geq n$, the fact that $(\gamma_n)_{n\in\mathbb{N}}$ is a maximizing sequence follows from the fact that, by assumption, $(c_n)_{n\in\mathbb{N}}$ is a maximizing sequence.

Now fix $T \in \mathbb{N}$ and observe that the argument developed after Definition 5.1 inductively shows that, for every $k \in \mathbb{N}$:

$$
\begin{aligned}
&\gamma_{T+k} = \overline{c}^{T}_{\nu_T(k)} \quad (\text{where } \nu_T(k) = \sigma_{T+1} \circ \cdots \circ \sigma_{T+k}(T+k)) \\
&\text{in } [0,T] \cap W^{T+k-1}_{T+k} \cap \bigcap_{p=1}^{k-1} W^{T+k-1-p}_{\bigodot_{j=0}^{p-1} \sigma_{T+k-(p-1-j)}}
\end{aligned}
\tag{32}
$$

Since by construction any set of the form $[0,T] \setminus W^{T+k-1-p}_m$, $p = 0, .., k-1$ has null Lebesgue measure, the above relation imply $\|\gamma_{T+k}\|_{\infty,[0,T]} = \left\|\overline{c}^{T}_{\nu_T(k)}\right\|_{\infty,[0,T]}$. This quantity is bounded above by $N(k_0, T)$, by the second and fourth condition in (29).

Moreover, the intersection for $k \in \mathbb{N}$ of the sets in (32) has negligible complementary in $[0,T]$; since ν_T is strictly increasing, this implies that $(\gamma_n)_{n\geq T}$, as a sequence, coincides almost everywhere in $[0,T]$ with a subsequence of $\left(\overline{c}^{T}_n\right)_{n\in\mathbb{N}}$. In particular $\gamma_n \rightharpoonup \gamma$ in $L^1([0,T], \mathbb{R})$ by the last condition in (29) and by the fact that $\gamma = c^T$ almost everywhere in $[0,T]$.

As this holds for every $T \in \mathbb{N}$, it is a consequence of Remark 5.1 that it must hold for every real number $T > 0$. □

The first step is then accomplished.

Step 2 The next step is to show that γ is admissible at k_0. For this purpose, it is enough to prove the following

Proposition 5.2 *Let $T > 0$. Hence $\gamma \geq 0$ almost everywhere in $[0,T]$, and, for every $t \in [0,T]$, $k(t; k_0, \gamma) \geq 0$.*

Proof It is well known that the weak convergence of $(\gamma_n)_{n\in\mathbb{N}}$ to γ in $L^1([0,T], \mathbb{R})$, ensured by Proposition 5.1, implies that

$$
\liminf_{n\to+\infty} \gamma_n(t) \leq \gamma(t) \leq \limsup_{n\to+\infty} \gamma_n(t) \ \text{ for almost every } t \in [0,T]. \tag{33}
$$

By Proposition 5.1 we also have

$$\forall n \in \mathbb{N} : \text{for almost every } t \in [0, T] : 0 \leq \gamma_n(t) \leq N(k_0, T). \tag{34}$$

We can interchange the quantifiers in the previous relation, since a numerable intersection of full-measure sets is a full-measure set. Consequently, taking the intersection with the set where (33) holds, we have

$$\begin{aligned}
&0 \leq \gamma \leq N(k_0, T) \quad \text{a.e. in } [0, T] \\
&k(\cdot; k_0, 0) \geq \kappa \geq k(\cdot; k_0, N(k_0, T)) \quad \text{in } [0, T] \\
&k(\cdot; k_0, 0) \geq \kappa_n \geq k(\cdot; k_0, N(k_0, T)) \quad \text{in } [0, T], \ \forall n \in \mathbb{N}
\end{aligned}$$

where $\kappa := k(\cdot; k_0, \gamma)$ and $\kappa_n := k(\cdot; k_0, \gamma_n)$; observe that the constant control $N(k_0, T)$ need not be admissible. The second relation follows from the first by Remark 3.2 and the third relation follows directly from (34). Hence:

$$|\kappa - \kappa_n| \leq k(\cdot; k_0, 0) - k(\cdot; k_0, N(k_0, T)) \quad \text{in } [0, T], \ \forall n \in \mathbb{N}. \tag{35}$$

Fix $n \in \mathbb{N}$. Subtracting the state equation for κ from the state equation for κ_n, we obtain, for every $t \in [0, T]$:

$$\begin{aligned}
\dot{\kappa_n}(t) - \dot{\kappa}(t) &= F(\kappa_n(t)) - F(\kappa(t)) - \left[\gamma_n(t) - \gamma(t)\right] \\
&= h_n(t)\left[\kappa_n(t) - \kappa(t)\right] - \left[\gamma_n(t) - \gamma(t)\right],
\end{aligned}$$

where h_n is the (continuous) function defined taking $k_1 = \kappa_n$ and $k_2 = \kappa$ in Remark 3.2.

Integrating both sides of this equation between 0 and t, then taking absolute values leads to:

$$|\kappa_n(t) - \kappa(t)| \leq \int_0^t |h_n(s)| \, |\kappa_n(s) - \kappa(s)| \, \mathrm{d}s + \left| \int_0^t \left[\gamma(s) - \gamma_n(s)\right] \mathrm{d}s \right|. \tag{36}$$

Observe that, for every $s \in [0, t]$:

$$|h_n(s)| \, |\kappa_n(s) - \kappa(s)| \leq \overline{M} \left[k(s; k_0, 0) - k(s; k_0, N(k_0, T))\right],$$

by Remark 3.3 and by (35).

This holds for every $n \in \mathbb{N}$ and for every fixed $t \in [0, T]$. Since the function of s on the right hand side obviously belongs to $L^1([0, t])$ we obtain from (36) (remembering that $\gamma_n \rightharpoonup \gamma$ in $L^1([0, t])$):

$$\begin{aligned}\limsup_{n\to+\infty}|\kappa_n(t)-\kappa(t)| &\le \limsup_{n\to+\infty}\int_0^t |h_n(s)|\,|\kappa_n(s)-\kappa(s)|\,\mathrm{d}s\\ &\le \int_0^t \limsup_{n\to+\infty}|h_n(s)|\,|\kappa_n(s)-\kappa(s)|\,\mathrm{d}s\\ &\le \int_0^t \overline{M}\limsup_{n\to+\infty}|\kappa_n(s)-\kappa(s)|\,\mathrm{d}s.\end{aligned} \tag{37}$$

Hence by Gronwall's inequality:

$$\limsup_{n\to+\infty}|\kappa_n(t)-\kappa(t)|=0,$$

for every $t\in[0,T]$. This is equivalent to

$$\lim_{n\to+\infty}\kappa_n=\kappa \quad \text{in } [0,T].$$

Since any κ_n is non-negative in $[0,T]$, the second assertion of the theorem is also proved. □

