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A trace regularity result for thermoelastic equations with application to optimal boundary control

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Abstract

We consider a mixed problem for a Kirchoff thermoelastic plate model with clamped boundary conditions. We establish a sharp regularity result for the outer normal derivative of the thermal velocity on the boundary. The proof, based upon interpolation techniques, benefits from the exceptional regularity of traces of solutions to the elastic Kirchoff equation. This result, which complements recent results obtained by the second and third authors, is critical in the study of optimal control problems associated with the thermoelastic system when subject to *thermal* boundary control. Indeed, the present regularity estimate can be interpreted as a suitable control-theoretic property of the corresponding abstract dynamics, which is crucial to guarantee well-posedness for the associated differential Riccati equations.

Keywords: Thermoelastic equations; Boundary regularity; Optimal control; Riccati equations

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1. Introduction and main result

The main goal of this paper is to provide a sharp regularity result for the boundary trace $\frac{\partial\theta_t}{\partial\nu}$ of the solution $(w(t,x),\theta(t,x))$ to a homogeneous version of a thermoelastic *mixed* (initial and boundary value) problem. We point out at the outset that the present investigation which continues and pushes further the one initiated in [6]—is motivated by the analysis of well-posedness of Differential Riccati Equations arising in the quadratic optimal control problem for thermoelastic plates subject to boundary thermal control. The mathematical description of the mixed PDE problem is given below.

Let Ω be a bounded domain of $\mathbb{R}^2$, with smooth boundary $\Gamma := \partial\Omega$ and let $T > 0$ be given. We consider the following PDE model of a thermoelastic plate in the variables $w(t,x)$ (vertical displacement) and $\theta(t,x)$ (temperature):

$$\begin{cases} w_{tt} - \rho\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0 & \text{in } Q := (0,T] \times \Omega, \\ \theta_t - \Delta\theta - \Delta w_t = 0 & \text{in } Q, \\ w = \frac{\partial w}{\partial\nu} = 0 \quad \text{(clamped B.C.)} & \text{on } \Sigma := (0,T] \times \Gamma, \\ \theta = u \quad \text{(Dirichlet boundary control)} & \text{on } \Sigma, \\ w(0,\cdot) = w^0,\ w_t(0,\cdot) = w^1;\ \theta(0,\cdot) = \theta^0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\rho > 0$, hence the elastic equation in w is of Kirchoff type.[1] In the description of the Boundary Conditions (B.C.) associated with the thermoelastic system (1.1)—here *clamped*, a physically relevant and technically challenging case—ν will denote the unit outward normal to the curve Γ. The dynamics of the plate is influenced by boundary control u acting on the temperature (thermal control).

With (1.1), we associate the following quadratic cost functional to be minimized over all $u \in L^2(0,T;L^2(\Gamma))$:

$$J(w,w_t,\theta;u) = \int_0^T \int_\Omega \left(|\Delta w|^2 + |\nabla w_t|^2 + |\theta|^2\right) dx\,dt + \int_0^T \int_\Gamma |u|^2\,ds\,dt. \tag{1.2}$$

As it is known, solvability of Riccati equations connected to PDEs with boundary/point control is a challenging issue, and the corresponding theories depend strongly on the type of dynamics involved. More precisely, since the control operator B arising in the abstract formulation

$$y'(t) = Ay(t) + By(t) \tag{1.3}$$

of the boundary value problem is intrinsically unbounded, the main difficulty is a good definition of the so called *gain operator* $B^*P(t)$ that appears as a nonlinear term in the Riccati equation, which is

$$\begin{aligned} &\frac{d}{dt}\big(P(t)x,y\big)_Y + \big(P(t)x,Ay\big)_Y + \big(P(t)Ax,y\big)_Y + (x,y)_Y \\ &\quad -\big(B^*P(t)x,B^*P(t)y\big)_Y = 0, \quad x,y \in \mathcal{D}(A). \end{aligned} \tag{1.4}$$

[1] The term $\rho\Delta w_{tt}$ in the elastic equation accounts for rotational forces, with the constant $\rho > 0$ proportional to the square of thickness of the plate; see [9].

We recall that while optimal results are achieved in the case of parabolic-like dynamics (i.e., when the strongly continuous semigroup e^{At} of the free dynamics is, moreover, analytic), there are simple hyperbolic PDE problems for which the gain operator is not even densely defined, and hence the Riccati equations have no meaning in their classical form. (There is a large literature on the linear-quadratic problem for PDEs with boundary/point control: for detailed expositions of the theories pertaining to both the abstract 'parabolic' and 'hyperbolic' classes, see the treatises [5,12]. More specifically, refinements of the Riccati theory pertaining to the 'hyperbolic' class, formerly developed in [7], are found in [4,18], and lastly in [19]; another related reference is [20].)

On the other hand, in recent years a distinct class of control systems has been singled out, for which the Riccati theory is also complete. This class is characterized by a singular behaviour at the origin of the operator $e^{At}B$, and is referred to as the class of systems which yield "singular estimates." Assuming this condition, a Riccati theory has been developed, which substantially extends the analytic set-up [2,10,14,15]. Typical models which satisfy singular estimates are coupled PDE systems consisting of a combination of hyperbolic and parabolic dynamics. We stress that these estimates reflect certain—hidden—regularity of the traces of solutions to the uncontrolled dynamics. The reference [10] specifically explores the mathematical properties of systems of coupled hyperbolic/parabolic PDEs which arise in modern technologies, along with the Riccati theory for the associated optimal control problems.

