Γ-convergence for infinite dimensional optimal control problems

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1 Introduction

Among the general goals in the various theories of variational convergence, an important one consists in singling out a notion of Γ-limit problem $(\mathcal{P})$ for a sequence of minimum problems $(\mathcal{P}_n)$ of the form

$$(\mathcal{P}_n) \qquad \min \{F_n(y,u) : (y,u) \in Y \times \mathcal{U}\}.$$

Loosely speaking, a good definition of Γ-limit problem should guarantee the following property:

"*If (y_n, u_n) is an optimal pair for problem $(\mathcal{P}_n)$, or simply a minimizing sequence, and if $(y_n, u_n) \to (y, u)$ in $Y \times \mathcal{U}$, then (y, u) is an optimal pair for the Γ-limit problem $(\mathcal{P})$.*"

The next Theorem indicates which are the relevant features that the Γ-limit problem should possess in order to satisfy this property (the theorem is proved in [7], Proposition 2.1).

Theorem 1.1 *Let Y and $\mathcal{U}$ be topological spaces and let $F_n : Y \times \mathcal{U} \to \overline{\mathbb{R}}$ be a sequence of functions. Let (y_n, u_n) be a minimum point for F_n, or simply a pair such that*

$$\lim_{n\to\infty} F_n(y_n, u_n) = \lim_{n\to\infty} \left[\inf_{Y\times\mathcal{U}} F_n \right].$$

Assume further that $(y_n, u_n) \to (y, u)$ in $Y \times \mathcal{U}$, and there exists

$$F(y,u) = \Gamma(\mathbb{N}, Y^-, \mathcal{U}^-) \lim_{n\to\infty} F_n(y,u). \tag{1.1}$$

Then (y, u) is a minimum point for F on $Y \times \mathcal{U}$, and

$$\lim_{n\to\infty} \left[\inf_{Y\times\mathcal{U}} F_n \right] = \min_{Y\times\mathcal{U}} F.$$

The definition of the Γ-limit (1.1) is given in Section 4 (Definition 4.1).
The above theorem motivates the following definition of Γ-limit problem.

Definition 1.2 *When (1.1) is satisfied, we say that the problem*

$$(\mathcal{P}) \qquad \min\left\{F(y,u) : (y,u) \in Y \times \mathcal{U}\right\},$$

is the Γ-limit *of problems* $(\mathcal{P}_n)$.

See [7], [8] for the explicit calculation of Γ-limits in various situations.
In this paper we consider the following family of optimal control problems: for fixed (small) $\varepsilon > 0$, we look for

$$\min_{u \in L^2(0,T;V)} J_\varepsilon(u), \tag{1.2}$$

where

$$J_\varepsilon(u) = \int_0^T \left(||u(s,\cdot)||_V^2 + ||y_\varepsilon(s,\cdot)||_H^2\right) ds; \tag{1.3}$$

here V is the space $H_{00}^{1/2}(\Gamma_0, \mathbb{C}^N) \times H^{-1/2}(\Gamma_1, \mathbb{C}^N)$, and for each control $u \in L^2(0,T;V)$ the state function y_ε is the solution of the second order regular parabolic initial-boundary value problem

$$\left\{\begin{array}{rcll} \frac{\partial y}{\partial t}(t,x) - A(x,D)y(t,x) & = & 0 & \text{in } [0,T] \times \overline{\Omega} \\ \vartheta_\varepsilon(x)\frac{\partial y}{\partial \nu_A}(t,x) + (1-\vartheta_\varepsilon(x))y(t,x) & = & u(t,x) & \text{in } [0,T] \times \partial\Omega \\ y(0,x) & = & y_0(x) & \text{in } \overline{\Omega}, \end{array}\right. \tag{1.4}$$

where the scalar function ϑ_ε is defined as follows:

$$\vartheta_\varepsilon(x) = \left\{\begin{array}{ll} 1-\varepsilon & \text{if } x \in \Gamma_1 \\ 1-\varepsilon - \frac{1-2\varepsilon}{\varepsilon}\, d(x,\Gamma_1) & \text{if } x \in \Gamma_0 \text{ and } d(x,\Gamma_1) < \varepsilon \\ \varepsilon & \text{if } x \in \Gamma_0 \text{ and } d(x,\Gamma_1) \geq \varepsilon. \end{array}\right. \tag{1.5}$$

Here Γ_0 and Γ_1 are disjoint smooth submanifolds of $\partial\Omega$, such that $\overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega$. Our main result is that the Γ-limit of this family as $\varepsilon \to 0^+$ is the following optimal control problem: look for

$$\min_{u \in L^2(0,T;U)} J(u) \tag{1.6}$$

where

$$J(u) = \int_0^T \left(||u(s,\cdot)||_U^2 + ||y(s,\cdot)||_H^2\right) ds; \tag{1.7}$$

the space U is $H^{1/2}(\Gamma_0, \mathbb{C}^N) \times [H_{00}^{1/2}(\Gamma_1, \mathbb{C}^N)]^*$, and for each control $u = (u_0, u_1) \in L^2(0,T;U)$ the state function y is the solution of the second order parabolic mixed initial-boundary value problem

$$\left\{\begin{array}{rcll} \frac{\partial y}{\partial t}(t,x) - A(x,D)y(t,x) = & 0 & \text{in } [0,T] \times \Omega \\ y(t,x) = & u_0(t,x) & \text{in } [0,T] \times \Gamma_0 \\ \frac{\partial y}{\partial \nu_A}(t,x) = & u_1(t,x) & \text{in } [0,T] \times \Gamma_1 \\ y(0,x) = & y_0(x) & \text{in } \Omega. \end{array}\right. \tag{1.8}$$

We note that the space of controls for problem (1.6) is larger than that of problems (1.2), and in the framework of Γ-convergence we are forced to choose the smaller one: thus we can only approximate the solutions of (1.6) which belong to the smaller space, with a little loss of generality (compare with Remark 2.7 in Section 2).
A special case of the above situation occurs when $\Gamma_1 = \emptyset$, so that the approximating boundary condition has the form

$$\varepsilon \frac{\partial y}{\partial \nu_A}(t,x) + y(t,x) = u(t,x) \qquad \text{in } [0,T] \times \partial\Omega;$$

in this case the Γ-limit problem contains the Dirichlet boundary condition $y = u$ in $[0,T] \times \partial\Omega$, and the natural space of controls is just $L^2(0,T; L^2(\partial\Omega, \mathbb{C}^N))$ (see [6] for details).
This kind of result is interesting also from the numerical point of view. Indeed, in many programs that calculate numerically the solution of partial differential equations, the Neumann and Dirichlet conditions are treated by considering a unique boundary conditions of type

$$a\,\frac{\partial y}{\partial \nu_A} + by = u \qquad \text{in } [0,T] \times \partial\Omega,$$

where the biggest constant appears near the condition one wants to consider (an example is the program [18]).
Let us shortly describe the content of the following sections. In order to apply the general result given in [8], we need to find a representation formula for the solutions of equations (1.4) and (1.8). Therefore, in Section 2 we analyze the mixed problem and we prove that the solution of (1.8) can be written as

$$y(t,\cdot) = e^{tA} y_0(\cdot) - \int_0^t A e^{(t-s)A} G u(s,\cdot)\, ds, \tag{1.9}$$

where $A{:}D(A) \to L^2(\Omega, \mathbb{C}^N)$ and $G{:}U \to D([-A]^\vartheta)$ (for $0 \le \vartheta < 1/4$) are suitable operators. Next, in Section 3 we prove the same kind of formula for the solution of (1.4), i.e.

$$y_\varepsilon(t,\cdot) = e^{tA_\varepsilon} y_0(\cdot) + \int_0^t A_\varepsilon e^{(t-s)A_\varepsilon} G_\varepsilon u(s,\cdot)\, ds, \tag{1.10}$$

where now A_ε: $D(A_\varepsilon) \to L^2(\Omega, \mathbb{C}^N)$ and $G_\varepsilon{:}V \to D([-A_\varepsilon]^\vartheta)$ (for $0 \le \vartheta < 1/2$). In Section 4 we finally prove our Γ-convergence result.
Finally we set some notations.
If $T > 0$ and X is a Banach space, we will use the standard spaces $L^p(0,T;X)$, $1 \le p \le \infty$, and $C^0([0,T];X)$, with their usual norms. As a rule, the space X will be a Sobolev space $H^m(A)$ of functions defined on some subset $A \subseteq \mathbb{R}^N$, with values in $\mathbb{R}^N$ or $\mathbb{C}^N$. We will simply write $H^m(A)$ instead of $H^m(A;\mathbb{R}^N)$ or $H^m(A;\mathbb{C}^N)$; here the number $m > 0$ may be integer or not. In particular, we

denote by $H_0^m(A)$ the closure of the space $C_0^\infty(A)$ in $H^m(A)$, where $C_0^\infty(A)$ is the set of infinitely differentiable functions with compact support contained in A; the set A may be an open set of $\mathbb{R}^N$, or a submanifold of the boundary of a smooth open set $\Omega \subset \mathbb{R}^N$.
In the latter case, we also define $H_{00}^{1/2}(A)$ as the Hilbert space of functions belonging to $H^{1/2}(A)$ whose trivial extension to $\partial\Omega$ is an element of $H^{1/2}(\partial\Omega)$. We recall that $H_{00}^{1/2}(A)$ is a proper, closed subspace of $H^{1/2}(A)$.
We denote by $(\cdot,\cdot)$ and $|\cdot|$ the scalar product and the norm in $\mathbb{C}^N$, whereas $\langle\cdot,\cdot\rangle_X$ denotes the duality pairing between a space X and its dual space X^*. In our estimates we will write C for a generic constant possibly varying from line to line.