Remark 5.2 The argument behind (37) goes as follows. Let

$$a_n:=\sup_{j\ge n}|h_j|\,|\kappa_j-\kappa|.$$

Then

$$\begin{aligned}&|a_n|=a_n\le\overline{M}\left[k(\cdot;k_0,0)-k(\cdot;k_0,N(k_0,T))\right] \quad \forall n\in\mathbb{N}\\ &a_n\downarrow_{n\to+\infty}\limsup_{m\to+\infty}|h_m|\,|\kappa_m-\kappa| \quad \text{in } [0,t].\end{aligned}$$

Then by Dominated Convergence:

$$\inf_{n\in\mathbb{N}}\int_0^t a_n(s)\,\mathrm{d}s=\int_0^t\limsup_{m\to+\infty}|h_m(s)|\,|\kappa_m(s)-\kappa(s)|\,\mathrm{d}s.$$

Moreover for every $n\in\mathbb{N}$ and every $i\ge n$:

$$a_n=\sup_{j\ge n}|h_j|\,|\kappa_j-\kappa|\ge|h_i|\,|\kappa_i-\kappa|,$$

which implies, passing to the integrals and then taking the sup for $i\ge n$:

$$\int_0^t a_n(s)\,\mathrm{d}s\ge\sup_{i\ge n}\int_0^t|h_i(s)|\,|\kappa_i(s)-\kappa(s)|\,\mathrm{d}s \quad \forall n\in\mathbb{N}.$$

Hence, passing to the inf for $n \in \mathbb{N}$:

$$\inf_{n\in\mathbb{N}} \int_0^t a_n(s)\,\mathrm{d}s \geq \limsup_{m\to+\infty} \int_0^t |h_m(s)|\,|\kappa_m(s)-\kappa(s)|\,\mathrm{d}s.$$

As a consequence of Proposition 5.2, γ is almost everywhere non-negative in $[0,+\infty)$ and $k(\cdot;k_0,\gamma)$ is everywhere non-negative in $[0,+\infty)$ - which precisely means that $\gamma \in \Lambda(k_0)$. Hence the second step is also ended.

Step 3 Now it is time to define the control which is optimal at k_0. In order to do this, we need to extract a subsequence from $(\gamma_n)_{n\in\mathbb{N}}$ because the weak convergence to γ in the intervals could not be enough to ensure that $\lim_{n\to+\infty} U(\gamma_n;k_0) = U(\gamma;k_0)$; we will also need the admissibility of γ. By the penultimate assertion stated in Proposition 5.1, and by the monotonicity of u, we have:

$$\|u(\gamma_n)\|_{\infty,[0,1]} \leq u(N(k_0,1)) \quad \forall n \in \mathbb{N}.$$

Hence by Lemma 5.1, there exists a function $f^1 \in L^1([0,1],\mathbb{R})$ and a sequence $\left(u\left(\gamma_{1,n}\right)\right)_{n\in\mathbb{N}}$ extracted from $(u(\gamma_n))_{n\in\mathbb{N}}$, such that

$$u\left(\gamma_{1,n}\right) \rightharpoonup f^1 \text{ in } L^1([0,1],\mathbb{R}).$$

Again by Proposition 5.1 and the monotonicity of u,

$$\left\|u\left(\gamma_{1,n}\right)\right\|_{\infty,[0,2]} \leq u(N(k_0,2)) \quad \forall n \in \mathbb{N}$$

which implies by Lemma 5.1 the existence of $f^2 \in L^1([0,2],\mathbb{R})$ and of a sequence $\left(u\left(\gamma_{2,n}\right)\right)_{n\in\mathbb{N}}$ extracted from $\left(u\left(\gamma_{1,n}\right)\right)_{n\in\mathbb{N}}$ such that

$$u\left(\gamma_{2,n}\right) \rightharpoonup f^2 \text{ in } L^1([0,2],\mathbb{R});$$

in particular $f^2 = f^1$ almost everywhere in $[0,1]$ by the essential uniqueness of the weak limit.

Going on this way we see that there exists a family $\left\{\left(u\left(\gamma_{T,n}\right)_{n\in\mathbb{N}}, f^T\right)/T \in \mathbb{N}\right\}$ satisfying, for every $T \in \mathbb{N}$:

$$\begin{aligned}
&\left\|u\left(\gamma_{T,n}\right)\right\|_{\infty,[0,T]} \leq u(N(k_0,T)) \quad \forall n \in \mathbb{N}\\
&\left(u\left(\gamma_{T+1,n}\right)\right)_{n\in\mathbb{N}} \text{ is extracted from } \left(u\left(\gamma_{T,n}\right)\right)_{n\in\mathbb{N}}\\
&f^{T+1} = f^T \text{ almost everywhere in } [0,T]\\
&u\left(\gamma_{T,n}\right) \rightharpoonup f^T \text{ in } L^1([0,T],\mathbb{R}).
\end{aligned}$$

Hence, for every $T \in \mathbb{N}$, the sequence $\left(u\left(\gamma_{n,n}\right)\right)_{n\geq T}$ is extracted from $\left(u\left(\gamma_{T,n}\right)\right)_{n\in\mathbb{N}}$. If we define $f(t) := f^{[t]+1}(t)$, then $f = f^T$ almost everywhere in $[0,T]$. So

$$u\left(\gamma_{n,n}\right) \rightharpoonup f \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0. \tag{38}$$

by construction and by Remark 5.1. This implies that

$$0 \leq \liminf_{n \to +\infty} u\left(\gamma_{n,n}(t)\right) \leq f(t)$$

for almost every $t \in \mathbb{R}$.

Now define $c^* : [0, +\infty) \to \mathbb{R}$ as

$$c^*(t) := \begin{cases} u^{-1}(f(t)) & \text{if } f(t) \geq 0 \\ 0 & \text{if } f(t) < 0. \end{cases}$$

Obviously $c^* \geq 0$ everywhere in $[0, +\infty)$. Moreover, again by the properties of the weak convergence, for any $T \in \mathbb{N}$ and for almost every $t \in [0, T]$:

$$f(t) \leq \limsup_{n \to +\infty} u\left(\gamma_{n,n}(t)\right) \leq u\left(N(k_0, T)\right).$$

This implies, together with the fact that u^{-1} is increasing, that c^* is bounded above by $N(k_0, T)$ almost everywhere in $[0, T]$. As this holds for every $T \in \mathbb{N}$,

$$c^* \in L^\infty_{loc}([0, +\infty), \mathbb{R}). \tag{39}$$

To complete the proof of the admissibility of c^*, we show that $c^* \leq \gamma$ almost everywhere in $[0, +\infty)$.