The system of thermoelasticity displays as well a hyperbolic/parabolic coupling. Unlike the case $\rho = 0$, where the system (1.1) becomes analytic [12,17], here, due to the presence of rotational inertia ($\rho > 0$), the free dynamics is 'predominantly hyperbolic' [13]. Nevertheless, when either *hinged* boundary conditions are associated with the thermoelastic model, or in the case the system is subject to Neumann—rather than Dirichlet—thermal control, singular estimates unexpectedly hold (see [6,11]). Thus, it is important to emphasize that the techniques used in the proofs of those references do not generalize to the present situation (clamped B.C./Dirichlet boundary control).

This particular difficulty became a motivation for seeking more general conditions than singular estimates. In [6] it is shown that in the case of problem (1.1) the operator $B^*e^{A^*t}$ can be represented as the sum of two operators $F(t) + G(t)$, where only $F(t)$ satisfies a singular estimate; more precisely

- $\|F(t)\|_{\mathcal{L}(Y,U)} \leqslant \frac{c_T}{t^{\frac{3}{4}+\delta}}$, $0 < t \leqslant T$ (with arbitrarily small $\delta > 0$),
- while $G(\cdot) \in \mathcal{L}(Y, L^p(0, T; U))$ for all $p \geqslant 1$, and
- $G(\cdot) \in \mathcal{L}(\mathcal{D}(A^{*\varepsilon}), C([0, T]; U))$ for any $\varepsilon \in (0, 1)$.

The significant result contained in this paper is that, moreover,

$$\text{there exists } \varepsilon \in (0,1) \quad \text{such that} \quad B^*e^{A^*t}A^{*\varepsilon} \in \mathcal{L}\big(Y, L^q(0, T; U)\big) \tag{1.5}$$

for some $q \in (1, 2)$. In fact, the above condition enables us to show that the Riccati equation associated with problem (1.1)–(1.2) is well posed; in particular, that the gain operator $B^*P(\cdot)$ is densely defined on $\mathcal{D}(A^{*\varepsilon})$.

A novel theory of the quadratic optimal control problem for abstract dynamics of the form (1.3), under the aforementioned assumptions, is developed in [1]. The present paper is focused on the crucial issue of establishing the trace regularity estimate which is equivalent—in PDEs terms—to the abstract condition (1.5). The trace estimate for the thermal component obtained in Theorem 1.1 is also of independent interest in PDE theory of thermoelasticity. The main result of this paper can be stated as follows.

Theorem 1.1. *Consider the thermoelastic problem* (1.1) *with* $u \equiv 0$, *and assume*

$$(w^0, w^1, \theta^0) \in \left[H^{3-\varepsilon}(\Omega) \cap H_0^2(\Omega)\right] \times H_0^{2-\varepsilon}(\Omega) \times \left[H^{2-2\varepsilon}(\Omega) \cap H_0^1(\Omega)\right],$$

with $\varepsilon \in (0, \frac{1}{4})$. *Then, the corresponding solution satisfies, for some* $q \in (1,2)$,

$$\left.\frac{\partial \theta_t}{\partial \nu}\right|_{\Gamma} \in L^q\big(0, T; L^2(\Gamma)\big)$$

continuously, that is

$$\left\| \frac{\partial \theta_t}{\partial \nu} \right\|_{L^q(0,T;L^2(\Gamma))} \leqslant C_T \left\{ \|w^0\|_{H^{3-\varepsilon}(\Omega)\cap H_0^2(\Omega)} + \|w^1\|_{H_0^{2-\varepsilon}(\Omega)} + \|\theta^0\|_{H^{2-2\varepsilon}(\Omega)\cap H_0^1(\Omega)} \right\}. \tag{1.6}$$

The exponent q *will depend on* ε: *more precisely, given* $\varepsilon \in (0, \frac{1}{4})$, *one has*

$$1 < q < \min\left\{ \frac{8}{7}, \frac{4}{3+4\varepsilon} \right\}. \tag{1.7}$$

Remark 1.2. It is important to emphasize that the *trace* regularity result provided by Theorem 1.1 does not follow from optimal *interior* regularity for the solutions to the (homogeneous) thermoelastic system. Indeed, the maximal interior regularity that can be achieved by exploiting the parabolic effect for the thermal component θ is $\nabla\theta_t \in L^2(0, T; H^{-1-\varepsilon}(\Omega))$, which does not allow us to give a meaning to $\frac{\partial \theta_t}{\partial \nu}$ on Γ. Even a formal application of trace theory would lead to $\frac{\partial \theta_t}{\partial \nu}\big|_\Gamma \in L^2(0, T; H^{-3/2-\varepsilon}(\Gamma))$. Thus, Theorem 1.1 establishes an 'additional' space regularity for the outer normal derivative of the thermal velocity on the boundary, with a gain in space variable which is strictly larger than $3/2$ of derivative.

2. Abstract setting. Preliminaries

In this section we recall from [6] the essential elements of the abstract set-up for the coupled system (1.1), and we introduce some preliminary material in view of the proof of the main result.

2.1. Abstract setting

Let A_D and $\mathcal{A}$ be the realizations of the operator $-\Delta$ with Dirichlet boundary conditions (B.C.) and of the bilaplacian Δ^2 with clamped B.C., respectively, i.e.,

$$A_D f := -\Delta f, \qquad \mathcal{D}(A_D) = H^2(\Omega) \cap H_0^1(\Omega); \tag{2.1}$$

$$\mathcal{A} f := \Delta^2 f, \qquad \mathcal{D}(\mathcal{A}) = \left\{ f \in H^4(\Omega)\colon\ f|_\Gamma = \frac{\partial f}{\partial \nu}\bigg|_\Gamma = 0 \right\}. \tag{2.2}$$

Then, the stiffness operator is given by

$$\mathcal{M} := I + \rho A_D. \tag{2.3}$$

It is well known that the operators A_D and $\mathcal{A}$, defined in (2.1) and (2.2) are selfadjoint, positive operators on $L^2(\Omega)$; a fortiori, the same holds for $\mathcal{M}$.