2 The mixed problem

We start with describing in a more precise way the optimal control problem we are going to approximate.
Let Ω be an open set of $\mathbb{R}^N$ with $\partial\Omega = \Gamma_0 \cup \Gamma_1$. For fixed $T > 0$, we consider the state space $L^2(0,T;H)$ and the control space $L^2(0,T;U)$, where $H := L^2(\Omega)$ and $U := H^{1/2}(\Gamma_0) \times [H_{00}^{1/2}(\Gamma_1)]^*$. We look for

$$\min_{u \in L^2(0,T;U)} J(u) \tag{2.1}$$

where

$$J(u) = \int_0^T \left(\|u(s,\cdot)\|_U^2 + \|y(s,\cdot)\|_H^2 \right) ds \tag{2.2}$$

and for each control $u = (u_0, u_1) \in L^2(0,T;U)$ the state function y is the solution of the parabolic mixed initial-boundary value problem

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - A(x,D)y(t,x) &= 0 & \text{in } [0,T] \times \Omega \\ y(t,x) &= u_0(t,x) & \text{in } [0,T] \times \Gamma_0 \\ \frac{\partial y}{\partial \nu_A}(t,x) &= u_1(t,x) & \text{in } [0,T] \times \Gamma_1 \\ y(0,x) &= y_0(x) & \text{in } \Omega. \end{cases} \tag{2.3}$$

Here

$$A(x,D) = \sum_{i,j=1}^n D_i(a_{ij}(x) \cdot D_j) \ , \qquad \frac{\partial}{\partial \nu_A} = \sum_{i,j=1}^n \nu_i(x) a_{ij}(x) \cdot D_j \ ,$$

and $\nu(x)$ is the unit outward normal vector at $x \in \partial\Omega$; Γ_0 and Γ_1 are suitable subsets of $\partial\Omega$. We interpret this problem in the variational sense: namely, we set $H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\}$, and introduce the bilinear form

$$a(u,v) = \sum_{i,j=1}^n \int_\Omega (a_{ij}(x) \cdot D_j u, D_i v) \, dx, \qquad u, v \in H^1_{\Gamma_0}(\Omega); \tag{2.4}$$

finally we fix a function $g_0 \in L^2(0,T;H^1(\Omega))$ such that $g_0|_{[0,T]\times\Gamma_0} = u_0$. Then, following [14], the variational version of problem (1.8) consists in looking for a function y such that $v := y - g_0 \in L^2(0,T;H^1_{\Gamma_0}(\Omega))$ and

$$\int_0^T [-\langle v(t),\varphi'(t)\rangle_{L^2(\Omega)} + a(v(t),\varphi(t))]\, dt = \langle y_0,\varphi(0)\rangle_{L^2(\Omega)} + F(\varphi) \tag{2.5}$$

for all test functions φ such that

$$\varphi \in L^2(0,T;H^1_{\Gamma_0}(\Omega)) \cap H^1(0,T;L^2(\Omega)), \qquad \varphi(T) = 0, \tag{2.6}$$

where the functional F is defined, for all functions φ of the above type, by

$$F(\varphi) = \int_0^T \langle g_0(t),\varphi'(t)\rangle_{L^2(\Omega)}\, dt - a(g_0,\varphi) + \langle u_1,\varphi|_{\Gamma_1}\rangle_{H^{1/2}_{00}(\Gamma_1)}. \tag{2.7}$$

Of course, any solution of system (1.8) solves equation (2.5) too, and any sufficiently smooth solution of equation (2.5) is also a solution of system (1.8).
We list now our basic assumptions.

(H0) Ω is a bounded connected open set of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^{1,1}$.

(H1) $y_0 \in L^2(\Omega)$.

(H2) $a_{ij} \in L^\infty(\Omega;\mathbb{C}^{N^2})$, with $M = \sum_{i,j=1}^n |a_{ij}(x)(\cdot)|_{L^\infty(\Omega;\mathbb{C}^{N^2})}$, and there exists $\nu > 0$ such that

$$Re \sum_{i,j=1}^n (a_{ij}(x)\cdot\eta_j,\eta_i) \geq \nu \sum_{i=1}^n |\eta_i|^2$$

for all $\eta_1,\dots,\eta_n \in \mathbb{C}^N$, and for all $x \in \Omega$.

(H3) Γ_0 and Γ_1 are $(n-1)$-manifolds of class C^2, such that $\Gamma_0 \subset \partial\Omega$, $\Gamma_1 = \partial\Omega\setminus\overline{\Gamma_0}$ and $\overline{\Gamma_0}\cap\overline{\Gamma_1} \neq \emptyset$.

For the solution of system (1.8) in the variational form (2.5) we have the following existence and uniqueness result.

Theorem 2.1 *We assume* (H0), (H1), (H2), (H3). *If* $u \in L^2(0,T;U)$, *then equation (2.5) has a unique solution* y *in the class* $L^2(0,T;H^1(\Omega))$; *moreover* $y \in C^0([0,T];L^2(\Omega)) \cap H^{1/2}(0,T;L^2(\Omega))$ *and*

$$\begin{aligned}&||y||_{C^0([0,T];L^2(\Omega))} + ||y||_{L^2(0,T;H^1(\Omega))} + ||y||_{H^{1/2}(0,T;L^2(\Omega))} \leq \\ &\quad \leq C\Big[||y_0||_{L^2(\Omega)} + ||u||_{L^2(0,T;H^{1/2}(\Gamma_0))} + ||u||_{L^2(0,T;[H^{1/2}_{00}(\Gamma_1)]^*)}\Big].\end{aligned}$$

Proof. For each $\lambda > 0$ the bilinear form $a(u,v) + \lambda\langle u,v\rangle_{L^2(\Omega)}$ is coercive on the space $H^1(\Omega)$; hence the same holds for the space $H^1_{\Gamma_0}(\Omega)$. Thus the result follows by Theorems 1.1 and 2.2, Chapter IV, in [14]. □

In order to obtain a representation for the solution of equation (2.5), we define the following operators A and G:

$$\begin{cases} D(A) := \{v \in H^1(\Omega) : & A(\cdot, D)v \in L^2(\Omega), \\ & v|_{\Gamma_0} = 0,\ \frac{\partial v}{\partial \nu_A}|_{\Gamma_1} = 0\} \\ Av := A(\cdot, D)v, \end{cases} \tag{2.8}$$

$$v := G(u_0, u_1) \qquad \Longleftrightarrow \qquad \begin{cases} A(\cdot, D)v &=& 0 & \text{in } \Omega \\ v &=& u_0 & \text{on } \Gamma_0 \\ \frac{\partial v}{\partial \nu_A} &=& u_1 & \text{on } \Gamma_1. \end{cases} \tag{2.9}$$

We prove now that A is the generator of an analytic semigroup in $L^2(\Omega)$.

Proposition 2.2 *Under assumptions* (H0), (H2), (H3)*, let the operator A be defined by (2.8). Then A is densely defined and generates an analytic semigroup in $H = L^2(\Omega)$. Moreover for any $\delta > 0$ we have the estimate*

$$||R(\lambda, A)f||_H \le \frac{C(\delta)}{|\lambda|}\ ||f||_H \qquad \forall \lambda \in S_{\vartheta_0}, \tag{2.10}$$

where $S_{\vartheta_0} := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \le \vartheta_0\}$, and $\vartheta_0 = \pi - \arctan\delta$.

Proof. Obviously, $C_0^\infty(\Omega) \subset D(A)$ so that $D(A)$ is dense in H. Next, we show that the resolvent set $\rho(A)$ contains the positive real half-line: indeed, by the coerciveness in $H^1(\Omega)$ of the form $a(u, \varphi) + \lambda\langle u, \varphi\rangle_H$ for all $\lambda > 0$, we get that for all $f \in H$ the problem

$$\begin{cases} u \in H^1_{\Gamma_0}(\Omega) \\ a(u, \varphi) + \lambda\langle u, \varphi\rangle_H = \langle f, \varphi\rangle_H \qquad \forall \varphi \in H^1_{\Gamma_0}(\Omega) \end{cases} \tag{2.11}$$

is uniquely solvable. Choosing $\varphi \in H^1_0(\Omega)$ we easily find $A(\cdot, D)u = f - \lambda u \in H$, i.e. $u \in D(A)$ and $\lambda u - Au = f$. This shows in particular that the solution u of (2.11) solves the problem

$$\begin{cases} \lambda u - A(\cdot, D)u &=& f & \text{in } \Omega \\ u &=& 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu_A} &=& 0 & \text{on } \Gamma_1. \end{cases} \tag{2.12}$$

Thus we only need to prove (2.10). Fix $\delta > 0$ and take $\lambda \in S_{\vartheta_0}$, with $\vartheta_0 = \pi - \arctan\delta$. Let $v \in D(A)$ and set $f = \lambda v - Av$. Multiplying this equation by v in H and integrating by parts we get

$$\int_\Omega \lambda|v|^2\ dx + \int_\Omega \sum_{i,j=1}^n (a_{ij}(x) \cdot D_j v, D_i v)\ dx = \int_\Omega (f, v)\ dx. \tag{2.13}$$

By taking the real part and recalling hypothesis (H2) we obtain

$$Re\lambda \int_{\Omega} |v|^2\ dx + \nu \int_{\Omega} |Dv|^2\ dx \leq ||f||_H ||v||_H\ ;$$

by taking the imaginary part we get

$$|Im\lambda| \int_{\Omega} |v|^2\ dx \leq ||f||_H\ ||v||_H + M \int_{\Omega} |Dv|^2\ dx.$$

Now if $|Re\lambda| \leq \delta |Im\lambda|$, then

$$\begin{aligned} |\lambda| \textstyle\int_{\Omega} |v|^2\ dx &\leq \sqrt{1+\delta^2}\ |Im\lambda| \textstyle\int_{\Omega} |v|^2\ dx \leq \\ &\leq \sqrt{1+\delta^2}\,(1+\tfrac{M}{\nu})\ ||f||_H ||v||_H\ , \end{aligned}$$

whereas if $Re\lambda > 0$ and $\delta|Im\lambda| < Re\lambda$ we have

$$\begin{aligned} |\lambda| \textstyle\int_{\Omega} |v|^2\ dx &\leq \sqrt{1+\tfrac{1}{\delta^2}}\ Re\lambda\ \textstyle\int_{\Omega} |v|^2\ dx \leq \\ &\leq \sqrt{1+\tfrac{1}{\delta^2}}\ ||f||_H ||v||_H\ . \end{aligned}$$

These two estimates show that for any $\delta > 0$ we have

$$|\lambda| ||v||_H \leq \sqrt{1+\delta^2} \left[\left(1+\frac{M}{\nu}\right) + \frac{1}{\delta} \right] ||f||_H \quad \forall \lambda \in S_{\vartheta_0}\ . \tag{2.14}$$

Note that the smaller is δ the larger is the sector S_{ϑ_0} but also the larger is the constant in the estimate.
As $\rho(A) \supseteq]0, \infty[$, by estimate (2.14) and standard arguments we deduce that $\rho(A)$ also contains the right half-plane; hence Proposition 2.1.11 in [16] implies that A is sectorial. Therefore A is the infinitesimal generator of an analytic semigroup; moreover, by (2.14) we immediately deduce (2.10). This completes the proof. □

The adjoint operator of A is defined as follows:

$$\begin{cases} D(A^*) := \{v \in H^1(\Omega) : & \overline{A(\cdot, D)}v \in L^2(\Omega), \\ & v|_{\Gamma_0} = 0,\ \frac{\partial v}{\partial \nu_{A^*}}|_{\Gamma_1} = 0\} \\ A^* v := \overline{A(\cdot, D)}v, & \end{cases} \tag{2.15}$$

where $\overline{A(x,D)}v := \sum_{i,j=1}^{n} D_i[\overline{a_{ij}(x)^t} \cdot D_j v]$ and $\overline{a_{ij}(x)^t}$ is the matrix whose elements are the conjugates of the elements of the transposed $a_{ij}(x)^t$ of $a_{ij}(x)$. Consequently it is clear that the following statement holds:

Proposition 2.3 *Under assumptions* (H0), (H2), (H3), *let the operator A^* be given by (2.15). Then A^* is densely defined and generates an analytic semigroup in $H = L^2(\Omega)$. Moreover for any $\delta > 0$ we have the estimate*

$$||R(\lambda, A^*) f||_H \leq \frac{C(\delta)}{|\lambda|}\ ||f||_H \qquad \forall \lambda \in S_{\vartheta_0}. \quad \square \tag{2.16}$$

Concerning the operator G, we have the following result:

Proposition 2.4 *Let* (H0), (H1), (H2), (H3) *be fulfilled. If A is defined in (2.8) then the operator G, given by (2.9), is well defined and continuous from* $U = H^{1/2}(\Gamma_0) \times [H_{00}^{1/2}(\Gamma_1)]^*$ *into* $D([-A]^\vartheta)$ *for* $0 \le \vartheta < \frac{1}{4}$.