Fix $T > 0$ and let $t_0 \in [0, T]$ be a Lebesgue point for both f and γ in $[0, T]$; then take $t_1 \in (t_0, T)$. By the concavity of u and by Jensen inequality:

$$\frac{\int_{t_0}^{t_1} u\left(\gamma_{n,n}(s)\right) \mathrm{d}s}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma_{n,n}(s)\, \mathrm{d}s}{t_1 - t_0}\right) \tag{40}$$

Observe that $\left(\gamma_{n,n}\right)_{n \geq 1}$ is a subsequence of $\left(\gamma_{1,n}\right)_{n \in \mathbb{N}}$, which is in its turn extracted from $(\gamma_n)_{n \in \mathbb{N}}$. Hence $\gamma_{n,n} \rightharpoonup \gamma$ in $L^1([0, T], \mathbb{R})$, which implies $\lim_{n \to +\infty} \int_{t_0}^{t_1} \gamma_{n,n}(s)\, \mathrm{d}s = \int_{t_0}^{t_1} \gamma(s)\, \mathrm{d}s$. So taking the limit for $n \to +\infty$ in (40), by the continuity of u and by (38), we have:

$$\frac{\int_{t_0}^{t_1} f(s)\, \mathrm{d}s}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma(s)\, \mathrm{d}s}{t_1 - t_0}\right).$$

As t_0 is a Lebesgue point for both f and γ in $[0, T]$, we can take the limit for $t_1 \to t_0$ in the previous inequality and get $f(t_0) \leq u(\gamma(t_0))$.

By the Lebesgue Point Theorem, this argument works for almost every $t_0 \in [0, T]$. So by the monotonicity of u^{-1} we deduce

$$c^* \leq \gamma \text{ almost everywhere in } [0, T].$$

Because T is generic, we have by (5): $k(t; k_0, c^*) \geq k(t; k_0, \gamma)$ for every $t \in \mathbb{R}$. Hence by the admissibility of γ at k_0, $k(\cdot; k_0, c^*) \geq 0$. This implies, together with (39) and $c^* \geq 0$ in $[0, +\infty)$,

$$c^* \in \Lambda(k_0) .$$

Finally, observe that by Lemma 4.1 we can apply the Dominated Convergence Theorem to the functions $t \to e^{-\rho t} \int_0^t u\left(\gamma_{n,n}(s)\right) \mathrm{d}s$, $n \in \mathbb{N}$.

Hence, using the functional form established in the same Lemma, part (iii), by Proposition 5.1, by the fact that $\left(\gamma_{n,n}\right)_{n\in\mathbb{N}}$ is extracted from $(\gamma_n)_{n\in\mathbb{N}}$, and by (38):

$$\begin{aligned} V(k_0) &= \lim_{n\to+\infty} U(\gamma_n; k_0) = \lim_{n\to+\infty} U\left(\gamma_{n,n}; k_0\right) \\ &= \lim_{n\to+\infty} \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u\left(\gamma_{n,n}(s)\right) \mathrm{d}s\mathrm{d}t \\ &= \rho \int_0^{+\infty} e^{-\rho t} \limsup_{n\to+\infty} \int_0^t u\left(\gamma_{n,n}(s)\right) \mathrm{d}s\mathrm{d}t \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t f(s) \,\mathrm{d}s\mathrm{d}t \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u\left(c^*(s)\right) \mathrm{d}s\mathrm{d}t = U\left(c^*; k_0\right) . \end{aligned}$$

So we have proved the following

Theorem 5.1 *For every $k_0 \geq 0$ there exists $c^* \in \Lambda(k_0)$ which is optimal at k_0 and everywhere positive in $[0, +\infty)$, satisfying:*

$$c^* \in L^{\infty}_{loc}([0, +\infty), \mathbb{R}).$$

6 Further Properties of the Value Function: Regularity and Monotonicity

Now it is possible to establish some regularity and monotonicity properties of the value function, with the help of optimal controls. The next theorem uses the monotonicity with respect to the first variable of the function defined in Lemma 3.2.

Theorem 6.1 *The value function $V : [0, +\infty) \to \mathbb{R}$ satisfies:*

(i) *V is strictly increasing in $[0, +\infty)$.*
(ii) *For every $k_0 > 0$, there exists $C(k_0)$, $\delta > 0$ such that for every $h \in (-\delta, \delta)$:*

$$\frac{V(k_0 + h) - V(k_0)}{h} \geq C(k_0)$$

(iii) *V is Lipschitz-continuous in every closed sub-interval of $(0, +\infty)$.*

Proof (i) Let $0 < k_1$. Set $c \in (0, F(k_1)]$ and $c_1 \equiv c$ in $[0, +\infty)$; hence by Proposition 3.1 and by Theorem 4.1,

$$V(0) = 0 < \frac{u(c)}{\rho} = U(c_1; k_1) \leq V(k_1).$$

The implication $0 < k_0 < k_1 \implies V(k_0) < V(k_1)$ follows from point (ii).

(ii) We split the proof in two parts.
First, take $k_0, h > 0$, c optimal at k_0 and set $k_1 := k_0 + h$. Because $k_1 > k_0$ we can choose $\underline{c}^{k_1 - k_0} = \underline{c}^h \in \Lambda(k_0 + h)$ as in Lemma 3.3. Hence

$$\begin{aligned} V(k_0 + h) - V(k_0) &\geq U\left(\underline{c}^h; k_0 + h\right) - U(c; k_0) \\ &\geq u'(N(k_0, h) + 1) \int_0^h e^{-\rho t} \mathrm{d}t \end{aligned}$$

Now, by the fact that $\lim_{h \to 0} \frac{1}{h} \int_0^h e^{-\rho t} \mathrm{d}t = 1$ and that $N(k_0, \cdot)$ is increasing, there exists $\delta > 0$ such that, for any $h \in (0, \delta)$:

$$\begin{aligned} \frac{V(k_0 + h) - V(k_0)}{h} &\geq u'(N(k_0, h) + 1) \frac{\int_0^h e^{-\rho t} \mathrm{d}t}{h} \geq \frac{u'(N(k_0, 1) + 1)}{2} \\ &=: C(k_0) \end{aligned}$$

In the second place, fix $k_0 > 0$, $h < 0$ and c optimal at $k_0 + h$.
Then again take $\underline{c}^{k_0 - (k_0 + h)} = \underline{c}^{-h} \in \Lambda(k_0)$ as in Lemma 3.3. Hence

$$\begin{aligned} V(k_0 + h) - V(k_0) &\leq U(c; k_0 + h) - U\left(\underline{c}^{-h}; k_0\right) \\ &\leq -u'(N(k_0 + h, -h) + 1) \int_0^{-h} e^{-\rho t} \mathrm{d}t. \end{aligned}$$

We can assume that $-\frac{1}{h} \int_0^{-h} e^{-\rho t} \mathrm{d}t \geq \frac{1}{2}$ for $-\delta < h < 0$. Hence, by the monotonicity of $N(\cdot, \cdot)$ in both variables, for every $h \in (-\delta, 0)$:

$$\frac{V(k_0 + h) - V(k_0)}{h} \geq \frac{u'(N(k_0 + h, -h) + 1)}{2} \geq \frac{u'(N(k_0, 1) + 1)}{2} = C(k_0).$$

(iii) Let $0 < k_0 < k_1$. We need a reverse inequality for $V(k_1) - V(k_0)$, so take $c_1 \in \Lambda(k_1)$ optimal at k_1. In order to define the proper $c_0 \in \Lambda(k_0)$, observe that the orbit $k = k(\cdot; k_0, 0)$ (with null control) satisfies $\dot{k} = F(k)$. With an argument similar to the one used in Proposition 3.1 we can see that $\dot{k}(t) > F(k_0) > 0$ for every $t > 0$, and so $\lim_{t \to +\infty} k(t) = +\infty$.
Then by Darboux's property there exists $\bar{t} > 0$ such that $k\left(\bar{t}\right) = k_1$. Observe that, since k and F are strictly increasing functions, $\dot{k}$ must also be strictly increasing.