Remark 2.1. By definition,

$$\mathcal{M}^{-1} A_D = (I + \rho A_D)^{-1} A_D \equiv \frac{1}{\rho} I - \frac{1}{\rho} \mathcal{M}^{-1},$$

and hence

$$\mathcal{M}^{-1} A_D \in \mathcal{L}\big(H^s(\Omega)\big) \quad \forall s \in \mathbb{R}.$$

This property will be used repeatedly throughout the paper.

The fractional powers of the operator A_D are well defined: for the characterization of domains of fractional powers of positive operators the reader is referred to [8]; see also [12].

With D we shall denote the Dirichlet map from $L^2(\Gamma)$ to $L^2(\Omega)$ [16]. Among the properties of this operator, we briefly recall the ones which will be critically used in the sequel. We have that

$$D \text{ is continuous:} \quad H^s(\Gamma) \to H^{s+\frac{1}{2}}(\Omega), \quad s \in \mathbb{R},$$

so that in particular

$$A_D^{\frac{1}{4}-\delta} D \in \mathcal{L}\big(L^2(\Gamma), L^2(\Omega)\big) \quad \text{for any } \delta > 0. \tag{2.4}$$

Moreover, the following trace result [12, Lemma 3.1.1, p. 181] holds true:

$$D^* A_D \varphi = \frac{\partial \varphi}{\partial \nu}\bigg|_\Gamma \quad \forall \varphi \in H^{\frac{3}{2}+\delta}(\Omega) \cap H_0^1(\Omega),\ \delta > 0. \tag{2.5}$$

With the above notation, the abstract representation of the PDE system (1.1) is the following second-order control system:

$$\mathcal{M} w_{tt} + \mathcal{A} w - A_D \theta = -A_D D u, \tag{2.6a}$$

$$\theta_t + A_D \theta + A_D w_t = A_D D u, \tag{2.6b}$$

$$w(0) = w^0, \quad w_t(0) = w^1; \qquad \theta(0) = \theta^0. \tag{2.6c}$$

The natural function spaces Y_w for the plate component $[w, w_t]$ and Y_θ for the thermal component of system (2.6) are given, respectively, by

$$Y_w := \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{M}^{1/2}) = H_0^2(\Omega) \times H_0^1(\Omega),$$

$$Y_\theta := L_2(\Omega).$$

Then, by introducing the state space Y for problem (2.6), namely

$$\begin{aligned} Y := Y_w \times Y_\theta &= \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{M}^{1/2}) \times L_2(\Omega) \\ &= H_0^2(\Omega) \times H_0^1(\Omega) \times L_2(\Omega), \end{aligned} \tag{2.7}$$

the coupled system (2.6) can be rewritten as a first-order system in the variable $y(t) = [w(t), w_t(t), \theta(t)]$:

$$\begin{cases} y'(t) = Ay(t) + Bu(t), & t \in [0, T], \\ y(0) = y_0, \end{cases} \tag{2.8}$$

with operators A (free dynamic operator) and B (control operator) explicitly identified. The expressions of the *unbounded* operators A and B are found, e.g., in both [6, Section 2] and [12, Appendix 3J], hence will be omitted. Since the adjoint operators A^* and B^* will be needed below, we rather recall from [6, Section 2] these. The Y-adjoint operator A^* of A is given by

$$A^* = \begin{pmatrix} 0 & -I & 0 \\ \mathcal{M}^{-1}\mathcal{A} & 0 & -\mathcal{M}^{-1}A_D \\ 0 & A_D & -A_D \end{pmatrix}, \tag{2.9}$$

with $\mathcal{D}(A^*) \equiv \mathcal{D}(A)$, i.e.,

$$\begin{aligned} \mathcal{D}(A^*) &= \mathcal{D}(\mathcal{A}^{3/4}) \times \big[\mathcal{D}(\mathcal{A}^{1/2}) \cap \mathcal{D}(A_D)\big] \times \mathcal{D}(A_D) \\ &= \big[H^3(\Omega) \cap H_0^2(\Omega)\big] \times H_0^2(\Omega) \times \big[H^2(\Omega) \cap H_0^1(\Omega)\big]. \end{aligned} \tag{2.10}$$

On the other hand, by using the trace result (2.5), it can be shown that the adjoint operator B^* is such that

$$B^* \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{\partial}{\partial \nu}(y_3 - y_2)\bigg|_\Gamma . \tag{2.11}$$

2.2. Preliminary observations

As explained in Section 1, motivated by the quadratic optimal control problem associated with the abstract dynamics (2.8), the goal of the present paper is to show that the following condition is satisfied:

> there exists $\varepsilon > 0$ such that the (closable) operator $B^* e^{A^* t} A^{*\varepsilon}$ admits a continuous extension (which may then be denoted by the same symbol) satisfying
>
> $$B^* e^{A^* t} A^{*\varepsilon} \text{continuous:} \quad Y \to L^q\big(0, T; L^2(\Gamma)\big) \quad \text{for some } q \in (1, 2),$$
>
> that is, there exists a constant $C > 0$ such that
>
> $$\|B^* e^{A^* \cdot} A^{*\varepsilon} y\|_{L^q(0,T;L^2(\Gamma))} \leqslant C\|y\|_Y, \quad y \in Y. \tag{2.12}$$

We may interpret the term $B^* e^{A^* t} A^{*\varepsilon} y$ as follows:

$$B^* e^{A^* t} A^{*\varepsilon} y = B^* \frac{d}{dt}(A^{*-1} e^{A^* t}) A^{*\varepsilon} y = B^* \frac{d}{dt}(e^{A^* t} A^{*\varepsilon - 1} y);$$

hence, we may equivalently focus on the term

$$B^* \frac{d}{dt} e^{A^* t} z, \quad \text{with } z \in \mathcal{D}(A^{*1-\varepsilon}).$$

Correspondingly, showing (2.12) becomes equivalent to prove that there exists $\varepsilon > 0$ such that

$$\left\| B^* \frac{d}{dt} e^{A^* t} z \right\|_{L^q(0,T;L^2(\Gamma))} \leqslant C \|z\|_{\mathcal{D}(A^{*1-\varepsilon})}, \quad z \in \mathcal{D}(A^{*1-\varepsilon}), \tag{2.13}$$

for some $q \in (1, 2)$.