Proof. Let $(u_0, u_1) \in U$ and fix a lifting $g_0 \in H^1(\Omega)$ of the datum u_0, with $\|g_0\|_{H^1(\Omega)} \le C\|u_0\|_{H^{1/2}(\Gamma_0)}$. Then the problem

$$\begin{cases} u - g_0 \in H^1_{\Gamma_0}(\Omega) \\ a(u,\varphi) = \langle u_1, \varphi\rangle_{H_{00}^{1/2}(\Gamma_1)} \qquad \forall \varphi \in H^1_{\Gamma_0}(\Omega) \end{cases}$$

has a unique solution $u \in H^1(\Omega)$. Hence $u = G(u_0, u_1)$ by definition of G, and, in particular, we have $G(u_0, u_1) \in H^{2\vartheta}(\Omega)$ for all $\vartheta \in [0, \frac{1}{2}]$.
On the other hand, by Theorem 3.1 of [5] we know that $D([-A]^\vartheta)$ coincides with $H^{2\vartheta}(\Omega)$ if and only if $\vartheta \in [0, \frac{1}{4}[$. This completes the proof. □

We are finally ready to prove the representation formula for the solution of system (1.8) in the variational form (2.5).

Theorem 2.5 *Under assumptions* (H0), (H1), (H2), (H3), *let* $u = (u_0, u_1)$ *be an element of* $L^2(0,T;U)$, *where* $U = H^{1/2}(\Gamma_0) \times [H_{00}^{1/2}(\Gamma_1)]^*$. *Then the solution of (2.5) is given for each* $t \in [0,T]$ *by*

$$y(t,\cdot) = e^{tA}y_0(\cdot) - \int_0^t Ae^{(t-s)A}Gu(s,\cdot)\, ds. \tag{2.17}$$

Proof. By Theorem 2.1 we know that equation (2.5) has a unique solution $y \in C^0([0,T];H) \cap H^{1/2}(0,T;H) \cap L^2(0,T;H^1(\Omega))$, where $H = L^2(\Omega)$. Set now

$$v(t) = e^{tA}y_0 - \int_0^t Ae^{(t-s)A}Gu(s)\, ds, \qquad t \in [0,T]; \tag{2.18}$$

by the standard properties of analytic semigroups and by Proposition 2.4 it is easily seen that $v \in C^0([0,T];H) \cap L^2(0,T;H^{2\vartheta}(\Omega))$ for any $\vartheta \in [0, \frac{1}{4}[$. We have to show that $y \equiv v$.
We suppose first that

$$\begin{cases} y_0 \in H^2(\Omega), \quad u = (u_0, u_1) \in C^\infty([0,T], U), \\ y_0 - Gu(0) \in D(A). \end{cases} \tag{2.19}$$

Under assumption (2.19) we can integrate by parts in (2.18):

$$\begin{aligned} v(t) &= e^{tA}y_0 + \left[e^{(t-s)A}Gu(s)\right]_0^t - \textstyle\int_0^t e^{(t-s)A}Gu'(s)\, ds = \\ &= e^{tA}y_0 + Gu(t) - e^{tA}Gu(0) - \textstyle\int_0^t e^{(t-s)A}Gu'(s)\, ds. \end{aligned} \tag{2.20}$$

Hence we can compute

$$v'(t) = Ae^{tA}[y_0 - Gu(0)] - \int_0^t Ae^{(t-s)A}Gu'(s)\, ds \qquad \forall t \in [0,T]; \tag{2.21}$$

thus we see that

$$v(t) - Gu(t) \in D(A) \qquad \forall t \in [0,T], \tag{2.22}$$

and in addition, by (2.8) and (2.9),

$$v'(t) = A[v(t) - Gu(t)] = A(\cdot, D)v(t, \cdot) \qquad \forall t \in [0,T]. \tag{2.23}$$

In particular, $v' \in C^0([0,T]; H)$. Moreover, $v(0) = y_0$ and, by (2.22), $v(t)$ behaves at $\partial\Omega$ just like $Gu(t)$, i.e.

$$v(t)|_{\Gamma_0} = u_0(t) \in H^{1/2}(\Gamma_0), \qquad \frac{\partial v(t)}{\partial \nu_A}|_{\Gamma_1} = u_1(t) \in [H^{1/2}_{00}(\Gamma_1)]^*.$$

This shows that v solves problem (1.8). In particular, v solves equation (2.5) too; by uniqueness, this implies $v \equiv y$, provided (2.19) holds.
Consider now the general case, i.e. $y_0 \in H$ and $u = (u_0, u_1) \in L^2(0,T;U)$; let y be the solution of equation (2.5) and let v be the function (2.18). There exist sequences $\{y_{0,n}\}$ and $\{u_n\}$, satisfying assumption (2.19) for all $n \in \mathbb{N}$, such that $y_{0,n} \to y_0$ in H and $u_n \to u$ in $L^2(0,T;U)$. Denoting by y_n the corresponding solution of (2.5) and by v_n the corresponding function (2.18), we have by the above argument $v_n \equiv y_n$ for all $n \in \mathbb{N}$. But since the $C^0([0,T];H)$-norm of both y_n and v_n depends continuosly on the H-norm of $y_{0,n}$ and on the $L^2(0,T;U)$-norm of g_n, as $n \to \infty$ we immediately deduce that $v \equiv y$. This proves the result. □

After the above preparations, we see that the control problem (1.6) fits in the abstract setting described in [12]: thus, by the results of [12] (see also Theorem 3.14 in [1], and Theorem 8.2 in [3]) we can characterize the optimal control through a feedback formula involving the Riccati operator. The synthesis of the optimal control problem is summarized in the following statement.

Theorem 2.6 *Let* (H0), (H1), (H2), (H3) *be fulfilled. Then:*

(i) *There exists a unique optimal pair* $(\hat{u}, \hat{y}) \in L^2(0,T;U) \times L^2(0,T;H))$ *for problem (1.6), where* $H = L^2(\Omega)$ *and* $U = H^{1/2}(\Gamma_0) \times [H^{1/2}_{00}(\Gamma_1)]^*$.

(ii) *The Riccati equation in integral form, i.e.*

$$P(t) = \int_t^T e^{(s-t)A^*}\, [I - P(s)AGG^*A^*P(s)]\, e^{(s-t)A}\, ds,$$

has a unique solution $P \in C^1([0,T[; \mathcal{L}(H)) \cap C^0([0,T]; \mathcal{L}(H))$.
Moreover, $P(t) = P(t)^* \geq 0$ *and* $P(t) \in D([-A^*]^{1-\vartheta})$ *for each* $\vartheta \in]0,1]$, *with* $\|[-A^*]^{1-\vartheta}P(t)\|_{\mathcal{L}(H)} \leq C(T-t)^{-(1-\vartheta)}$ *for all* $t \in [0,T[$*; in addition it holds*

$$J(\hat{u}) = \langle P(0)y_0, y_0\rangle_H \ .$$

(iii) *We have the feedback formula for* $\hat{u}$*:*

$$\hat{u}(t,\cdot) = G^* A^* P(t)\hat{y}(t,\cdot), \qquad t \in [0,T[.$$

(iv) *The optimal trajectory* $\hat{y}$ *is expressed by* $\hat{y}(t,\cdot) = \Phi(t,0)y_0(\cdot)$*, where* $\Phi(t,s)$ *is defined by the integral equation*

$$\Phi(t,s) = e^{(t-s)A} - \int_s^t Ae^{(t-r)A}GG^*A^*P(r)\Phi(r,s)\, dr, \quad t \in [s,T].$$

The expressions $P(s)AG$ *and* $G^*A^*P(s)$ *are shorter forms relative to the well defined operator* $\left[-[(-A^*)^{1-\vartheta}P(s)]^*(-A)^{\vartheta}G\right]$ *and to its adjoint, with fixed* $\vartheta \in]0, \frac{1}{4}[$. □

Remark 2.7 As noted in the Introduction, in order to obtain our Γ-convergence result we will need to restrict somewhat the control space U, replacing $H^{1/2}(\Gamma_0) \times [H_{00}^{1/2}(\Gamma_1)]^*$ by its closed subspace $V = H_{00}^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_1)$. Of course, all results proved in this Section still hold if we use the restricted control space.

3 The approximating problems

Our goal is to approach problem (1.6) by a family of more regular problems which we now describe.
Fix $\varepsilon > 0$. In the same open set Ω as in the preceding section, consider again the state space $L^2(0,T;H)$, with $H = L^2(\Omega)$, and take as control space $L^2(0,T;V)$, with

$$V = \{u \in \mathcal{D}'(\partial\Omega) : u|_{\Gamma_0} \in H_{00}^{1/2}(\Gamma_0),\ u|_{\Gamma_1} \in H^{-1/2}(\Gamma_1)\}, \tag{3.1}$$

endowed with its natural norm

$$||u||_V = ||u|_{\Gamma_0}||_{H_{00}^{1/2}(\Gamma_0)} + ||u|_{\Gamma_1}||_{H^{-1/2}(\Gamma_1)}\ .$$

We remind that a distribution $u \in \mathcal{D}'(\partial\Omega)$ belongs to V if the restrictions of the functional u to the subspaces $\mathcal{D}(\Gamma_0)$ and $\mathcal{D}(\Gamma_1)$ verify respectively

$$\begin{aligned} |\langle u, \varphi\rangle| &\le C_0||\varphi||_{H^{1/2}(\Gamma_1)} \quad \forall\varphi \in \mathcal{D}(\Gamma_1), \\ |\langle u, \varphi\rangle| &\le C_1||\varphi||_{[H_{00}^{1/2}(\Gamma_0)]^*} \quad \forall\varphi \in \mathcal{D}(\Gamma_0); \end{aligned} \tag{3.2}$$

by density, the above equalities are true for all $\varphi \in H^{1/2}(\Gamma_1)$ and for all $\varphi \in [H_{00}^{1/2}(\Gamma_0)]^*$ respectively.