Hence applying Lagrange's theorem to k gives for some $\xi \in \left(0, \bar{t}\right)$:

$$k_1 - k_0 = k\left(\bar{t}\right) - k\left(0\right) = \bar{t} \cdot \dot{k}\left(\xi\right) > \bar{t}\dot{k}\left(0\right) = \bar{t}F\left(k_0\right) \tag{41}$$

Now define

$$c_0\left(t\right) := \begin{cases} 0 & \text{if } t \in \left[0, \bar{t}\right] \\ c_1\left(t - \bar{t}\right) & \text{if } t > \bar{t} \end{cases}$$

It is easy to check that $c_0 \in \Lambda\left(k_0\right)$, because

$$\begin{aligned} k\left(t; k_0, c_0\right) &= k\left(t; k_0, 0\right) > 0 \quad \forall t \in \left[0, \bar{t}\right] \\ k\left(t + \bar{t}; k_0, c_0\right) &= k\left(t; k_1, c_1\right) \geq 0 \quad \forall t \geq 0 \end{aligned}$$

by the uniqueness of the orbit; as far as the second equality is concerned, observe that both orbits pass through $(0, k_1)$ and satisfy the differential equation controlled with c_1 for $t > 0$. Hence by (41):

$$\begin{aligned} V\left(k_1\right) - V\left(k_0\right) \leq U\left(c_1; k_1\right) - U\left(c_0; k_0\right) &= \int_0^{+\infty} e^{-\rho t}\left[u\left(c_1\left(t\right)\right) - u\left(c_0\left(t\right)\right)\right] \mathrm{d}t \\ &= \int_0^{+\infty} e^{-\rho t} u\left(c_1\left(t\right)\right) \mathrm{d}t - \int_{\bar{t}}^{+\infty} e^{-\rho t} u\left(c_1\left(t - \bar{t}\right)\right) \mathrm{d}t \\ &= \int_0^{+\infty} e^{-\rho t} u\left(c_1\left(t\right)\right) \mathrm{d}t - \int_0^{+\infty} e^{-\rho\left(s + \bar{t}\right)} u\left(c_1\left(s\right)\right) \mathrm{d}s \\ &= \left(1 - e^{-\rho \bar{t}}\right) U\left(c_1; k_1\right) \\ &= \left(1 - e^{-\rho \bar{t}}\right) V\left(k_1\right) \leq \rho \bar{t} V\left(k_1\right) < \rho V\left(k_1\right) \frac{k_1 - k_0}{F\left(k_0\right)} \end{aligned}$$

So by the monotonicity of V and F we have, for $a \leq k_0 < k_1 \leq b$:

$$V\left(k_1\right) - V\left(k_0\right) \leq \rho \frac{V\left(b\right)}{F\left(a\right)}\left(k_1 - k_0\right).$$

□

7 Dynamic Programming

7.1 Dynamic Programming Principle and Characterization of Optimal Controls

In this section we study the properties of the value function as a solution to Bellman and HJB equations.

First observe that we can translate an orbit by translating the control, according to the next remark.

Remark 7.1 (Translation of the orbit) For every $k_0, \tau \geq 0$ and every $c \in L^1_{loc}[(0, +\infty), \mathbb{R})$:

$$k(\cdot; k(\tau; k_0, c), c(\cdot + \tau)) = k(\cdot + \tau; k_0, c)$$

by the uniqueness of the orbit. In particular, if $c \in \Lambda(k_0)$ then $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$.

The first step consists in proving a suitable version of Dynamic Programming Principle.

Theorem 7.1 *For every $\tau > 0$, the value function $V : [0, +\infty) \to \mathbb{R}$ satisfies the following functional equation:*

$$\forall k_0 \geq 0 : \mathrm{v}(k_0) = \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t))\, dt + e^{-\rho\tau} \mathrm{v}(k(\tau; k_0, c)) \right\} \tag{42}$$

in the unknown $\mathrm{v} : [0, +\infty) \to \mathbb{R}$.

Proof Fix $\tau > 0$ and $k_0 \geq 0$, and set

$$\sigma(\tau, k_0) := \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u(c(t))\, \mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c)) \right\}.$$

We prove that

$$\sigma(\tau, k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0).$$

In the first place, we show that $\sigma(\tau, k_0)$ is an upper bound of $\{U(c; k_0) \,/\, c \in \Lambda(k_0)\}$.

Fix $c \in \Lambda(k_0)$; then by Remark 7.1 $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$; hence

$$\begin{aligned}
\sigma(\tau, k_0) &\geq \int_0^\tau e^{-\rho t} u(c(t))\, \mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c)) \\
&\geq \int_0^\tau e^{-\rho t} u(c(t))\, \mathrm{d}t + e^{-\rho\tau} U(c(\cdot + \tau); k(\tau; k_0, c)) \\
&= \int_0^\tau e^{-\rho t} u(c(t))\, \mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u(c(t + \tau))\, \mathrm{d}t \\
&= \int_0^\tau e^{-\rho t} u(c(t))\, \mathrm{d}t + \int_\tau^{+\infty} e^{-\rho s} u(c(s))\, \mathrm{d}t = U(c; k_0)
\end{aligned}$$

In the second place, fix $\epsilon > 0$, and take

$$0 < \epsilon' \leq \frac{2\epsilon}{\left(1 + e^{-\rho\tau}\right)}.$$

Hence there exists $\tilde{c}_\epsilon \in \Lambda(k_0)$ and $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$ such that

$$
\begin{aligned}
\sigma(\tau, k_0) - \epsilon &\le \sigma(\tau, k_0) - \frac{\epsilon'}{2}\left(1 + e^{-\rho\tau}\right) \\
&\le \int_0^\tau e^{-\rho t} u\left(\tilde{c}_\epsilon(t)\right) \mathrm{d}t + e^{-\rho\tau} V\left(k\left(\tau; k_0, \tilde{c}_\epsilon\right)\right) - e^{-\rho\tau}\frac{\epsilon'}{2} \\
&\le \int_0^\tau e^{-\rho t} u\left(\tilde{c}_\epsilon(t)\right) \mathrm{d}t + e^{-\rho\tau} U\left(\tilde{\tilde{c}}_\epsilon; k\left(\tau; k_0, \tilde{c}_\epsilon\right)\right) \\
&= \int_0^\tau e^{-\rho t} u\left(\tilde{c}_\epsilon(t)\right) \mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u\left(\tilde{\tilde{c}}_\epsilon(t)\right) \mathrm{d}t
\end{aligned}
$$