Remark 2.2. More precisely, we will prove that given any $\varepsilon \in (0, 1/4)$, inequality (2.13) is satisfied, with $q \in (1, 2)$ explicitly determined by ε (see (1.7)). This will yield, as a consequence, the statement of Theorem 1.1.

2.3. Notation

We will use $\|\cdot\|_{0,\Omega}$ and $(\cdot,\cdot)_{0,\Omega}$ to denote the norm and the inner product in $H^0(\Omega) = L^2(\Omega)$.

3. Proof of Theorem 1.1

We proceed in several steps.

Step 1 (*Start*). Since we seek to compute explicitly $B^* \frac{d}{dt} e^{A^* t} z$, with $z = (w^0, w^1, \theta^0)$ in $\mathcal{D}(A^{*1-\varepsilon})$, we preliminary observe that $\frac{d}{dt} e^{A^* t} z$ represents the time derivative of the solution to the abstract Cauchy problem

$$\begin{cases} y' = A^* y, \\ y(0) = z \end{cases}$$

(this is well defined at least as an element of $[\mathcal{D}(A)]'$, the dual space of $\mathcal{D}(A)$ with respect to Y). Then, as already described in [6], by using the definition of the adjoint operator A^*, it is simple to show that $e^{A^* t} z = (w(t), -w_t(t), \theta(t))$, where $(w(t), w_t(t), \theta(t))$ solves the uncontrolled version of the thermoelastic system (2.6), with a bit different initial condition, namely

$$w_{tt} + \mathcal{M}^{-1} \mathcal{A} w - \mathcal{M}^{-1} A_D \theta = 0, \tag{3.1a}$$

$$\theta_t + A_D \theta + A_D w_t = 0, \tag{3.1b}$$

$$w(0) = w^0, \quad w_t(0) = -w^1; \qquad \theta(0) = \theta^0. \tag{3.1c}$$

Thus, applying the definition (2.11) of B^* and taking into account that $\frac{\partial w_{tt}}{\partial \nu} = 0$ on Γ, as it follows from the *clamped* B.C. in (1.1), we find that

$$B^*\frac{d}{dt}e^{A^*t}z = \frac{\partial}{\partial\nu}\theta_t(t)\bigg|_\Gamma \tag{3.2}$$

$$= \frac{\partial}{\partial\nu}\left[e^{-A_Dt}\theta_t(0) - \int_0^t A_D e^{-A_D(t-s)}\mathcal{M}^{-1}\big(A_D\theta(s) - \mathcal{A}w(s)\big)\,ds\right]\bigg|_\Gamma. \tag{3.3}$$

In the last step, we have used both Eqs. (3.1a) and (3.1b) to compute the right-hand side of (3.2). In (3.3) we integrate by parts in t, and exploit once more the *clamped* B.C., thus obtaining

$$\frac{\partial}{\partial\nu}\theta_t(t)\bigg|_\Gamma = \frac{\partial}{\partial\nu}\Big[e^{-A_Dt}\big(\theta_t(0) + w_{tt}(0)\big)\Big]\bigg|_\Gamma$$
$$+ \frac{\partial}{\partial\nu}\left[\int_0^t e^{-A_D(t-s)}\mathcal{M}^{-1}\big(A_D\theta_t(s) - \mathcal{A}w_t(s)\big)\,ds\right]\bigg|_\Gamma. \tag{3.4}$$

Notice now that the term $\mathcal{A}w_t$ at the right of (3.4) can be rewritten as follows:

$$\mathcal{A}w_t = \Delta(\Delta w_t) = \Delta\big(\Delta w_t - D(\Delta w_t|_\Gamma)\big) = -A_D\big(\Delta w_t - D(\Delta w_t|_\Gamma)\big).$$

With this, combining (3.2) with (3.4), and invoking once more the trace theorem (2.5), we finally get

$$\begin{aligned} B^*\frac{d}{dt}e^{A^*t}z &= D^*A_D\bigg\{e^{-A_Dt}\big(\theta_t(0) + w_{tt}(0)\big) \\ &\quad + \int_0^t e^{-A_D(t-s)}\mathcal{M}^{-1}A_D\big(\theta_t(s) + \Delta w_t(s)\big)\,ds \\ &\quad - \int_0^t e^{-A_D(t-s)}(\mathcal{M}^{-1}A_D)D\big(\Delta w_t(s)|_\Gamma\big)\,ds\bigg\} \\ &=: T_1(t) + T_2(t) - T_3(t). \end{aligned} \tag{3.5}$$

The formula (3.5) above is the key of the subsequent analysis.