Lemma 3.1 *The space* V*, difined in (3.1), is isomorphic to* $H_{00}^{1/2}(\Gamma_0)\times H^{-1/2}(\Gamma_1)$ *by the map* $j(u) = (u|_{\Gamma_0}, u|_{\Gamma_1})$*.*

Proof. We just verify that the map j is onto, the other properties being quite easy. Fix $(u_0, u_1) \in H^{1/2}_{00}(\Gamma_0) \times H^{-1/2}(\Gamma_1)$, and set $u = U + V$, where U is the trivial extension of u_0 to $\partial\Omega$ and V is the element of $H^{-1/2}(\partial\Omega)$ defined by $V(\varphi) = u_1(\varphi)$ for all $\varphi \in H^{1/2}(\partial\Omega) \subset H^{1/2}(\Gamma_1)$. Then it is straightforward to check that that $u|_{\Gamma_0} = u_0$ and $u|_{\Gamma_1} = u_1$, i.e. $j(u) = (u_0, u_1)$. $\Box$

We recall here our approximating control problem: in view of Lemma 3.1, we set

$$V = H^{1/2}_{00}(\Gamma_0) \times H^{-1/2}(\Gamma_1), \tag{3.3}$$

and we look for

$$\min_{u \in L^2(0,T;V)} J_\varepsilon(u), \tag{3.4}$$

where

$$J_\varepsilon(u) = \int_0^T \left(||u(s,\cdot)||^2_V + ||y_\varepsilon(s,\cdot)||^2_H \right) ds \tag{3.5}$$

and for each control $u \in L^2(0,T;V)$ the state function y_ε is the solution of the regular parabolic initial-boundary value problem

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - A(x,D)y(t,x) &= 0 & \text{in } [0,T] \times \overline{\Omega} \\ \vartheta_\varepsilon(x) \frac{\partial y}{\partial \nu_A}(t,x) + (1 - \vartheta_\varepsilon(x))y(t,x) &= u(t,x) & \text{in } [0,T] \times \partial\Omega \\ y(0,x) &= y_0(x) & \text{in } \overline{\Omega}. \end{cases} \tag{3.6}$$

Here the scalar function ϑ_ε is defined as follows:

$$\vartheta_\varepsilon(x) = \begin{cases} 1 - \varepsilon & \text{if } x \in \Gamma_1 \\ 1 - \varepsilon - \frac{1-2\varepsilon}{\varepsilon}\, d(x, \Gamma_1) & \text{if } x \in \Gamma_0 \text{ and } d(x,\Gamma_1) < \varepsilon \\ \varepsilon & \text{if } x \in \Gamma_0 \text{ and } d(x, \Gamma_1) \geq \varepsilon. \end{cases} \tag{3.7}$$

The classical results on regular elliptic and parabolic problems guarantee that, at least for smooth data, a unique solution of the state equation (1.4) exists; however the standard estimates (see e.g. [4]) depend on ε, so that the solution might be unbounded in certain spaces with respect to ε as $\varepsilon \to 0$.

On the other hand, problem (1.4) has also a variational formulation: for fixed $u \in L^2(0,T;V)$, problem (1.4) consists in looking for a function $y \in L^2(0,T;H^1(\Omega))$ such that

$$\begin{aligned} &\textstyle\int_0^T \big[- \langle y(t), \varphi'(t) \rangle_{L^2(\Omega)} + a(y(t), \varphi(t)) + \\ &\qquad + \langle \frac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon}\, y(t), \varphi(t) \rangle_{H^{1/2}(\partial\Omega)} \big]\, dt = \\ &\qquad = \langle y_0, \varphi(0) \rangle_{L^2(\Omega)} + \textstyle\int_0^T \langle u(t), \varphi(t) \rangle_{H^{1/2}(\partial\Omega)}\, dt \end{aligned} \tag{3.8}$$

for all test functions φ such that

$$\varphi \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)), \qquad \varphi(T) = 0. \tag{3.9}$$

Our first task is to give an existence and uniqueness result for the variational problem (3.8), with an estimate not depending on ε.

Theorem 3.2 *We assume* (H0), (H1), (H2), (H3). *If* $u \in L^2(0,T;V)$, *with* V *defined in (3.3), then equation (3.8) has a unique solution* y_ε *in the class* $C^0([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap H^{1/2}(0,T;L^2(\Omega))$. *Moreover there exists* $C \geq 0$, *independent of* ε, *such that*

$$||y_\varepsilon||_{C^0([0,T];L^2(\Omega))} + ||y_\varepsilon||_{L^2(0,T;H^1(\Omega))} + ||y_\varepsilon||_{H^{1/2}(0,T;L^2(\Omega))} \leq$$
$$\leq C\Big[||y_0||_{L^2(\Omega)} + ||u||_{L^2(0,T;H^{1/2}_{00}(\Gamma_0))} + ||u||_{L^2(0,T;H^{-1/2}(\Gamma_1))}\Big].$$

Proof. The bilinear form $a(y,v) + \langle \frac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon} y, v\rangle_{H^{1/2}(\partial\Omega)}$ is clearly weakly coercive on $H^1(\Omega)$; in fact it is even coercive, since the norm $||u||_{L^2(\partial\Omega)} + ||Du||_{L^2(\Omega)}$ is equivalent to the usual norm of $H^1(\Omega)$, as shown e.g. by Lemma 4.4 in [2]. Hence the result follows by Theorems 1.1 and 2.2, Chapter IV in [14]. □

As we did for the case of the mixed problem, we want to find now a representation formula for the solution y_ε of problem (1.4). We start with defining the operators

$$\begin{cases} D(A_\varepsilon) := \Big\{v \in H^2(\Omega) : \ \vartheta_\varepsilon \frac{\partial v}{\partial \nu_A} + (1-\vartheta_\varepsilon)v = 0 \text{ on } \partial\Omega\Big\} \\ A_\varepsilon v := A(\cdot, D)v, \end{cases} \tag{3.10}$$

and

$$w := G_\varepsilon h \qquad \Longleftrightarrow \qquad \begin{cases} A(\cdot,D)w = 0 & \text{in } \Omega \\ \vartheta_\varepsilon \frac{\partial w}{\partial \nu_A} + (1-\vartheta_\varepsilon)w = h & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

As (3.10) is a regular elliptic operator, it is clear that A_ε is the infinitesimal generator of an analytic semigroup; we show now that this property holds "uniformly" with respect to ε.

Proposition 3.3 *Let* (H0), (H1), (H2) *be fulfilled. Then the operator* A_ε *defined by (3.10) is densely defined and generates an analytic semigroup in* $L^2(\Omega)$. *Moreover for any* $\delta > 0$ *we have the estimate*

$$||R(\lambda, A_\varepsilon)f||_{L^2(\Omega)} \leq \frac{C(\delta)}{|\lambda|}\ ||f||_{L^2(\Omega)} \qquad \forall \lambda \in S_{\vartheta_0}\ , \tag{3.12}$$

where $S_{\vartheta_0} := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \vartheta_0\}$, *and* $\vartheta_0 = \pi - \arctan\delta$. *In particular, the constant* $C(\delta)$ *does not depend on* ε.

Proof. We just need to prove (3.12).
Let $f \in L^2(\Omega)$ and set $v_\varepsilon = R(\lambda, A_\varepsilon)f$: then v_ε solves

$$\begin{cases} \lambda v_\varepsilon - A(\cdot,D)v_\varepsilon &= f & \text{in } \Omega \\ \vartheta_\varepsilon \frac{\partial v_\varepsilon}{\partial \nu_A} + (1-\vartheta_\varepsilon)v_\varepsilon &= 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Let us multiply by v_ε, integrate over Ω and use the boundary condition: we get

$$\begin{aligned} &\lambda \textstyle\int_\Omega |v_\varepsilon|^2\ dx + a(v_\varepsilon, v_\varepsilon) + \\ &\quad + \textstyle\int_{\Gamma_0} \frac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon} |\frac{\partial v_\varepsilon}{\partial \nu_A}|^2\ d\sigma + \int_{\Gamma_1} \frac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon} |v_\varepsilon|^2\ d\sigma = \int_\Omega |(f, v_\varepsilon)|\ dx. \end{aligned} \tag{3.14}$$

This estimate plays the role of (2.13) in the proof of Proposition 2.2: thus, just repeating the argument used in that proof, we get for any $\delta > 0$

$$||v_\varepsilon||_{L^2(\Omega)} \le \frac{C(\delta)}{|\lambda|} \, ||f||_{L^2(\Omega)} \qquad \forall \lambda \in S_{\vartheta_0} \ ,$$

where $\vartheta_0 = \pi - \arctan \delta$. This proves the result. $\square$

Remark 3.4 By (3.14) it follows in particular the useful estimate, which holds true for all $\lambda \in S_{\vartheta_0}$:

$$||v_\varepsilon||_{H^1(\Omega)} + \int_{\Gamma_0} \frac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon} \left|\frac{\partial v_\varepsilon}{\partial \nu_A}\right|^2 d\sigma + \int_{\Gamma_1} \frac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon} |v_\varepsilon|^2 \, d\sigma \le \frac{C(\delta)}{|\lambda|^{1/2}} \, ||f||_{L^2(\Omega)} \ .$$

The adjoint operator of A_ε is defined by:

$$\begin{cases} D(A_\varepsilon^*) := \left\{ v \in H^2(\Omega) : \ \vartheta_\varepsilon \frac{\partial v}{\partial \nu_{A^*}} + (1-\vartheta_\varepsilon) v = 0 \text{ on } \partial\Omega \right\} \\ A_\varepsilon^* v := \overline{A(\cdot, D)} v. \end{cases} \tag{3.15}$$

Hence it is clear that the following result, parallel to that of Proposition 3.3, holds:

Proposition 3.5 *Let* (H0), (H1), (H2) *be fulfilled and fix* $\varepsilon > 0$. *Then the operator* A_ε^* *given by (3.15) is densely defined and generates an analytic semigroup in* $L^2(\Omega)$. *Moreover for any* $\delta > 0$ *we have the estimate*

$$||R(\lambda, A_\varepsilon^*) f||_{L^2(\Omega)} \le \frac{C(\delta)}{|\lambda|} \, ||f||_{L^2(\Omega)} \qquad \forall \lambda \in S_{\vartheta_0} \ , \tag{3.16}$$

where $S_{\vartheta_0} := \{\lambda \in \mathbb{C} : |\arg(\lambda)| \le \vartheta_0\}$, *and* $\vartheta_0 = \pi - \arctan \delta$. *In particular, the constant* $C(\delta)$ *does not depend on* ε. $\square$

Concerning the operator G_ε, we have:

Proposition 3.6 *Let* (H0), (H1), (H2), (H3) *be fulfilled; fix* $\varepsilon > 0$ *and let* A_ε *be defined by (3.10). The operator* G_ε , *given by (3.11), is well defined from the space* V, *defined by (3.3), into the space* $D([-A_\varepsilon]^\vartheta)$ *for each* $\vartheta \in]0, \frac{1}{2}[$. *In addition, there exists* $C > 0$, *independent of* ε, *such that*

$$\begin{aligned} C(\varepsilon)^{-1} ||[-A_\varepsilon]^\vartheta G_\varepsilon h||^2_{L^2(\Omega)} &\le ||D G_\varepsilon h||^2_{L^2(\Omega)} \le \\ &\le C \left[||h|_{\Gamma_0}||^2_{H_{00}^{1/2}(\Gamma_0)} + ||h|_{\Gamma_1}||^2_{H^{-1/2}(\Gamma_1)} \right] \qquad \forall h \in V. \end{aligned}$$

Proof. Let first $w = G_\varepsilon h$, where $h \in C^\infty(\partial\Omega)$ and h vanishes in a neighbourhood of $\overline{\Gamma_0} \cap \overline{\Gamma_1}$; then w is smooth and solves problem (3.11). We repeat the argument used in the proof of Proposition 3.3. Multiply by $w(x)$ the equation $A(x, D) w = 0$

and integrate by parts: using Hypothesis (H2) and the boundary condition, we get

$$\begin{aligned}
&\nu \int_\Omega |Dw|^2\ dx + \int_{\Gamma_0} \tfrac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon}\ |\tfrac{\partial w}{\partial \nu_A}|^2\ d\sigma + \int_{\Gamma_1} \tfrac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon}\ |w|^2\ d\sigma \leq \\
&\qquad \leq \left| \int_{\Gamma_0} (h, \tfrac{\partial w}{\partial \nu_A})\ \tfrac{1}{1-\vartheta_\varepsilon}\ d\sigma \right| + \left| \int_{\Gamma_1} (h, w)\ \tfrac{1}{\vartheta_\varepsilon}\ d\sigma \right|.
\end{aligned} \tag{3.17}$$