Now set

$$
c_\epsilon(t) := \begin{cases} \tilde{c}_\epsilon(t) & \text{if } t \in [0, \tau] \\ \tilde{\tilde{c}}_\epsilon(t - \tau) & \text{if } t > \tau \end{cases}
$$

Hence $c_\epsilon \in L^1_{loc}([0, +\infty), \mathbb{R})$ and $\forall t > 0 : c_\epsilon(t + \tau) = \tilde{\tilde{c}}_\epsilon(t)$. So:

$$
\sigma(\tau, k_0) - \epsilon \le \int_0^{+\infty} e^{-\rho t} u\left(c_\epsilon(t)\right) \mathrm{d}t \tag{43}
$$

Finally, it is easy to show that $c_\epsilon \in \Lambda(k_0)$. Observe that $k(\cdot; k_0, c_\epsilon) = k(\cdot; k_0, \tilde{c}_\epsilon)$ in $[0, \tau]$ by definition of c_ϵ and by uniqueness. In particular $k(\tau; k_0, c_\epsilon) = k(\tau; k_0, \tilde{c}_\epsilon)$, so that $k(\cdot + \tau; k_0, c_\epsilon)$ and $k\left(\cdot; k(\tau; k_0, \tilde{c}_\epsilon), \tilde{\tilde{c}}_\epsilon\right)$ have the same initial value; moreover, these two orbits satisfy the same state equation (i.e. the equation associated to the control $c_\epsilon(\cdot + \tau)$) and so they coincide, again by uniqueness. Recalling that by definition $\tilde{c}_\epsilon \in \Lambda(k_0)$ and $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$, we have $k(t; k_0, c_\epsilon) \ge 0$ for all $t \ge 0$. Hence by (43) we can write

$$
\sigma(\tau, k_0) - \epsilon \le U(c_\epsilon; k_0)
$$

and the assertion is proven. □

Equation (42) is called *Bellman Functional Equation*.

A consequence of the above theorem is that every control which is optimal respect to a state, is also optimal respect to every following optimal state. But Theorem 7.1 also suggests and partially imply a useful characterization of optimal controls as solutions of a certain integral equation.

Theorem 7.2 *Let $k_0 \ge 0$, $c^* \in \Lambda(k_0)$. Hence the following are equivalent:*

(i) *c^* is optimal at k_0*
(ii) *For every $\tau > 0$:*

$$
V(k_0) = \int_0^\tau e^{-\rho t} u\left(c^*(t)\right) dt + e^{-\rho\tau} V\left(k\left(\tau; k_0, c^*\right)\right)
$$

Moreover, (i) or (ii) imply that for every $\tau > 0$, $c^*(\cdot + \tau)$ is admissible and optimal at $k(\tau; k_0, c^*)$.

Proof (i) $\Rightarrow$ (ii) Let us assume that c^* is admissible and optimal at $k_0 \geq 0$ and fix $\tau > 0$. Observe that $c^*(\cdot + \tau)$ is admissible at $k(\tau; k_0, c^*)$ by Remark 7.1. Hence, by Theorem 7.1:

$$\begin{aligned} V(k_0) &\geq \int_0^{\tau} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + e^{-\rho\tau} V\left(k\left(\tau; k_0, c^*\right)\right) \\ &\geq \int_0^{\tau} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + e^{-\rho\tau} U\left(c^*(\cdot + \tau); k\left(\tau; k_0, c^*\right)\right) \\ &= \int_0^{+\infty} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t \;=\; U\left(c^*; k_0\right) \;=\; V(k_0). \end{aligned} \tag{44}$$

Hence

$$V(k_0) = \int_0^{\tau} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + e^{-\rho\tau} V\left(k\left(\tau; k_0, c^*\right)\right). \tag{45}$$

(ii) $\Rightarrow$ (i) Suppose that $c^* \in \Lambda(k_0)$ and (45) holds for every $\tau > 0$. For every $\epsilon > 0$ pick $\hat{c}_\epsilon \in \Lambda\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right)$ such that:

$$V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon \leq U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right). \tag{46}$$

Then define

$$c_\epsilon(t) := \begin{cases} c^*(t) & \text{if } t \in \left[0, \frac{1}{\epsilon}\right] \\ \hat{c}_\epsilon\left(t - \frac{1}{\epsilon}\right) & \text{if } t > \frac{1}{\epsilon} \end{cases}$$

By the same arguments we used in the proof of Theorem 7.1, $c_\epsilon \in \Lambda(k_0)$ and, obviously, $c_\epsilon\left(t + \frac{1}{\epsilon}\right) = \hat{c}_\epsilon(t)$ for every $t > 0$.

Hence, taking $\tau = 1/\epsilon$ in (45), we have by (46):

$$\begin{aligned} V(k_0) - \epsilon e^{-\rho/\epsilon} &= \int_0^{1/\epsilon} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + e^{-\rho/\epsilon}\left[V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon\right] \\ &\leq \int_0^{1/\epsilon} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + e^{-\rho/\epsilon} U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) \\ &= \int_0^{1/\epsilon} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + \int_0^{+\infty} e^{-\rho\left(t + \frac{1}{\epsilon}\right)} u\left(c_\epsilon\left(t + \frac{1}{\epsilon}\right)\right) \mathrm{d}t \\ &= \int_0^{1/\epsilon} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + \int_{1/\epsilon}^{+\infty} e^{-\rho s} u\left(c_\epsilon(s)\right) \mathrm{d}s \end{aligned} \tag{47}$$

Now we show that the second addend tends to 0 as $\epsilon \to 0$. First, using Jensen inequality and the properties of the function ψ_{k_0} established in Lemma 4.1, we see that for every $T \geq 1/\epsilon$:

$$\begin{aligned}\int_{1/\epsilon}^{T} e^{-\rho s} u\left(c_\epsilon(s)\right) \mathrm{d}s &= \left[e^{-\rho s} \int_{1/\epsilon}^{s} u\left(c_\epsilon(\tau)\right) \mathrm{d}\tau\right]_{s=1/\epsilon}^{s=T} \\ &\quad + \rho \int_{1/\epsilon}^{T} e^{-\rho s} \int_{1/\epsilon}^{s} u\left(c_\epsilon(\tau)\right) \mathrm{d}\tau \mathrm{d}s \\ &\leq e^{-\rho T} \int_{0}^{T} u\left(c_\epsilon(\tau)\right) \mathrm{d}\tau + \rho \int_{1/\epsilon}^{T} e^{-\rho s} \int_{0}^{s} u\left(c_\epsilon(\tau)\right) \mathrm{d}\tau \mathrm{d}s \\ &\leq \psi_{k_0}(T) + \rho \int_{1/\epsilon}^{T} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) \mathrm{d}s \\ &\to \rho \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) \mathrm{d}s \text{ as } T \to +\infty \end{aligned} \tag{48}$$