Step 2 (*Interior regularity*). Before we proceed with the analysis of each summand at the right of (3.5), it is necessary to determine the interior regularity for each component of the solution $(w(t), w_t(t), \theta(t))$ corresponding to initial data $(w_0, w_1, \theta_0) = z \in \mathcal{D}({A^*}^{1-\varepsilon})$. Interpolation between (2.7) and (2.10) yields, for $\theta \in (1/2, 1)$,

$$\mathcal{D}({A^*}^\theta) = \big[H^{2+\theta}(\Omega) \cap H_0^2(\Omega)\big] \times H_0^{1+\theta}(\Omega) \times \big[H^{2\theta}(\Omega) \cap H_0^1(\Omega)\big]. \tag{3.6}$$

Therefore, standard semigroup theory—combined with interpolation arguments—implies the following regularity for the solution $(w(t), w_t(t), \theta(t))$ corresponding to initial data in $\mathcal{D}({A^*}^{1-\varepsilon})$:

$$w \in C\big([0,T]; H^{3-\varepsilon}(\Omega) \cap H_0^2(\Omega)\big), \tag{3.7a}$$

$$w_t \in C\big([0,T]; H_0^{2-\varepsilon}(\Omega)\big) \subset C\big([0,T]; \mathcal{D}\big(A_D^{1-\varepsilon/2}\big)\big), \tag{3.7b}$$

$$\theta \in C\big([0,T]; H^{2-2\varepsilon}(\Omega) \cap H_0^1(\Omega)\big) \equiv C\big([0,T]; \mathcal{D}\big(A_D^{1-\varepsilon}\big)\big). \tag{3.7c}$$

Notice that, moreover,

$$\Delta w|_\Gamma \in C\big([0,T]; H^{\frac{1}{2}-\varepsilon}(\Gamma)\big), \tag{3.8}$$

as it readily follows from (3.7a) via trace theory.

Remark 3.1. The regularity result in (3.8) follows, via trace theorem, from the interior regularity of the mechanical variable w. A much stronger version of related result will be given later in Lemma 3.4. Indeed, the trace regularity in Lemma 3.4 is a "hidden regularity" type of result that is not implied by the interior regularity of solutions to thermoelasticity.

Because T_1 in (3.5) involves the terms $\theta_t(0)$ and $w_{tt}(0)$, a separate analysis of the velocity terms' regularity is also required. In fact, the next step will rely on the following lemma.

Lemma 3.2. *Let* $(w(t), w_t(t), \theta(t))$ *be the solution to the system* (3.1) *corresponding to initial data* $(w^0, w^1, \theta^0) = z \in D(A^{*1-\varepsilon})$. *Then, we have*

$$w_{tt} \in C\big([0,T]; H^{\varepsilon}(\Omega)\big) \subseteq C\big([0,T]; \mathcal{D}\big(A_D^{\varepsilon/2}\big)\big), \tag{3.9}$$

$$\|w_{tt}\|_{C([0,T];H^\varepsilon(\Omega))} \leqslant C\|z\|_{D(A^{*1-\varepsilon})}, \quad 0 < \varepsilon < \frac{1}{2}; \tag{3.10}$$

and

$$\theta_t \in C\big([0,T]; \big[\mathcal{D}\big(A_D^{\varepsilon}\big)\big]'\big), \tag{3.11}$$

$$\|\theta_t\|_{C([0,T];[\mathcal{D}(A_D^\varepsilon)]')} \leqslant C\|z\|_{D(A^{*1-\varepsilon})}, \quad 0 < \varepsilon < \frac{1}{2}. \tag{3.12}$$

Proof. Our starting point is the plate equation (3.1a), that is

$$w_{tt} = -\mathcal{M}^{-1}\Delta^2 w - \mathcal{M}^{-1}\Delta\theta. \tag{3.13}$$

We multiply (3.13) by $\varphi \in L^2(\Omega)$ and integrate by parts, by using Green's formulas. Then, exploiting that $\mathcal{M}^{-1}\varphi \equiv 0$ on Γ, it is not difficult to obtain the following identity:

$$\begin{aligned}\big(w_{tt}(t), \varphi\big)_{0,\Omega} = {}& \left(\Delta w(t), \frac{\partial}{\partial \nu}\mathcal{M}^{-1}\varphi\right)_{0,\Gamma} - \big(\Delta w(t), \Delta\mathcal{M}^{-1}\varphi\big)_{0,\Omega} \\ & + \big(\nabla\theta(t), \nabla\mathcal{M}^{-1}\varphi\big)_{0,\Omega}.\end{aligned} \tag{3.14}$$

From a careful analysis of each of the three summands in the right-hand side of (3.14), on the basis of the regularity asserted in (3.7a) and (3.7c), it follows that $w_{tt}(t)$ can be extended from $L^2(\Omega)$ to the dual space $[H^{-\varepsilon}(\Omega)]'$, that is $H^\varepsilon(\Omega)$. More precisely, the interior regularity result in (3.9) holds, continuously with respect to initial data, that is (3.10) is satisfied.

As for the thermal velocity, its regularity follows at once by rewriting equation (3.1b) as follows:

$$\theta_t = -A_D^{\varepsilon}\big(A_D^{1-\varepsilon}\theta\big) - A_D^{\varepsilon/2}\big(A_D^{1-\varepsilon/2}w_t\big). \tag{3.15}$$

Thus, by using the memberships in (3.7b) and (3.7c), it is immediately seen that (3.11) holds true, together with (3.12). □

Step 3 (*Analysis of terms T_1 and T_2*). We are now able to establish the following result.

Proposition 3.3. *With reference to the summands T_1 and T_2 at the right of* (3.5)*, the following estimates hold true* (*the former for arbitrarily small $\delta > 0$*):

$$\big\|T_1(t)\big\|_{0,\Gamma} \leqslant \frac{C}{t^{3/4+\varepsilon+\delta}}\|z\|_{D(A^{*1-\varepsilon})}, \quad \textit{so that} \tag{3.16}$$

$$\forall\varepsilon < \frac{1}{4},\ \exists q \in (1,2): \quad \|T_1\|_{L^q(0,T;L^2(\Gamma))} \leqslant C_\varepsilon\|z\|_{D(A^{*1-\varepsilon})}; \tag{3.17}$$

$$\|T_2\|_{L^\infty(0,T;L^2(\Gamma))} \leqslant C_\varepsilon\|z\|_{D(A^{*1-\varepsilon})} \quad \forall\varepsilon \in \left(0,\frac{1}{4}\right). \tag{3.18}$$