Let us estimate separately the addenda of the last term in (3.17). We start with the second one, which is easier to handle. As $\vartheta_\varepsilon(x) \equiv 1-\varepsilon$ on Γ_1 we have

$$\begin{aligned}
\left| \int_{\Gamma_1} (h,w) \tfrac{1}{\vartheta_\varepsilon}\ d\sigma \right| &= \tfrac{1}{1-\varepsilon}\ \left| \langle h, w\rangle_{H^{1/2}(\Gamma_1)} \right| \leq \\
&\leq \tfrac{1}{1-\varepsilon}\ \|h\|_{H^{-1/2}(\Gamma_1)}\ \|w\|_{H^{1/2}(\Gamma_1)}\ \leq \\
&\leq \tfrac{1}{1-\varepsilon}\ \|h\|_{H^{-1/2}(\Gamma_1)}\ \|w\|_{H^{1/2}(\partial\Omega)}\ \leq \\
&\leq \tfrac{\nu}{2}\ \|h\|^2_{H^{-1/2}(\Gamma_1)} + C\ \|w\|^2_{H^1(\Omega)}.
\end{aligned} \tag{3.18}$$

Concerning the first term on the last member of (3.17), set

$$\Gamma_\varepsilon = \{x \in \Gamma_0 : d(x, \Gamma_1) \leq \varepsilon\}, \tag{3.19}$$

and denote by $\overline{h}$ the function which coincides with h on Γ_0 and vanishes on Γ_1: then $\overline{h} \in H^{1/2}(\partial\Omega)$ and we can write

$$\begin{aligned}
&\left| \int_{\Gamma_0} (h, \tfrac{\partial w}{\partial \nu_A})\ \tfrac{1}{1-\vartheta_\varepsilon}\ d\sigma \right| \leq \\
&= \left| \int_{\Gamma_0} (h, \tfrac{\partial w}{\partial \nu_A})\ \tfrac{1}{1-\varepsilon}\ d\sigma \right| + \int_{\Gamma_\varepsilon} |(h, \tfrac{\partial w}{\partial \nu_A})|\ (\tfrac{1}{1-\vartheta_\varepsilon} - \tfrac{1}{1-\varepsilon})\ d\sigma \leq \\
&\leq \left| \int_{\partial\Omega} (\overline{h}, \tfrac{\partial w}{\partial \nu_A})\ \tfrac{1}{1-\varepsilon}\ d\sigma \right| + \int_{\Gamma_\varepsilon} |(h, \tfrac{\partial w}{\partial \nu_A})|\ \tfrac{1}{1-\vartheta_\varepsilon}\ d\sigma \\
&= I + II.
\end{aligned} \tag{3.20}$$

On one hand, we get for all $\eta > 0$

$$\begin{aligned}
I &= \tfrac{1}{1-\varepsilon}\ \left| \int_{\partial\Omega} (\overline{h}, \tfrac{\partial w}{\partial \nu_A})\ d\sigma \right| \leq \\
&\leq C\ \|\tfrac{\partial w}{\partial \nu_A}\|_{H^{-1/2}(\partial\Omega)}\ \|\overline{h}\|_{H^{1/2}(\partial\Omega)}\ \leq \\
&\leq \eta\ \|w\|^2_{H^1(\Omega)} + C(\eta)\ \|h|_{\Gamma_0}\|^2_{H^{1/2}_{00}(\Gamma_0)};
\end{aligned} \tag{3.21}$$

on the other hand, we have

$$\begin{aligned}
II &= \int_{\Gamma_\varepsilon} |(h, \tfrac{\partial w}{\partial \nu_A})|\ \tfrac{1}{1-\vartheta_\varepsilon}\ d\sigma \leq \\
&\leq \int_{\Gamma_\varepsilon} \tfrac{1}{2}\ \tfrac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon}\ |\tfrac{\partial w}{\partial \nu_A}|^2\ d\sigma + \int_{\Gamma_\varepsilon} \tfrac{2}{\vartheta_\varepsilon(1-\vartheta_\varepsilon)}\ |h|^2\ d\sigma,
\end{aligned} \tag{3.22}$$

and the second addendum in the last member of (3.22) is estimated by

$$\begin{aligned}
&\int_{\Gamma_\varepsilon} \tfrac{2}{\vartheta_\varepsilon(1-\vartheta_\varepsilon)}\ |h|^2\ d\sigma \leq C \int_{\Gamma_\varepsilon} \tfrac{1}{\varepsilon}\ |h|^2\ d\sigma \leq \\
&\qquad \leq C \int_{\Gamma_0} |h|^2\ \tfrac{1}{d(x,\Gamma_1)}\ d\sigma \leq C \|h|_{\Gamma_0}\|^2_{H^{1/2}_{00}(\Gamma_0)}.
\end{aligned} \tag{3.23}$$

By (3.17), (3.18), (3.20), (3.21), (3.22) and (3.23), we deduce for a sufficiently small η:

$$\begin{aligned}\int_\Omega |Dw|^2\ dx + \int_{\Gamma_0} \tfrac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon}\ |\tfrac{\partial w}{\partial \nu_A}|^2\ d\sigma + \int_{\Gamma_1} \tfrac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon}\ |w|^2\ d\sigma \le \\ \le C\left[||h|_{\Gamma_0}||^2_{H^{1/2}_{00}(\Gamma_0)} + ||h|_{\Gamma_1}||^2_{H^{-1/2}(\Gamma_1)} \right].\end{aligned} \tag{3.24}$$

Now the general case, i.e. the case $h \in V$, follows by a density argument, since we may approximate h by a sequence $\{h_n\} \subset C^\infty(\partial\Omega)$ of functions vanishing in a neighbourhood of $\overline{\Gamma_0} \cap \overline{\Gamma_1}$. Thus estimate (3.24) holds for $w = G_\varepsilon h$ too; this means that the map G_ε is bounded, uniformly with respect to ε, from $H^{1/2}_{00}(\Gamma_0) \times H^{-1/2}(\Gamma_1)$ to $H^1(\Omega)$.
We invoke finally a result due to Fujiwara (Theorem 2 in [10]), according to which we have $D([-A_\varepsilon]^\vartheta) = H^{2\vartheta}(\Omega)$ for all $\vartheta \in [0, 3/4[$ and

$$||f||_{D([-A_\varepsilon]^\vartheta)} \le C(\varepsilon)||f||_{H^{2\vartheta}(\Omega)} \qquad \forall f \in H^{2\vartheta}(\Omega), \tag{3.25}$$

with $C(\varepsilon)$ possibly depending on ε. This estimate, together with (3.24), implies our result. □

Remark 3.7 In the proof of Proposition 3.6 we have proved in particular the following useful estimate for $w = G_\varepsilon h$:

$$\begin{aligned}\int_{\Gamma_0} \tfrac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon}\ |\tfrac{\partial w}{\partial \nu_A}|^2\ d\sigma + \int_{\Gamma_1} \tfrac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon}\ |w|^2\ d\sigma \le \\ \le C\left[||h|_{\Gamma_0}||^2_{H^{1/2}_{00}(\Gamma_0)} + ||h|_{\Gamma_1}||^2_{H^{-1/2}(\Gamma_1)} \right].\end{aligned} \tag{3.26}$$

Remark 3.8 An estimate similar to (3.26) is valid for the solution y_ε of the parabolic problem (3.8), namely

$$\begin{aligned}\int_0^T \int_{\Gamma_0} \tfrac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon} \left|\tfrac{\partial y_\varepsilon}{\partial \nu_A}\right|^2 d\sigma dt + \int_0^T \int_{\Gamma_1} \tfrac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon} |y|^2\ d\sigma dt \le \\ \le C \int_0^T \left[||u_0(t,\cdot)||^2_{H^{1/2}(\Gamma_0)} + ||u_1(t,\cdot)||^2_{H^{-1/2}(\Gamma_1)} \right] dt.\end{aligned} \tag{3.27}$$

The proof requires the same argument used in the proof of Proposition 3.6, and we can omit it.

We are finally ready to prove the representation formula for the solution of problem (1.4).

Theorem 3.9 *Assume* (H0), (H1), (H2), (H3). *If $u \in L^2(0,T;V)$, with V defined by (3.3), then the solution y_ε of (1.4) is given for each $t \in [0,T]$ by*

$$y_\varepsilon(t,\cdot) = e^{tA_\varepsilon} y_0(\cdot) + \int_0^t A_\varepsilon e^{(t-s)A_\varepsilon} G_\varepsilon u(s,\cdot)\ ds. \tag{3.28}$$

Proof. As (1.4) is a regular parabolic initial-boundary value problem, the result follows by adapting the proof of Proposition 2.13 in [1]; otherwise, one may repeat the same argument used in proving Theorem 2.5 above. We omit the details. □

Finally we state the result on the synthesis of the optimal control problem (1.2), whose proof is parallel to that of Theorem 2.6 (compare again with [12], [1], and [3]).

Theorem 3.10 *Let* (H0), (H1), (H2), (H3) *be fulfilled. Then:*

(i) *There exists a unique optimal pair* $(\hat{u}_\varepsilon, \hat{y}_\varepsilon) \in L^2(0,T;V) \times L^2(0,T;H))$ *for problem (1.6), where* $H = L^2(\Omega)$ *and* V *is defined in (3.3).*

(ii) *The Riccati equation in integral form, i.e.*

$$P_\varepsilon(t) = \int_t^T e^{(s-t)A_\varepsilon^*} \left[I - P_\varepsilon(s) A_\varepsilon G_\varepsilon G_\varepsilon^* A_\varepsilon^* P_\varepsilon(s)\right] e^{(s-t)A_\varepsilon} \, ds,$$

has a unique solution $P_\varepsilon \in C^1([0,T[;\mathcal{L}(H)) \cap C^0([0,T];\mathcal{L}(H))$, *such that* $P_\varepsilon(t) = P_\varepsilon(t)^* \geq 0$ *and* $P_\varepsilon(t) \in D([-A^*]^{1-\vartheta})$ *for each* $\vartheta \in]0,1]$, *with* $\|[-A_\varepsilon^*]^{1-\vartheta} P_\varepsilon(t)\|_{\mathcal{L}(H)} \leq C(T-t)^{-(1-\vartheta)}$ *for all* $t \in [0,T[$; *in addition it holds*

$$J(\hat{u}_\varepsilon) = \langle P_\varepsilon(0) y_0, y_0 \rangle_H \ .$$

(iii) *We have the feedback formula for* $\hat{u}_\varepsilon$:

$$\hat{u}_\varepsilon(t,\cdot) = G_\varepsilon^* A_\varepsilon^* P_\varepsilon(t) \hat{y}_\varepsilon(t,\cdot), \qquad t \in [0,T[.$$