(remembering that c_ϵ is admissible at k_0). By point (i) of Lemma 4.1, for every $\epsilon < 1$ and every $s \geq 1/\epsilon$:

$$\begin{aligned} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) &\leq s e^{-\rho s} u\left(M(k_0)\left[1 + e^{(L+\epsilon_0)s}\right] + \frac{M(k_0)}{s(L+\epsilon_0)}\right) \\ &\leq s e^{-\rho s}\left\{u\left(M(k_0)\right) + M(k_0)\, u\left(e^{(L+\epsilon_0)s}\right) + u\left(\frac{M(k_0)}{L+\epsilon_0}\right)\right\} \end{aligned}$$

(remembering that u is increasing and has the properties in Remark 4.2) which implies, together with (48), for every $\epsilon < 1$:

$$\begin{aligned} 0 \leq \int_{1/\epsilon}^{+\infty} e^{-\rho s} u\left(c_\epsilon(s)\right) \mathrm{d}s &\leq \rho \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) \mathrm{d}s \\ &\leq \rho\left[u\left(M(k_0)\right) + u\left(\frac{M(k_0)}{L+\epsilon_0}\right)\right] \int_{1/\epsilon}^{+\infty} s e^{-\rho s} \mathrm{d}s \\ &\quad + \rho M(k_0) \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(e^{(L+\epsilon_0)s}\right) \mathrm{d}s. \end{aligned}$$

By Remark 2.1 the last integral converges, hence the upper bound tends to 0 as $\epsilon \to 0$.

Hence, letting $\epsilon \to 0$ in (47), we find:

$$V(k_0) \leq \int_0^{+\infty} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t = U\left(c^*; k_0\right)$$

which implies that c^* is optimal at k_0.

Finally, if (i) holds, then by (44):

$$V\left(k\left(\tau; k_0, c^*\right)\right) = U\left(c^*(\cdot + \tau); k\left(\tau; k_0, c^*\right)\right).$$

□

7.2 The Value Function as a Viscosity Solution of HJB

In many interesting cases the value function V is non-differentiable. Moreover, in general it is not possible to prove the differentiability of V relying only on the fact that it solves the Bellman Functional Equation, or BFE (in our case, Eq. (42)), since the latter needs not have a unique regular solution. Of course such equation has a natural "infinitesimal version" (usually called Hamilton–Jacobi–Bellman equation, or HJB, which is in general a first order non-linear PDE), and it can be proven that any continuously differentiable solution to BFE is indeed a solution of HJB. This is of no help without information about the regularity of V; furthermore, HJB could have no classical solution (see e.g. [13]).

This is why the theory of viscosity solutions plays a key role in Dynamic Programming methods: one wonders if the value function is a solution of HJB in a weaker sense. As pointed out in the introduction, our case is a bit special meaning that the problem itself of the value function being a viscosity solution of HJB equation must be proven to be well-posed. Indeed the "right" equation involves an Hamiltonian function whose domain is not $\mathbb{R}^N$ (in our case $\mathbb{R}^2$), so the test functions involved in the definition of viscosity solution must match this restriction. This is ensured by asking that the candidate solution has a special property, stronger than monotonicity.

Definition 7.1 Let $f \in \mathcal{C}^0((0,+\infty),\mathbb{R})$; we say that $f \in \mathcal{C}^+((0,+\infty),\mathbb{R})$ if, and only if, for every $k_0 > 0$ there exist $\delta, C^+, C^- > 0$ such that

$$\frac{f(k_0+h)-f(k_0)}{h} \geq C^+ \quad \forall h \in (0,\delta)$$
$$\frac{f(k_0+h)-f(k_0)}{h} \geq C^- \quad \forall h \in (-\delta,0)$$

We note that by Theorem 6.1, (ii) the value function V satisfies

$$V \in \mathcal{C}^+((0,+\infty),\mathbb{R}). \tag{49}$$

Definition 7.2 The function $H : [0,+\infty) \times (0,+\infty) \to \mathbb{R}$ defined by

$$H(k,p) := -\sup\{[F(k)-c]\cdot p + u(c) \;/\; c \in [0,+\infty)\}$$

is called *Hamiltonian*.

The equation

$$\rho \mathrm{v}(k) + H\left(k, \mathrm{v}'(k)\right) = 0 \quad \forall k > 0 \tag{50}$$

in the unknown $\mathrm{v} \in \mathcal{C}^+((0,+\infty),\mathbb{R}) \cap \mathcal{C}^1((0,+\infty),\mathbb{R})$ is called HJB equation.

Observe that any solution of (50) must be strictly increasing, by Definition 7.1.

Remark 7.2 The Hamiltonian is always finite. Indeed

$$-\sup_{c\in[0,+\infty)}\{[F(k)-c]\cdot p + u(c)\} > -\infty \iff p > 0.$$

If $p > 0$, since $\lim_{c\to+\infty} u'(c) = 0$ we can choose $c_p \geq 0$ such that $u'\left(c_p\right) \leq p$; this implies by the concavity of u:

$$\forall c \geq 0 : u(c) - cp \leq u(c) - u'\left(c_p\right) c \leq u\left(c_p\right) - u'\left(c_p\right) c_p,$$

so that

$$-F(k)\, p - \sup_{c\in[0,+\infty)} \{u(c) - cp\} \geq -F(k)\, p - u\left(c_p\right) + u'\left(c_p\right) c_p > -\infty.$$

Otherwise, when $p \leq 0$, since $\lim_{c\to+\infty} u(c) = +\infty$ we have

$$-F(k)\, p - \sup_{c\in[0,+\infty)} \{u(c) - cp\} \leq -F(k)\, p - \sup_{c\in[0,+\infty)} u(c) = -\infty.$$

Definition 7.3 A function $v \in \mathcal{C}^+((0, +\infty), \mathbb{R})$ is called a *viscosity subsolution* [*supersolution*] *of* (HJB) if, and only if:

for every $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$ and for every local maximum [minimum] point $k_0 > 0$ of $v - \varphi$:

$$\begin{aligned} \rho v(k_0) - \sup\left\{[F(k_0) - c] \cdot \varphi'(k_0) + u(c) \;/\; c \in [0, +\infty)\right\} = \\ \rho v(k_0) + H\left(k_0, \varphi'(k_0)\right) \;\leq\; 0 \\ [\geq 0] \end{aligned}$$

If v is both a viscosity subsolution of (HJB) and a viscosity supersolution of (HJB), then we say that v is a *viscosity solution of* (HJB).