In particular, given $\varepsilon \in (0,\frac{1}{4})$, the regularity in (3.17) *is valid with any exponent q such that*

$$1 < q < \frac{4}{3+4\varepsilon}.$$

Proof. Let us recall from (3.5) that

$$T_1(t) = D^*A_D\big(e^{-A_Dt}\big[\theta_t(0) + w_{tt}(0)\big]\big). \tag{3.19}$$

We will examine the summands at the right of (3.19) separately. The term $T_{11}(t) := D^*A_De^{-A_Dt}\theta_t(0)$ can be split as

$$T_{11}(t) = \big(D^*A_D^{1/4-\delta}\big)A_D^{3/4+\delta}e^{-A_Dt}A_D^{\varepsilon}\big[A_D^{-\varepsilon}\theta_t(0)\big].$$

Then, using (2.4) first, and combining the regularity estimate (3.12) with the usual estimates pertaining to analytic semigroups, we get, for any $\delta \in (0,1/4)$,

$$\begin{aligned}\big\|T_{11}(t)\big\|_{0,\Gamma} &\leqslant c\big\|A_D^{3/4+\varepsilon+\delta}e^{-A_Dt}\big[A_D^{-\varepsilon}\theta_t(0)\big]\big\|_{0,\Omega} \leqslant \frac{c}{t^{3/4+\varepsilon+\delta}}\big\|A_D^{-\varepsilon}\theta_t(0)\big\|_{0,\Omega}\\ &\leqslant \frac{C}{t^{3/4+\varepsilon+\delta}}\|z\|_{D(A^{*1-\varepsilon})}.\end{aligned} \tag{3.20}$$

Therefore, since we aim to obtain that $T_1 \in L^q(0,T;L^2(\Gamma))$ for some $q \in (1,2)$, we need to assume $\varepsilon < 1/4$ and we can take any $\delta \in (0,1/4-\varepsilon)$.

Similarly, from (3.9) and (3.10) it readily follows that

$$\begin{aligned}\big\|T_{12}(t)\big\|_{0,\Gamma} &:= \big\|D^*A_De^{-A_Dt}w_{tt}(0)\big\|_{0,\Gamma}\\ &= \big\|\big(D^*A_D^{1/4-\delta}\big)A_D^{3/4+\delta}e^{-A_Dt}A_D^{-\varepsilon/2}\big[A_D^{\varepsilon/2}w_{tt}(0)\big]\big\|_{0,\Gamma}\\ &\leqslant c\big\|A_D^{3/4-\varepsilon/2+\delta}e^{-A_Dt}\big[A_D^{\varepsilon/2}w_{tt}(0)\big]\big\|_{0,\Omega}\end{aligned}$$

$$\leqslant \frac{c}{t^{3/4-\varepsilon/2+\delta}} \left\| A_D^{\varepsilon/2} w_{tt}(0) \right\|_{0,\Omega}$$
$$\leqslant \frac{C}{t^{3/4-\varepsilon/2+\delta}} \|z\|_{D(A^{*1-\varepsilon})}. \tag{3.21}$$

Summing up (3.20) and (3.21) results in the (pointwise in t) estimate (3.16), which in turn implies the regularity in (3.17).

Next, let us recall from (3.5) that

$$T_2(t) = D^* A_D \int_0^t e^{-A_D(t-s)} \mathcal{M}^{-1} A_D \big(\theta_t(s) + \Delta w_t(s)\big)\, ds.$$

We shall use both the regularity of θ_t previously established in (3.11), and that of Δw_t which can be derived from (3.7b), namely

$$\Delta w_t \in C\big([0,T]; H^{-\varepsilon}(\Omega)\big) \equiv C\big([0,T]; \big[\mathcal{D}\big(A_D^{\varepsilon/2}\big)\big]'\big).$$

Thus, it is useful to rewrite T_2 as follows:

$$\begin{aligned}
T_2(t) &= D^* A_D \int_0^t e^{-A_D(t-s)} (\mathcal{M}^{-1} A_D) \big[A_D^{\varepsilon} \big(A_D^{-\varepsilon} \theta_t(s) \big) \\
&\quad + A_D^{\varepsilon/2} \big(A_D^{-\varepsilon/2} \Delta w_t(s) \big) \big]\, ds \\
&= D^* A_D^{1/4-\delta} (\mathcal{M}^{-1} A_D) \int_0^t e^{-A_D(t-s)} \big[A_D^{3/4+\varepsilon+\delta} \big(A_D^{-\varepsilon} \theta_t(s) \big) \\
&\quad + A_D^{3/4+\varepsilon/2+\delta} \big(A_D^{-\varepsilon/2} \Delta w_t(s) \big) \big]\, ds,
\end{aligned}$$

with arbitrarily small $\delta > 0$. Therefore, using that $D^* A_D^{1/4-\delta}$ and $\mathcal{M}^{-1} A_D$ are bounded operators, by the usual analytic estimates it follows

$$\begin{aligned}
\left\| T_2(t) \right\|_{0,\Gamma} &\leqslant c I_{\delta,\varepsilon}(t) \big(\|\theta_t\|_{C([0,T];[\mathcal{D}(A_D^{\varepsilon})]')} + \|\Delta w_t\|_{C([0,T];[\mathcal{D}(A_D^{\varepsilon/2})]')} \big) \\
&\leqslant C I_{\delta,\varepsilon}(t) \|z\|_{D(A^{*1-\varepsilon})},
\end{aligned} \tag{3.22}$$

where we have set

$$I_{\varepsilon,\delta}(t) := \int_0^t \frac{1}{(t-s)^{3/4+\varepsilon+\delta}}\, ds.$$

Then, the conclusion in (3.18) immediately follows from (3.22), provided that $\varepsilon < 1/4$. $\square$

Step 4 (*Analysis of the term* T_3). Establishing the regularity of summand T_3 defined in (3.5) is by far the most challenging and difficult issue. The corresponding result is stated as Proposition 3.5 below. Preliminarily, we record—for the reader's convenience—a sharp trace regularity result pertaining to the *elastic* component of the solutions to the thermoelastic problem, since it will be invoked in the proof of Proposition 3.5.