(iv) *The optimal trajectory* $\hat{y}_\varepsilon$ *is given by* $\hat{y}_\varepsilon(t,\cdot) = \Phi_\varepsilon(t,0) y_0(\cdot)$, *where* $\Phi_\varepsilon(t,s)$ *is defined by the integral equation*

$$\Phi_\varepsilon(t,s) = e^{(t-s)A_\varepsilon} - \int_s^t A_\varepsilon e^{(t-r)A_\varepsilon} G_\varepsilon G_\varepsilon^* A_\varepsilon^* P_\varepsilon(r) \Phi_\varepsilon(r,s) \, dr, \quad t \in [s,T].$$

The expressions $P_\varepsilon(s) A_\varepsilon G_\varepsilon$ *and* $G_\varepsilon^* A_\varepsilon^* P_\varepsilon(s)$ *are shorter forms relative to the well defined operator* $\left[-[(-A_\varepsilon^*)^{1-\vartheta} P_\varepsilon(s)]^* (-A_\varepsilon)^\vartheta G_\varepsilon\right]$ *and to its adjoint, with fixed* $\vartheta \in]0, \frac{1}{2}[$. □

4 Γ-convergence

First of all, we observe that all Γ-convergence results in the literature deal with a discrete parameter n tending to $+\infty$. So, from now on we will consider a fixed subsequence of our approximating problems with parameter $\varepsilon = \varepsilon_n$ such that $\varepsilon_n \to 0^+$ for $n \to \infty$.
We remark that, by (1.5),

$$\lim_{n\to\infty} \vartheta_{\varepsilon_n}(x) = \begin{cases} 1 & \text{if } x \in \Gamma_1 \\ 0 & \text{if } x \in \Gamma_0. \end{cases} \tag{4.1}$$

In order to apply the abstract result of [8], we need to rewrite the optimal control problems in a different way. Therefore, we set

$$H = L^2(\Omega), \qquad V = H^{1/2}_{00}(\Gamma_0) \times H^{-1/2}(\Gamma_1), \tag{4.2}$$

and

$$Y = L^2(0,T;H), \qquad \mathcal{U} = L^2(0,T;V). \tag{4.3}$$

Define the operators $M_{\varepsilon_n}, M : Y \to Y$ and $B_{\varepsilon_n}, B : \mathcal{U} \to Y$ as follows:

$$\left\{ \begin{array}{lcl} M_{\varepsilon_n}(y) & := & y - e^{tA_{\varepsilon_n}} y_0 \\ M(y) & := & y - e^{tA} y_0 \end{array} \right. \qquad \forall y \in Y, \tag{4.4}$$

$$\left\{ \begin{array}{lcl} B_{\varepsilon_n}(u) & := & \int_0^t A_{\varepsilon_n} e^{(t-s)A_{\varepsilon_n}} G_{\varepsilon_n}(s) u(s)\, ds, \\ B(u) & := & \int_0^t A e^{(t-s)A} G(s) u(s)\, ds \end{array} \right. \qquad \forall u \in \mathcal{U}. \tag{4.5}$$

As we already noted, the space of controls for problems (1.2) is smaller than that of problem (1.6). See the remark in the Introduction and compare with Remark 2.7.

Due to Theorems 2.5 and 3.9 above, the state equations of the approximating problems and of the mixed problem can be written as

$$M_{\varepsilon_n}(y) = B_{\varepsilon_n}(u) \qquad \text{and} \qquad M(y) = B(u), \tag{4.6}$$

respectively. We also observe that the corresponding cost functionals J_{ε_n} and J, defined by (1.3) and (1.7), are in fact the same (they differed only in the choice of the space of controls, which are equal to V now). In order to stress their dependence on y, we relabel both $J_{\varepsilon_n}(u)$ and $J(u)$ as $J(y,u)$, i.e. we set

$$J(y,u) := \int_0^T \left(||u(s,\cdot)||_V^2 + ||y(s,\cdot)||_H^2 \right) ds. \tag{4.7}$$

Now our approximating problem (1.2) has the following equivalent formulation:

$$(\mathcal{P}_n) \qquad \min \left\{ J(y,u) + \chi_{\{M_{\varepsilon_n}(y) = B_{\varepsilon_n}(u)\}} :\ (y,u) \in Y \times \mathcal{U} \right\},$$

where for each set A the function χ_A is given by

$$\chi_A(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A. \end{array} \right.$$

Similarly, we rewrite the mixed problem (1.6) (with U replaced by V) as

$$(\mathcal{P}) \qquad \min \left\{ J(y,u) + \chi_{\{M(y) = B(u)\}} :\ (y,u) \in Y \times \mathcal{U} \right\}.$$

Thus, setting

$$F_n(y,u) := J(y,u) + \chi_{\{M_{\varepsilon_n}(y) = B_{\varepsilon_n}(u)\}}(y,u), \tag{4.8}$$

the sequence of optimal control problems we consider is

$$(\mathcal{P}_n) \qquad \min \{F_n(y,u) : (y,u) \in Y \times \mathcal{U}\},$$

with Y and $\mathcal{U}$ given by (4.3).
Of course, a point $(y_n, u_n) \in Y \times \mathcal{U}$ is an optimal pair for problem $(\mathcal{P}_n)$ if

$$F_n(y_n, u_n) = \min_{Y \times \mathcal{U}} F_n(y,u).$$

We recall now the general definition of multiple Γ-limits.

Definition 4.1 *Let X and W be topological spaces and let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of functions from $X \times W$ to $\overline{\mathbb{R}}$; we denote by $Z(+)$ and $Z(-)$ the* sup *and* inf *operators respectively. For every $x \in X$ and $w \in W$ we set*

$$\Gamma(\mathbb{N}^\alpha, X^\beta, W^\gamma) \lim_{n\to\infty} F_n(x,w) = \underset{\{w_n\}\in S(w)}{\mathrm{Z}(\gamma)} \ \underset{\{x_n\}\in S(x)}{\mathrm{Z}(\beta)} \ \underset{k\in\mathbb{N}}{\mathrm{Z}(-\alpha)} \ \underset{n\geq k}{\mathrm{Z}(\alpha)} F_n(x_n, w_n)$$

where $\alpha, \beta, \gamma \in \{+,-\}$, and $S(x)$, $S(w)$ denote the set of all sequences $x_n \to x$ in X and $w_n \to w$ in W respectively. Note that when the Γ-limit does not depend on the signs + or − in one (or more) of its variables, the corresponding sign is customarily omitted.

Our aim is now to apply Proposition 2.1 in [7] (see also Theorem 1.1 in the Introduction). Therefore we take as F_n the functionals defined in (4.8). Our goal is to prove that the functional F corresponding to problem $(\mathcal{P})$, i.e.

$$F(y,u) = J(y,u) + \chi_{\{M(y)=B(u)\}}, \tag{4.9}$$

coincides precisely with the multiple Γ-limit (1.1):

$$F(y,u) = \Gamma(\mathbb{N}, Y^-, \mathcal{U}^-) \lim_{n\to\infty} F_n(y,u). \tag{4.10}$$

This will mean that in fact problem $(\mathcal{P})$ is the Γ*-limit problem* of the sequence of problems $\{(\mathcal{P}_n)\}$, in the sense given by Definition 1.2 in the Introduction.
In order to prove (4.10), we are going to apply an abstract result in [8]. We first recall the definition of G-convergence of operators.

Definition 4.2 *Let Z and W be topological spaces and for all $n \in \mathbb{N}$ let D_n be operators from Z to W. We say that the sequence $\{D_n\}$ G-*converges *to the operator $D : Z \to W$ if*

$$\Gamma(\mathbb{N}, W, Z^-) \lim_{n\to\infty} \chi_{\{D_n(z)=w\}}(w,z) = \chi_{\{D(z)=w\}}(w,z) \qquad \forall w \in W,\ \forall z \in Z,$$

that is, if the following conditions are satisfied:

(i) *if $z_n \to z$ in Z, $w_n \to w$ in W and $D_n(z_n) = w_n$ for infinitely many $n \in \mathbb{N}$, then $D(z) = w$;*

(ii) *if $z \in Z$ and $w \in W$ are such that $D(z) = w$ and $w_n \to w$ in W, then there exists $\{z_n\} \subseteq Z$ such that $z_n \to z$ in Z and $D_n(z_n) = w_n$ for all sufficiently large $n \in \mathbb{N}$.*

The following general result holds true (it is a special case of a result proved in [8], Proposition 3.3 and Section 3).

Theorem 4.3 *Let Z, W be topological spaces, let $G_n : W \times Z \to \mathbb{R}$ be a functional and let $D, D_n : Z \to Z$, $K, K_n : W \to Z$ be operators. Consider a sequence of control problems of the following form:*

$$\min_{Z\times W} \{G_n(z,w) + \chi_{\{D_n(z)=K_n(w)\}}\}.$$

Assume that:

(i) *the sequence $\{D_n\}$ G-converges to D;*

(ii) *if $w_n \to w$ in W, then $K_n(w_n) \to K(w)$ in W;*

(iii) *there exist a function $\Psi : W \to \mathbb{R}$, bounded on bounded sets of W, and a function $\omega : Z \times Z \to \mathbb{R}$, with $\lim_{v\to z} \omega(z,v) = 0$ for all $z \in Z$, such that*

$$G_n(z,w) \leq G_n(v,w) + \Psi(w)\omega(z,v). \qquad \forall w \in W,\ \forall z,v \in Z,\ \forall n \in \mathbb{N}.$$

Then

$$\Gamma(\mathbb{N}, Z^-, W^-) \lim_{n\to\infty} [G_n + \chi_{\{D_n(z)=K_n(w)\}}](z.w) = [G + \chi_{\{D(z)=K(w)\}}](z,w),$$

where

$$G(z,w) = \Gamma(\mathbb{N}, W^-) \lim_{n\to\infty} G_n(z,w). \tag{4.11}$$

We will apply this theorem with $Z = Y$, $W = \mathcal{U}$ (defined in (4.3)), $G_n = G = J$ (given by (4.7)), and $D_n = M_{\varepsilon_n}$, $D = M$, $K_n = B_{\varepsilon_n}$, $K = B$ (see (4.4) and (4.5)). Thus what we have to do is to verify that assumptions **(i)-(iii)** of Theorem 4.3 are in fact satisfied.
We start with proving hypothesis **(i)**.