Remark 7.3 The latter definition is well posed. Indeed, let $v \in \mathcal{C}^+((0, +\infty), \mathbb{R})$ and $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$. If k_0 is a local maximum for $v - \varphi$ in $(0, +\infty)$, then for $h < 0$ big enough we have:

$$\begin{aligned} & v(k_0) - v(k_0 + h) \geq \varphi(k_0) - \varphi(k_0 + h) \implies \\ & 0 < C^- \leq \frac{v(k_0) - v(k_0 + h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0 + h)}{h}. \end{aligned}$$

If k_0 is a local minimum for $v - \varphi$ in $(0, +\infty)$, then for $h > 0$ small enough we have:

$$\begin{aligned} & v(k_0) - v(k_0 + h) \leq \varphi(k_0) - \varphi(k_0 + h) \implies \\ & 0 < C^+ \leq \frac{v(k_0) - v(k_0 + h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0 + h)}{h}. \end{aligned}$$

In both cases, we have $\varphi'(k_0) > 0$, so the quantity $H\left(k_0, \varphi'(k_0)\right)$ involved in the definition is well-defined.

Thus we see that the value function is a good candidate to be a viscosity solution of HJB. We are now going to prove that this is indeed the case. As pointed out in the introduction, this will be done without any regularity assumption on H; nevertheless, this function can be easily shown to be continuous, since for every $k \geq 0$, $p > 0$:

$$H(k, p) = F(k)\, p + (-u)^*(p),$$

where $(-u)^*$ is the (convex) conjugate function of the convex function $-u$.

Lemma 7.1 *Let $k_0 > 0$ and $(c_T)_{T>0} \subseteq \Lambda(k_0)$ satisfying:*

$$\|c_T\|_{\infty,[0,T]} \leq N(k_0, T) \quad \forall T > 0.$$

where N is the function defined in Lemma 3.2. *Hence*

$$\forall T \in [0, 1] : \forall t \in [0, T] : |k(t; k_0, c_T) - k_0| \leq T e^{\bar{M}t} [F(k_0) + N(k_0, 1)].$$

In particular $k(T; k_0, c_T) \to k_0$ as $T \to 0$.

Proof Set k_0 and $(c_T)_{T>0}$ as in the hypothesis and fix $0 \leq T \leq 1$. Hence integrating both sides of the state equation we get, for every $t \in [0, T]$:

$$k(t; k_0, c_T) - k_0 = \int_0^t [F(k_0) - c_T(s)]\, \mathrm{d}s + \int_0^t [F(k(s; k_0, c_T)) - F(k_0)]\, \mathrm{d}s$$

which implies by Remark 3.3:

$$\begin{aligned} |k(t; k_0, c_T) - k_0| &\leq \int_0^t |F(k_0) - c_T(s)|\, \mathrm{d}s + \int_0^t |F(k(s; k_0, c_T)) - F(k_0)|\, \mathrm{d}s \\ &\leq \int_0^T |F(k_0) - c_T(s)|\, \mathrm{d}s + \bar{M} \int_0^t |k(s; k_0, c_T) - k_0|\, \mathrm{d}s \end{aligned}$$

Hence by Gronwall's inequality and by the monotonicity of $N(k_0, \cdot)$, for every $T \in [0, 1]$ and every $t \in [0, T]$:

$$\begin{aligned} |k(t; k_0, c_T) - k_0| &\leq e^{\bar{M}t} \int_0^T |F(k_0) - c_T(s)|\, \mathrm{d}s \\ &\leq T e^{\bar{M}t} [F(k_0) + N(k_0, T)] \\ &\leq T e^{\bar{M}t} [F(k_0) + N(k_0, 1)]. \end{aligned}$$

□

Theorem 7.3 *The value function $V : [0, +\infty) \to \mathbb{R}$ is a viscosity solution of* (HJB).

Consequently, if $V \in \mathcal{C}^1([0, +\infty), \mathbb{R})$, then V is strictly increasing and is a solution of (HJB)- (50) *in the classical sense.*

Proof In the first place, we show that V is a viscosity supersolution of (HJB).

Let $\varphi \in \mathcal{C}^1((0,+\infty),\mathbb{R})$ and $k_0 > 0$ be a local minumum point of $V - \varphi$, so that

$$V(k_0) - V \leq \varphi(k_0) - \varphi \tag{51}$$

in a proper neighbourhood of k_0. Now fix $c \in [0,+\infty)$ and set $k := k(\cdot; k_0, c)$. As $k_0 > 0$, there exists $T_c > 0$ such that $k > 0$ in $[0, T_c]$. Hence the control

$$\tilde{c}(t) := \begin{cases} c & \text{if } t \in [0, T_c] \\ 0 & \text{if } t > T_c \end{cases}$$

is admissible at k_0. Then by Theorem 7.1, for every $\tau \in [0, T_c]$:

$$\begin{aligned} V(k_0) - V(k(\tau)) &\geq \int_0^\tau e^{-\rho t} u(\tilde{c}(t))\,\mathrm{d}t + V(k(\tau))\left[e^{-\rho\tau} - 1\right] \\ &= u(c)\int_0^\tau e^{-\rho t}\mathrm{d}t + V(k(\tau))\left[e^{-\rho\tau} - 1\right]. \end{aligned}$$

Hence by (51) and by the continuity of k, we have for every $\tau > 0$ sufficiently small:

$$\frac{\varphi(k(0)) - \varphi(k(\tau))}{\tau} \geq u(c)\frac{\int_0^\tau e^{-\rho t}\mathrm{d}t}{\tau} + V(k(\tau))\frac{\left[e^{-\rho\tau} - 1\right]}{\tau}.$$

Letting $\tau \to 0$ and using the continuity of V and k:

$$-\varphi'(k_0)\left[F(k_0) - c\right] \geq u(c) - \rho V(k_0)$$

which implies, taking the sup for $c \geq 0$:

$$\rho V(k_0) + H\left(k_0, \varphi'(k_0)\right) \geq 0$$

Secondly we show that V is a viscosity subsolution of HJB.

Let $\varphi \in \mathcal{C}^1((0,+\infty),\mathbb{R})$ and $k_0 > 0$ be a local maximum point of $V - \varphi$, so that

$$V(k_0) - V \geq \varphi(k_0) - \varphi \tag{52}$$

in a proper neighborhood $\mathcal{N}(k_0)$ of k_0.

Fix $\epsilon > 0$ and, using the definition of V, define a family of controls $\left(c_{T,\epsilon}\right)_{T>0} \subseteq \Lambda(k_0)$ such that for every $T > 0$:

$$V(k_0) - T\epsilon \leq U\left(c_{T,\epsilon}; k_0\right). \tag{53}$$

Now take $\left(c_{T,\epsilon}\right)^T$ as in Lemma 3.2 and set $\bar{c}_{T,\epsilon} := \left(c_{T,\epsilon}\right)^T$ for simplicity of notation (so that $\bar{c}_{T,\epsilon} \in \Lambda(k_0)$). We have:

$$\begin{aligned} V(k_0) - T\epsilon \le U\left(c_{T,\epsilon}; k_0\right) \le U\left(\bar{c}_{T,\epsilon}; k_0\right) \\ &= \int_0^T e^{-\rho t} u\left(\bar{c}_{T,\epsilon}(t)\right) \mathrm{d}t + e^{-\rho T} \int_T^{+\infty} e^{-\rho(s-T)} u\left(\bar{c}_{T,\epsilon}(s-T+T)\right) \mathrm{d}s \\ &= \int_0^T e^{-\rho t} u\left(\bar{c}_{T,\epsilon}(t)\right) \mathrm{d}t + e^{-\rho T} U\left(\bar{c}_{T,\epsilon}(\cdot + T); k\left(T; k_0, \bar{c}_{T,\epsilon}\right)\right) \\ &\le \int_0^T e^{-\rho t} u\left(\bar{c}_{T,\epsilon}(t)\right) \mathrm{d}t + e^{-\rho T} V\left(k\left(T; k_0, \bar{c}_{T,\epsilon}\right)\right) \end{aligned}$$

where we have used Remark 7.1.