Lemma 3.4 [3]. *With reference to the solution* $(w(t), w_t(t), \theta(t))$ *to the homogeneous thermoelastic problem* (3.1)*, the following boundary regularity result holds true*:

$$\int\limits_{\Sigma} \left|\Delta w(t,x)\right|^2 d\sigma\, dt \leqslant C\|z\|_Y^2, \quad z \in Y. \tag{3.23}$$

The inequality above implies, as well,

$$\int\limits_{\Sigma} \left|\Delta w_t(t,x)\right|^2 d\sigma\, dt \leqslant C\|A^* z\|_Y^2, \quad z \in \mathcal{D}(A^*). \tag{3.24}$$

Proposition 3.5. *Assume* $\varepsilon \in (0, 1/4)$*, and let* $z \in \mathcal{D}({A^*}^{1-\varepsilon})$. *Then* $T_3 \in L^q(0,T; L^2(\Gamma))$ *for all* $q \in [1, 8/7)$*, with* q *independent of* ε*, and the following estimate holds true*:

$$\|T_3\|_{L^q(0,T;L^2(\Gamma))} \leqslant C\|z\|_{D({A^*}^{1-\varepsilon})}. \tag{3.25}$$

Proof. The estimate in (3.25) is proved via interpolation theory. First of all, since the operator $\mathcal{M}^{-1}A_D$ commutes with the semigroup $e^{A_D t}$, let us rewrite the integral T_3 as follows:

$$\begin{aligned} T_3(t) &:= D^* A_D \int\limits_0^t e^{-A_D(t-s)}(\mathcal{M}^{-1}A_D) D\Delta w_s(s)|_\Gamma\, ds \\ &= D^*(\mathcal{M}^{-1}A_D)A_D \int\limits_0^t e^{-A_D(t-s)} D\Delta w_s(s)|_\Gamma\, ds. \end{aligned} \tag{3.26}$$

Next, setting

$$v(t) := A_D \int\limits_0^t e^{-A_D(t-s)} D\Delta w_s(s)|_\Gamma\, ds, \tag{3.27}$$

we see that (3.26) simply reads as

$$T_3(t) = D^*(\mathcal{M}^{-1}A_D)v(t), \tag{3.28}$$

whereas v (defined by (3.27)) solves the Cauchy problem

$$\begin{cases} v'(t) = -A_D(v - Dg), \\ v(0) = 0, \end{cases} \tag{3.29}$$

with $g = \Delta w_t|_\Gamma$. Let us now invoke Lemma 3.4. When initial data z belong to $D(A^*)$, (3.24) applies, i.e., $\Delta w_t|_\Gamma \in L^2(\Sigma)$. Hence, with $g \in L_2(\Sigma)$, the classical parabolic regularity yields

$$v \in H^{1/4}\big(0,T; L^2(\Omega)\big) \cap L^2\big(0,T; H^{1/2}(\Omega)\big) \tag{3.30}$$

(see [16]). If it were possible to deduce the regularity of v directly from the expression (3.27) even when $z \in Y$, then the regularity of v corresponding to $z \in \mathcal{D}({A^*}^{1-\varepsilon})$ would follow by interpolation. However, this is not the case.

On the other hand, integrating initially by parts in t, it is not difficult to deduce from (3.27) the following identity:

$$v(t) = \frac{d}{dt}\int_0^t A_D e^{-A_D(t-s)} D\Delta w(s)|_\Gamma \, ds - A_D e^{-A_D t} D\Delta w(0)|_\Gamma, \tag{3.31}$$

or, equivalently,

$$v(t) + \underbrace{A_D e^{-A_D t} D\Delta w(0)|_\Gamma}_{F(t)} = \psi'(t), \tag{3.32}$$

where we have introduced

$$\psi(t) := A_D \int_0^t e^{-A_D(t-s)} D\Delta w(s)|_\Gamma \, ds.$$

At this point, we turn our attention to the term $\psi'(t)$, whose regularity can be explored both when $z \in \mathcal{D}(A^*)$ and when we just have $z \in Y$. Indeed, when $z \in \mathcal{D}(A^*)$, the basic regularity of solutions to the thermoelastic problem yields $w \in C([0,T]; H^3(\Omega))$, so that $\Delta w \in C([0,T]; H^1(\Omega))$ and $\Delta w(0)|_\Gamma \in H^{1/2}(\Gamma)$. Then, with $\delta \in (0, 1/2)$, we have that $H^{1/2-\delta}(\Omega) \equiv \mathcal{D}(A_D^{1/4-\delta/2})$ so that

$$\begin{aligned}
\|F(t)\|_{H^{1/2-\delta}(\Omega)} &= \|A_D e^{-A_D t} D\Delta w(0)|_\Gamma\|_{H^{1/2-\delta}(\Omega)} \\
&\leqslant \|A_D^{5/4-\delta/2} e^{-A_D t} D\Delta w(0)|_\Gamma\|_{L^2(\Omega)} \\
&\leqslant \frac{c}{t^{1-\delta/4}} \|A_D^{(1-\delta)/4} D\Delta w(0)|_\Gamma\|_{L^2(\Omega)} \\
&\leqslant \frac{c}{t^{1-\delta/4}} \|\Delta w(0)|_\Gamma\|_{L^2(\Gamma)} \leqslant \frac{c}{t^{1-\delta/4}} \|\Delta w(0)|_\Gamma\|_{H^{1/2}(\Gamma)},
\end{aligned} \tag{3.33}$$

which implies

$$F \in L^q\big(0,T; H^{1/2-\delta}(\Omega)\big) \quad \forall \delta \in \left(0, \frac{1}{2}\right),\ \forall q \in \left[1, \frac{4}{4-\delta}\right), \tag{3.34a}$$

$$\|F\|_{L^q(0,T;H^{1/2-\delta}(\Omega))} \leqslant C\|A^* z\|_Y, \quad z \in \mathcal{D}(A^*). \tag{3.34b}$$