Lemma 4.4 *Let* (H0), (H1), (H2), (H3) *be fulfilled. If M_{ε_n} and M are defined by (4.4), then the sequence $\{M_{\varepsilon_n}\}$ G-converges to M as $n \to \infty$.*

Proof. Suppose we have shown that

$$e^{tA_{\varepsilon_n}} y_0 \to e^{tA} y_0 \quad \text{in } Y \qquad \forall y_0 \in H; \tag{4.12}$$

then we easily deduce the following properties. Firstly, if $\{y_n\}, \{v_n\} \subset Y$ are such that $y_n \to y$ in Y and $v_n = M_{\varepsilon_n}(y_n) = y_{\varepsilon_n} - e^{tA_{\varepsilon_n}} y_0$ for infinitely many $n \in \mathbb{N}$, then letting $n \to \infty$ we get $v = y - e^{tA} y_0 = M(y)$.
Secondly, if $v = M(y) = y - e^{tA} y_0$, and $\{v_n\} \subset Y$ is a sequence such that $v_n \to v$

in Y, then setting $y_n := v_n + e^{tA_{\varepsilon_n}} y_0$ we have $M_{\varepsilon_n}(y_n) = v_n$ for infinitely many n and $y_n \to y$ in Y.
Recalling Definition 4.2, these facts imply that M_{ε_n} G-converges to M; hence the proof of Lemma 4.4 is achieved provided we show (4.12).
Let us prove now (4.12).
To begin with, fix $f \in L^2(\Omega)$ and $\lambda \in \rho(A_{\varepsilon_n}) \cap \rho(A)$. If we set

$$v_{\varepsilon_n} := R(\lambda, A_{\varepsilon_n}), \qquad v := R(\lambda, A), \tag{4.13}$$

then by (3.13) and (2.12) the function $v - v_{\varepsilon_n}$ solves the equation

$$\lambda(v - v_{\varepsilon_n}) - A(\cdot, D)(v - v_{\varepsilon_n}) = 0 \quad \text{in } \Omega, \tag{4.14}$$

whereas at the boundary v and v_{ε_n} verify

$$\begin{cases} \vartheta_{\varepsilon_n} \frac{\partial v_{\varepsilon_n}}{\partial \nu_A} + (1 - \vartheta_{\varepsilon_n}) v_{\varepsilon_n} = 0 & \text{in } \partial\Omega \\ v = 0 & \text{in } \Gamma_0 \\ \frac{\partial v}{\partial \nu_A} = 0 & \text{in } \Gamma_1. \end{cases} \tag{4.15}$$

Hence multiplying by $v - v_{\varepsilon_n}$ and integrating by parts we easily get

$$\begin{aligned} &|\lambda| \|v - v_{\varepsilon_n}\|^2_{L^2(\Omega)} + \nu \|D(v - v_{\varepsilon_n})\|^2_{L^2(\Omega)} \leq \\ &\leq \int_{\partial\Omega} (v - v_{\varepsilon_n}) \tfrac{\partial}{\partial \nu_A}(v - v_{\varepsilon_n}) d\sigma = \\ &= -\int_{\Gamma_0} v_{\varepsilon_n} \tfrac{\partial}{\partial \nu_A}(v - v_{\varepsilon_n}) d\sigma - \int_{\Gamma_1} (v - v_{\varepsilon_n}) \tfrac{\partial v_{\varepsilon_n}}{\partial \nu_A} d\sigma = \\ &= -\int_{\partial\Omega} v_{\varepsilon_n} \tfrac{\partial}{\partial \nu_A}(v - v_{\varepsilon_n}) d\sigma - \int_{\partial\Omega} (v - v_{\varepsilon_n}) \tfrac{\partial v_{\varepsilon_n}}{\partial \nu_A} d\sigma - \\ &- \int_{\partial\Omega} v_{\varepsilon_n} \tfrac{\partial v_{\varepsilon_n}}{\partial \nu_A} d\sigma = I + II + III. \end{aligned} \tag{4.16}$$

Let us estimate separately each term. We have

$$I \leq c \|v_{\varepsilon_n}\|_{H^{1/2}(\partial\Omega)} \left\| \frac{\partial}{\partial \nu_A}(v - v_{\varepsilon_n}) \right\|_{H^{-1/2}(\partial\Omega)} \leq C \|v_{\varepsilon_n}\|_{H^1(\Omega)} \|v - v_{\varepsilon_n}\|_{H^1(\Omega)} \ ,$$

and similarly

$$II \leq C \left\| \frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \right\|_{H^{-1/2}(\partial\Omega)} \|v - v_{\varepsilon_n}\|_{H^{1/2}(\partial\Omega)} \leq C \|v_{\varepsilon_n}\|_{H^1(\Omega)} \|v - v_{\varepsilon_n}\|_{H^1(\Omega)} \ ;$$

hence

$$I + II \leq C \|v_{\varepsilon_n}\|_{H^1(\Omega)} \|v - v_{\varepsilon_n}\|_{H^1(\Omega)} \leq \frac{\nu}{4} \|v - v_{\varepsilon_n}\|^2_{H^1(\Omega)} + \frac{C}{|\lambda|} \|v_{\varepsilon_n}\|^2_{H^1(\Omega)}$$

and by Remark 3.4 we obtain

$$I + II \leq \frac{\nu}{4} \|v - v_{\varepsilon_n}\|^2_{H^1(\Omega)} + \frac{C}{|\lambda|} \|f\|^2_{L^2(\Omega)} \ . \tag{4.17}$$

Concerning the third term, we get as before

$$III \le C\|v_{\varepsilon_n}\|_{H^{1/2}(\partial\Omega)} \left\|\frac{\partial v_{\varepsilon_n}}{\partial \nu_A}\right\|_{H^{-1/2}(\partial\Omega)} \le C\|v_{\varepsilon_n}\|^2_{H^1(\Omega)} \le \frac{C}{|\lambda|}\|f\|^2_{L^2(\Omega)} \ . \tag{4.18}$$

Thus, by (4.16), (4.17) and (4.18) it follows that

$$|\lambda|\|v - v_{\varepsilon_n}\|^2_{L^2(\Omega)} + \nu\|D(v - v_{\varepsilon_n})\|^2_{L^2(\Omega)} \le \frac{C}{|\lambda|}\|f\|^2_{L^2(\Omega)}. \tag{4.19}$$

By compactness there exist a subsequence, that we still call $v - v_{\varepsilon_n}$, and a function $w \in H^1(\Omega)$ such that

$$v - v_{\varepsilon_n} \rightharpoonup w \quad \text{in } H^1(\Omega) \quad \text{as } n \to \infty \tag{4.20}$$

and

$$v - v_{\varepsilon_n} \to w \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty. \tag{4.21}$$

Since the operator $A(\cdot, D)$ is closed on $L^2(\Omega)$ and $v - v_{\varepsilon_n}$ solves (4.14), we deduce that

$$\lambda w - A(\cdot, D)w = 0 \qquad \text{in } \Omega. \tag{4.22}$$

Now, using Remark 3.4, we can rewrite (4.19) as follows

$$\begin{aligned} &|\lambda|\|v - v_{\varepsilon_n}\|^2_{L^2(\Omega)} + \nu\|D(v - v_{\varepsilon_n})\|^2_{L^2(\Omega)} + \\ &\quad + \textstyle\int_{\Gamma_0} \frac{\vartheta_\varepsilon}{1-\vartheta_\varepsilon}|\frac{\partial v_\varepsilon}{\partial \nu_A}|^2 d\sigma + \int_{\Gamma_1} \frac{1-\vartheta_\varepsilon}{\vartheta_\varepsilon}|v_\varepsilon|^2 \, d\sigma \le \frac{C}{|\lambda|}\|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.23}$$

Using again compactness, passing possibly to another subsequence, we get the existence of two functions $\gamma \in L^2(\Gamma_0)$ and $\mu \in L^2(\Gamma_1)$ such that

$$z_{\varepsilon_n} := \sqrt{\tfrac{1-\vartheta_{\varepsilon_n}}{\vartheta_{\varepsilon_n}}}\; v_{\varepsilon_n} \rightharpoonup \gamma \text{ in } L^2(\Gamma_0), \qquad q_{\varepsilon_n} := \sqrt{\tfrac{\vartheta_{\varepsilon_n}}{(1-\vartheta_{\varepsilon_n})}}\; \tfrac{\partial v_{\varepsilon_n}}{\partial \nu_A} \rightharpoonup \mu \text{ in } L^2(\Gamma_1). \tag{4.24}$$

Now, since Ω is bounded, the $(N-1)$-dimensional measure of the set Γ_ε introduced in (3.19) is bounded by $C\varepsilon$. Consequently, by Remark 3.4 we get

$$\|v_{\varepsilon_n}\|_{L^1(\Gamma_{\varepsilon_n})} \le \|1\|_{L^2(\Gamma_{\varepsilon_n})}\|v_{\varepsilon_n}\|_{L^2(\partial\Omega)} \le \frac{C}{|\lambda|^{1/2}} \, \|f\|_{L^2(\Omega)} \, \sqrt{\varepsilon_n} \tag{4.25}$$

Thus, recalling (1.5), we deduce

$$\begin{aligned} \|v_{\varepsilon_n}\|_{L^1(\Gamma_0)} &= \|v_{\varepsilon_n}\|_{L^1(\Gamma_{\varepsilon_n})} + \|v_{\varepsilon_n}\|_{L^1(\Gamma_0\setminus\Gamma_{\varepsilon_n})} \le \\ &\le C\sqrt{\varepsilon_n} + \|\tfrac{\vartheta_{\varepsilon_n}}{1-\vartheta_{\varepsilon_n}}\|_{L^2(\Gamma_0\setminus\Gamma_{\varepsilon_n})}\|z_{\varepsilon_n}\|_{L^2(\Gamma_0)} \le \\ &\le C\sqrt{\varepsilon_n} + C\sqrt{\tfrac{\varepsilon_n}{1-\varepsilon_n}} \ , \end{aligned} \tag{4.26}$$

so that

$$v_{\varepsilon_n} \to 0 \qquad \text{in } L^1(\Gamma_0). \tag{4.27}$$

Similarly, as

$$\left\| \frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \right\|_{L^1(\Gamma_1)} \le \sqrt{\frac{\varepsilon_n}{1-\varepsilon_n}}\ \|q_{\varepsilon_n}\|_{L^2(\Gamma_1)} \le C\sqrt{\frac{\varepsilon_n}{1-\varepsilon_n}}\ , \tag{4.28}$$

we get

$$\frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \to 0 \qquad \text{in } L^1(\Gamma_1). \tag{4.29}$$

Comparing (4.20), (4.27) and (4.15), we see that

$$w = 0 \qquad \text{in } \Gamma_0. \tag{4.30}$$

Similarly, by the closedness of the operator $\frac{\partial}{\partial \nu_A}$ from $H^1(\Omega)$ into $H^{-1/2}(\partial\Omega)$, (4.20) implies that

$$\frac{\partial(v - v_{\varepsilon_n})}{\partial \nu_A} \rightharpoonup \frac{\partial w}{\partial \nu_A} \qquad \text{in } H^{-1/2}(\partial\Omega),$$

so that by (4.29) and (4.15) we obtain

$$\frac{\partial w}{\partial \nu_A} = 0 \qquad \text{in } \Gamma_1. \tag{4.31}$$

By (4.22), (4.30) and (4.31) we deduce that w solves the mixed problem (2.12) with homogeneous data, so that Theorem 2.1 allows us to conclude that $w = 0$. As a result, by (4.21), ve get $v_{\varepsilon_n} \to v$ in $L^2(\Omega)$. Now an easy argument by contradiction proves that in fact the whole sequence $\{v_{\varepsilon_n}\}$ converges to v in $L^2(\Omega)$. Thus, recalling (4.13), we have proved that

$$R(\lambda, A_{\varepsilon_n})f \to R(\lambda, A)f \quad \text{in } L^2(\Omega) \qquad \forall f \in L^2(\Omega). \tag{4.32}$$

Let us show now that

$$e^{tA_{\varepsilon_n}} f \to e^{tA} f \quad \text{in } L^2(\Omega) \qquad \forall f \in L^2(\Omega),\ \forall t > 0. \tag{4.33}$$