By Lemma 7.1 we have for $T > 0$ sufficiently small (say $T < \hat{T}$),

$$k\left(T; k_0, \bar{c}_{T,\epsilon}\right) \in \mathcal{N}(k_0).$$

Hence, setting $\bar{k}_{T,\epsilon} := k\left(\cdot; k_0, \bar{c}_{T,\epsilon}\right)$, for every $T < \hat{T}$, we have by (52):

$$\begin{aligned} \varphi(k_0) - \varphi\left(\bar{k}_{T,\epsilon}(T)\right) - e^{-\rho T} V\left(\bar{k}_{T,\epsilon}(T)\right) &\le V(k_0) - V\left(\bar{k}_{T,\epsilon}(T)\right) - e^{-\rho T} V\left(\bar{k}_{T,\epsilon}(T)\right) \\ &\le \int_0^T e^{-\rho t} u\left(\bar{c}_{T,\epsilon}(t)\right) \mathrm{d}t - V\left(\bar{k}_{T,\epsilon}(T)\right) + T\epsilon \end{aligned}$$

which implies

$$\begin{aligned} &\int_0^T - \left\{\varphi'\left(\bar{k}_{T,\epsilon}(t)\right)\left[F\left(\bar{k}_{T,\epsilon}(t)\right) - \bar{c}_{T,\epsilon}(t)\right] + e^{-\rho t} u\left(\bar{c}_{T,\epsilon}(t)\right)\right\} \mathrm{d}t \\ &\le V\left(\bar{k}_{T,\epsilon}(T)\right)\left[e^{-\rho T} - 1\right] + T\epsilon. \end{aligned} \tag{54}$$

Observe that the integral at the left hand member is bigger than:

$$\begin{aligned} &\int_0^T - \left\{\left[\varphi'(k_0) + \omega_1(t)\right]\left[F(k_0) - \bar{c}_{T,\epsilon}(t) + \omega_2(t)\right] + u\left(\bar{c}_{T,\epsilon}(t)\right)\right\} \mathrm{d}t \\ &= \int_0^T - \left\{\varphi'(k_0)\left[F(k_0) - \bar{c}_{T,\epsilon}(t)\right] + u\left(\bar{c}_{T,\epsilon}(t)\right)\right\} \mathrm{d}t \\ &\quad + \int_0^T - \left\{\varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t + \omega_1(t)\left[\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)\right]\right\} \mathrm{d}t \end{aligned} \tag{55}$$

where ω_1, ω_2 are functions which are continuous in a neighborhood of 0 and satisfy:

$$\omega_1(0) = \omega_2(0) = 0.$$

This implies, for $T < 1$:

$$\begin{aligned}
&\left|\int_0^T \varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t + \int_0^T \omega_1(t)\left[\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)\right]\mathrm{d}t\right| \\
&\le \left|\varphi'(k_0)\right| o_1(T) + o_2(T) + \int_0^T |\omega_1(t)|\left[F(k_0) + \bar{c}_{T,\epsilon}(t)\right]\mathrm{d}t \\
&\le \left|\varphi'(k_0)\right| o_1(T) + o_2(T) + \left[F(k_0) + N(k_0, T)\right] o_3(T) \\
&\le \left|\varphi'(k_0)\right| o_1(T) + o_2(T) + \left[F(k_0) + N(k_0, 1)\right] o_3(T)
\end{aligned}$$

where

$$\lim_{T\to 0} \frac{o_i(T)}{T} = 0$$

for $i = 1, 2, 3$. Observe that this is true even if the o_is depend on T, by Lemma 7.1. For instance,

$$\begin{aligned}
|o_1(T)| &= \left|\int_0^T \omega_2(t)\,\mathrm{d}t\right| \le T \max_{[0,T]} |\omega_2| = T\,|\omega_2(\tau_T)| \\
&= T\left|F\left(\bar{k}_{T,\epsilon}(\tau_T)\right) - F(k_0)\right| \\
&\le \overline{M} T \left|\bar{k}_{T,\epsilon}(\tau_T) - k_0\right| \le \overline{M} T^2 e^{\bar{M}\tau_T}\left[F(k_0) + N(k_0, 1)\right]
\end{aligned}$$

Moreover, by the fact that $V \in \mathcal{C}^+([0, +\infty), \mathbb{R})$ and by Remark 7.3, we have for any $t \in [0, T]$:

$$\begin{aligned}
-\left\{\varphi'(k_0)\left[F(k_0) - \bar{c}_{T,\epsilon}(t)\right] + u\left(\bar{c}_{T,\epsilon}(t)\right)\right\} &\ge -\sup_{c\ge 0}\left\{\varphi'(k_0)\left[F(k_0) - c\right] + u(c)\right\} \\
&= H\left(k_0, \varphi'(k_0)\right) > -\infty,
\end{aligned}$$

by which we can write:

$$\int_0^T -\left\{\varphi'(k_0)\left[F(k_0) - \bar{c}_{T,\epsilon}(t)\right] + u\left(\bar{c}_{T,\epsilon}(t)\right)\right\}\mathrm{d}t \;\ge\; T \cdot H\left(k_0, \varphi'(k_0)\right).$$

Hence, by (54) and (55):

$$\begin{aligned}
&V\left(\bar{k}_{T,\epsilon}(T)\right)\left[e^{-\rho T} - 1\right] + T\epsilon \\
&\ge -\int_0^T \left\{\varphi'(k_0)\left[F(k_0) - \bar{c}_{T,\epsilon}(t)\right] + u\left(\bar{c}_{T,\epsilon}(t)\right)\right\}\mathrm{d}t \\
&\quad + \int_0^T -\left\{\varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t + \omega_1(t)\left[\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)\right]\mathrm{d}t\right\} \\
&\ge T \cdot H\left(k_0, \varphi'(k_0)\right) + o_{T\to 0}(T)
\end{aligned}$$

for any $0 < T < 1, \hat{T}$. Hence dividing by T, and then letting $T \to 0$, again by Lemma 7.1 and the continuity of V we obtain:

$$-\rho V(k_0) + \epsilon \geq H\left(k_0, \varphi'(k_0)\right)$$

which proves the assertion since ϵ is arbitrary. □

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