Thus, by using the decomposition (3.32) and taking into account (3.30) and (3.34), we obtain

$$z \in D(A^*) \quad \Longrightarrow \quad \psi' \in L^q\big(0,T; H^{1/2-\delta}(\Omega)\big) \quad \forall \delta \in \left(0, \frac{1}{2}\right),$$
$$\forall q \in \left[1, \frac{4}{4-\delta}\right). \tag{3.35}$$

When $z \in Y$, the exceptional trace regularity of solutions to the thermoelastic system gives (3.23), i.e., $\Delta w|_\Gamma \in L^2(\Sigma)$. Notice that ψ solves, as well, the Cauchy problem (3.29), this time with $g \equiv \Delta w|_\Gamma$. Hence, $\psi \in L^2(0,T; H^{1/2}(\Omega))$ and we have simultaneously

$$\psi, D\Delta w|_\Gamma \in L^2\big(0,T; H^{1/2-\delta}(\Omega)\big) \quad \forall\delta>0,$$

so that in particular

$$A_D(\psi - D\Delta w|_\Gamma) \in L^2\big(0,T; H^{-3/2-\delta}(\Omega)\big).$$

Therefore, we obtain for the right-hand side of (3.32),

$$z\in Y \quad\Longrightarrow\quad \psi' \in L^2\big(0,T; H^{-3/2-\delta}(\Omega)\big) \quad \forall\delta>0. \tag{3.36}$$

Interpolation between $D(A^*)$ (where we have validity of (3.35)) and Y (where (3.36) holds true) yields, for $z\in D({A^*}^{1-\varepsilon})$,

$$\begin{aligned} &z\in D({A^*}^{1-\varepsilon}) \quad\Longrightarrow\quad \psi' \in L^q\big(0,T; H^{1/2-2\varepsilon-\delta}(\Omega)\big),\\ &\quad\forall\delta\in\left(0,\frac{1}{2}\right),\ \forall q\in\left[1,\frac{4}{4-\delta}\right). \end{aligned} \tag{3.37}$$

Since ultimately we will have to apply D^*, which is a bounded operator from $H^{-1/2+\sigma}(\Omega)$ to $L^2(\Gamma)$) for any $\sigma>0$, we need to take $0<\delta<1-2\varepsilon$. Notice that this constraint is always fulfilled, as we had initially $0<\varepsilon<1/4$, while $0<\delta<1/2$. Consistently, we get $1\leqslant q<8/7$.

With $z\in D({A^*}^{1-\varepsilon})$, we return to $T_3(t)$ as given by (3.28). Taking into account once more the decomposition (3.32), we see that

$$T_3(t) = D^*(\mathcal{M}^{-1}A_D)\psi'(t) - D^*(\mathcal{M}^{-1}A_D)F(t), \tag{3.38}$$

where from (3.37) we know that

$$D^*(\mathcal{M}^{-1}A_D)\psi' \in L^q\big(0,T; L^2(\Gamma)\big) \quad \forall q\in\left[1,\frac{8}{7}\right),\ \forall x\in D({A^*}^{1-\varepsilon}), \tag{3.39}$$

after using as well that $\mathcal{M}^{-1}A_D$ is a bounded operator (cf. Remark 2.1).

Therefore, in order to conclude the proof it remains to establish the regularity of the second summand on the right-hand side of (3.38), assuming this time $z\in D({A^*}^{1-\varepsilon})$. Indeed, when $z\in D({A^*}^{1-\varepsilon})$, the regularity of solutions to the thermoelastic problem in (3.8) yields $\Delta w(0)|_\Gamma \in H^{1/2-\varepsilon}(\Gamma)$. Then, readily

$$\begin{aligned} \big\|D^*(\mathcal{M}^{-1}A_D)F(t)\big\|_{0,\Gamma} &= \big\|D^*A_De^{-A_Dt}(\mathcal{M}^{-1}A_D)D\Delta w(0)|_\Gamma\big\|_{0,\Gamma}\\ &\leqslant \frac{c}{t^{1/2+\delta}}\big\|\Delta w(0)|_\Gamma\big\|_{0,\Gamma} \leqslant \frac{C}{t^{1/2+\delta}}\|{A^*}^{1-\varepsilon}z\|_Y \end{aligned}$$

with arbitrarily small $\delta>0$, so that

$$D^*(\mathcal{M}^{-1}A_D)F \in L^q\big(0,T; L^2(\Gamma)\big) \quad \forall q\in[1,2),\ \forall z\in D({A^*}^{1-\varepsilon}). \tag{3.40}$$

Combining (3.39) with (3.40) yields the desired conclusion in (3.25). □

Step 5 (*Conclusion*). Let us return to the representation (3.5). Combining the estimates (3.17) and (3.18) (obtained in Proposition 3.3) with the estimate (3.25) (from Proposition 3.5), it is immediately seen that the soughtafter abstract condition (2.13) is satisfied, provided that $\varepsilon<\frac{1}{4}$. Moreover, the final range for the exponent q (i.e. (1.7)) follows as a consequence of the constraints given by the respective propositions. Thus, using (3.6)—which describes the domains of fractional powers of the dynamic operator A in terms of Sobolev spaces—and the key equality (3.2), we finally interpret the abstract condition (2.13) as the trace regularity estimate (1.6), concluding the proof of Theorem 1.1.

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