We recall that, by the usual representation of analytic semigroups as Dunford integrals, we have

$$e^{tA_{\varepsilon_n}} f - e^{tA} f = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\,[R(\lambda, A_{\varepsilon_n}) - R(\lambda, A)]\,f\ d\lambda \qquad \forall f \in L^2(\Omega),$$

where γ is a path from $+\infty e^{-i\vartheta}$ to $+\infty e^{i\vartheta}$, with $0 < \vartheta < \vartheta_0$, contained in S_{ϑ_0} (see Proposition 2.2) and leaving 0 on its left side. Using this representation and a simple change of variable, as the integrand is a holomorphic function of λ, we have for all $f \in L^2(\Omega)$:

$$\begin{aligned} &\|e^{tA_{\varepsilon_n}} f - e^{tA} f\|^2_{L^2(\Omega)} = \\ &\quad = \left\| \tfrac{1}{2\pi i}\textstyle\int_\gamma e^{\sigma}\left[R(\tfrac{\sigma}{t}, A_{\varepsilon_n}) - R(\tfrac{\sigma}{t}, A)\right] f\ \tfrac{d\sigma}{t} \right\|_{L^2(\Omega)} \le \\ &\quad \le Ct^{-1}\textstyle\int_\gamma e^{Re\ \sigma}\|[R(\tfrac{\sigma}{t}, A_{\varepsilon_n}) - R(\tfrac{\sigma}{t}, A)]f\|_{L^2(\Omega)}\ |d\sigma|. \end{aligned} \tag{4.34}$$

Now fix $\delta \in]1/2, 1[$: by (2.10) and (3.12) we can write

$$||e^{tA_{\varepsilon_n}} f - e^{tA} f||_{L^2(\Omega)} \leq$$
$$\leq C(\delta) ||f||^{\delta}_{L^2(\Omega)} t^{\delta-1} \int_\gamma e^{Re\ \sigma} \sigma^{-\delta} ||[R(\tfrac{\sigma}{t}, A_{\varepsilon_n}) - R(\tfrac{\sigma}{t}, A)] f||^{1-\delta}_{L^2(\Omega)} \ |d\sigma|.$$

We observe that

$$\frac{e^{Re\ \sigma}}{\sigma^\delta} \left\| \left[R\left(\frac{\sigma}{t}, A_{\varepsilon_n}\right) - R\left(\frac{\sigma}{t}, A\right) \right] f \right\|^{1-\delta}_{L^2(\Omega)} \leq C \ \frac{e^{Re \sigma}}{\sigma} T^{1-\delta} ||f||_{L^2(\Omega)} \ ,$$

and that the right-hand side has finite integral over γ; thus, recalling (4.32), by Lebesgue Theorem we get (4.33).
Moreover, since

$$||e^{tA_{\varepsilon_n}} f - e^{tA} f||_{L^2(\Omega)} \leq C(\delta) t^{\delta-1} \ ||f||^{\delta}_{L^2(\Omega)} \left[\int_\gamma e^{Re\ \sigma} \frac{|d\sigma|}{|\sigma|} \right]^{1/2} T^{1-\delta} \qquad (4.35)$$

and $\delta > 1/2$, applying again the Lebesgue Theorem we conclude that (4.12) holds true. This completes the proof of Lemma 4.4. □

We prove now that hypothesis **(ii)** in Theorem 4.3 is fulfilled too.

Lemma 4.5 *Assume* (H0), (H1), (H2), (H3), *let* B_{ε_n} *and* B *be defined by (4.5) and let* Y *and* $\mathcal{U}$ *be given by (4.3). If* $u_{\varepsilon_n} \to u$ *in* $\mathcal{U}$, *then* $B_{\varepsilon_n}(u) \to B(u)$ *in* Y.

Proof. Set $v_{\varepsilon_n} := B_{\varepsilon_n}(u_{\varepsilon_n})$ and write $u := (u_0, u_1)$. By Theorem 3.9, v_{ε_n} is the solution of

$$\begin{cases} \frac{\partial v_{\varepsilon_n}}{\partial t} - A(\cdot, D) v_{\varepsilon_n} &= 0 \qquad \text{in } [0,T] \times \overline{\Omega} \\ \vartheta_{\varepsilon_n} \frac{\partial v_{\varepsilon_n}}{\partial \nu_A} + (1 - \vartheta_{\varepsilon_n}) v_{\varepsilon_n} &= u_{\varepsilon_n} \qquad \text{in } [0,T] \times \partial\Omega \\ v_{\varepsilon_n}(0, \cdot) &= 0 \qquad \text{in } \overline{\Omega}. \end{cases} \qquad (4.36)$$

Then, by Theorem 3.2 and Remark 3.8, v_{ε_n} satisfies

$$\sup_{s \in [0,T]} ||v_{\varepsilon_n}(s, \cdot)||^2_{L^2(\Omega)} + \int_0^T ||Dv_{\varepsilon_n}||^2_{L^2(\Omega)} \ dt +$$
$$+ \int_0^T \int_{\Gamma_0} \frac{\vartheta_{\varepsilon_n}}{1 - \vartheta_{\varepsilon_n}} \left| \frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \right|^2 d\sigma \ dt + \int_0^T \int_{\Gamma_1} \frac{1 - \vartheta_{\varepsilon_n}}{\vartheta_{\varepsilon_n}} |v_{\varepsilon_n}|^2 \ d\sigma \ dt \leq \qquad (4.37)$$
$$\leq C \int_0^T \left(||u_0||^2_{H^{1/2}_{00}(\Gamma_0)} + ||u_1||^2_{H^{-1/2}(\Gamma_1)} \right) \ dt.$$

By compactness, passing possibly to a subsequence still denoted by $\{v_{\varepsilon_n}\}$, there exists a function $v \in L^2(0, T; H^1(\Omega))$ such that

$$v_{\varepsilon_n} \rightharpoonup v \qquad \text{in } L^2(0, T; H^1(\Omega)). \qquad (4.38)$$

Since this implies $v_{\varepsilon_n} \rightharpoonup v$ in $L^2(0, T; L^2(\partial\Omega))$, we also have

$$\vartheta_{\varepsilon_n} \frac{\partial v_{\varepsilon_n}}{\partial \nu_A} = u_{\varepsilon_n} - (1 - \vartheta_{\varepsilon_n}) v_{\varepsilon_n} \rightharpoonup u_0 - v \qquad \text{in } L^2(0, T; L^2(\Gamma_0)). \qquad (4.39)$$

Now, using again compactness, (4.37) implies that there exists a function $w \in L^2(0,T;L^2(\Gamma_0))$ such that

$$\vartheta_{\varepsilon_n}\frac{\partial v_{\varepsilon_n}}{\partial \nu_A} = \sqrt{(1-\vartheta_{\varepsilon_n})\vartheta_{\varepsilon_n}}\,\sqrt{\frac{\vartheta_{\varepsilon_n}}{1-\vartheta_{\varepsilon_n}}}\,\frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \rightharpoonup 0\cdot w = 0 \qquad \text{in } L^2(0,T;L^2(\Gamma_0)).$$

Then, by uniqueness, (4.39) implies

$$v = u_0 \qquad \text{on } \Gamma_0. \tag{4.40}$$

Moreover, since the operator $\frac{\partial}{\partial \nu_A}$ is closed from $H^1(\Omega)$ into $H^{-1/2}(\partial\Omega)$, by (4.38) we have

$$\frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \rightharpoonup \frac{\partial v}{\partial \nu_A} \qquad \text{in } L^2(0,T;H^{-1/2}(\partial\Omega)). \tag{4.41}$$

Then, recalling that $\vartheta_{\varepsilon_n} = 1-\varepsilon_n$ on Γ_1 we obtain

$$\vartheta_{\varepsilon_n}\frac{\partial v_{\varepsilon_n}}{\partial \nu_A} \rightharpoonup \frac{\partial v}{\partial \nu_A} \qquad \text{in } L^2(0,T;[H^{1/2}_{00}(\Gamma_1)]^*),$$

so that, by (4.41), we get

$$\frac{\partial v}{\partial \nu_A} = u_1 \qquad \text{on } \Gamma_1. \tag{4.42}$$

Finally, letting $n\to\infty$ in problem (4.36), by (4.38), (4.40) and (4.42) we conclude that v solves the equation

$$\begin{cases} \frac{\partial v}{\partial t} - A(\cdot,D)v &= 0 & \text{in } [0,T]\times\overline{\Omega} \\ v &= u_0 & \text{in } [0,T]\times\Gamma_0 \\ \frac{\partial v}{\partial \nu_A} &= u_1 & \text{in } [0,T]\times\Gamma_1 \\ v(0,\cdot) &= 0 & \text{in } \overline{\Omega}. \end{cases} \tag{4.43}$$

Therefore, by uniqueness and by Theorem 2.5, we deduce $v = B(u)$; hence we have proved that $B_{\varepsilon_n}(u_{\varepsilon_n}) \rightharpoonup B(u)$ in $L^2(0,T;H^1(\Omega))$. By Rellich Theorem we also get $B_{\varepsilon_n}(u_{\varepsilon_n}) \to B(u)$ in Y. □

We are finally ready to prove our main result.

Theorem 4.6 *Assume* (H0), (H1), (H2), (H3). *In the spaces Y, $\mathcal{U}$ introduced in (4.3), let M_{ε_n}, M, B_{ε_n} and B be defined by (4.4) and (4.5) respectively, and let J be given by (4.7). Then the sequence of optimal control problems*

$$(\mathcal{P}_n) \qquad \min\left\{J(y,u) + \chi_{\{M_{\varepsilon_n}(y)=B_{\varepsilon_n}(u)\}}(y,u) : (y,u)\in Y\times\mathcal{U}\right\}$$

Γ-converges to the optimal control problem

$$(\mathcal{P}) \qquad \min\left\{J(y,u) + \chi_{\{M(y)=B(u)\}}(y,u) : (y,u)\in Y\times\mathcal{U}\right\}.$$

Proof. We apply Theorem 4.3. In Lemma 4.4 and in Lemma 4.5 we proved that, under our present assumptions, hypotheses **(i)** and **(ii)** in Theorem 4.3 hold true. On the other hand it is easy to see that the cost functional verifies **(iii)**. Thus by Theorem 4.3 and Theorem 1.1 we conclude that the sequence of problems $(\mathcal{P}_n)$ Γ-converges to the optimal control problem $(\mathcal{P})$. $\square$

The above theorem holds for an arbitrary sequence $\varepsilon_n \to 0^+$; this allows us to say that problems $(\mathcal{P}_\varepsilon)$ Γ-converge to problem $(\mathcal{P})$ as $\varepsilon \to 0^+$.

Remark 4.7 For the sake of simplicity we considered a cost functional with a very simple form. One can also deal with more general cost functionals such as

$$\int_0^T \left(||y(s,\cdot)||_H^2 + ||u(s,\cdot)||_V^2\right) ds + \langle P_T y(T), y(T)\rangle_{L^2(\Omega)}$$

with $P_T \in \mathcal{L}(L^2(\Omega))$, provided the operator P_T is regular enough in order to have existence of the optimal controls and to satisfy condition **(iii)** of Theorem 4.3 (see [1] and [2]). The cost functional might also depend explicitly on ε: in that case one has in addition to determine the Γ-limit in (4.11).

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