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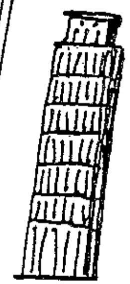

BOUNDARY CONTROL FOR NON-AUTONOMOUS PARABOLIC EQUATIONS IN NON-CYLINDRICAL DOMAINS

Paolo Acquistapace

§0. The problem

We consider the following initial-boundary value problem:

$$\begin{cases} y_t - D_i\big(a_{ij}(t,\cdot)D_j y\big) + b_j(t,\cdot)D_j y - D_j\big(c_j(t,\cdot)y\big) + h(t,\cdot)y = 0 \text{ in } \Omega_t,\ t>0, \\ a_{ij}(t,\cdot)D_j y\,\nu_i(t,\cdot) + c_j(t,\cdot)\nu_j(t,\cdot)y = u(t,\cdot) \quad \text{on } \partial\Omega_t,\ t>0, \\ y(0,\cdot) = y_0(\cdot) \quad \text{in } \Omega_0; \end{cases} \tag{0.1}$$

we assume that:

$$\Omega_0 \text{ is a bounded open set of } \mathbb{R}^n \text{ with } \partial\Omega\in C^2, \tag{0.2}$$

whereas Ω_t, the moving domain, is defined by

$$\Omega_t = T_t(\Omega_0), \tag{0.3}$$

where the map $T_t:\mathbb{R}^n\to\mathbb{R}^n$ solves

$$\begin{cases} \frac{d}{dt} T_t(x) = V(t,T_t(x)), \quad t>0 \\ T_0(x) = x \end{cases} \text{, with } V\in C^2\big([0,\infty[\times\mathbb{R}^n,\mathbb{R}^n\big); \tag{0.4}$$

$$a_{ij}, c_j \in C^1(\overline{Q}),\ b_j, h\in C^0(\overline{Q}), \text{ where } Q = \bigcup_{t>0}\big(\{t\}\times\Omega_t\big); \tag{0.5}$$

$$a_{ij}(t,\xi)\eta_i\eta_j \ge \nu(t)|\eta|^2 \quad \forall\eta\in\mathbb{R}^n,\ \forall(t,\xi)\in\overline{Q}; \quad \nu(t)>0 \quad \forall t\ge 0; \tag{0.6}$$

$$\nu(t,\xi) \text{ is the unit outward normal vector to } \partial\Omega_t \text{ at } \xi\in\partial\Omega_t; \tag{0.7}$$

$$y_0\in L^2(\Omega_0);\ u\in L^2(\Sigma), \text{ where } \Sigma = \bigcup_{t>0}\big(\{t\}\times\partial\Omega_t\big). \tag{0.8}$$

We want to study the following control problem:

$$\begin{cases} \text{minimize} \\ \qquad I(u) = \int_Q m(t,\xi)|y(t,\xi|^2 dt d\xi + \int_\Sigma n(t,\xi)|u(t,\xi)|^2 d\Sigma(t,\xi) \\ \text{among all } u\in L^2(\Sigma) \text{ subject to the state equation (0.1);} \end{cases} \tag{0.9}$$

here

$$m\in C^0(\overline{Q}),\ n\in C^0(\overline{\Sigma}),\ \text{and } m\ge 0 \text{ in } \overline{Q},\ n>0 \text{ in } \overline{\Sigma}. \tag{0.10}$$

The case of distributed control in (0.1) was studied in [7] (finite time horizon) and in [9] (infinite time horizon); the case of Dirichlet boundary control was treated in [10]. In all these papers however $a_{ij}\equiv\delta_{ij}$ and $b_j, c_j, h\equiv 0$. Non-autonomous parabolic equations in moving domains were also considered in [8].

We rewrite the state equation in weak form:

$$\int_0^\infty\int_{\Omega_t}\left[y_t\Phi + a_{ij}D_jy\,D_i\Phi + c_jy\,D_j\Phi + b_jD_jy\,\Phi + h\,y\,\Phi\right]d\xi dt = \int_0^\infty\int_{\partial\Omega_t} u\,\Phi\,dH^{n-1}(\xi)dt \quad \forall\Phi\in C^\infty\cap W^{1,2}(\overline{Q}). \tag{0.11}$$

Following [11] we are going to transform equation (0.11) into a similar one in the cylindrical domain $Q_0=]0,\infty[\times\Omega_0$, by a suitable change of variable. The latter problem will be studied in abstract form, using the results of [2,5].

§1. The change of variable

We set

$$\begin{cases} \xi = T_t(x),\ x\in\overline{\Omega_0} \text{ (for fixed } t>0), \\ z(t,x) = y(t,T_t(x)),\ (t,x)\in\overline{Q_0}, \\ v(t,x) = u(t,T_t(x)),\ (t,x)\in\overline{Q_0}. \end{cases} \tag{1.1}$$

Then a routine calculation shows that y satisfies (0.11) if and only if z satisfies

$$\int_0^\infty\int_{\Omega_t}\left[J_tz_t\varphi + A_{hk}D_kz\,D_h\varphi + C_hz\,D_h\varphi + J_tB_hD_hz\,\varphi + J_tH\,z\,\varphi\right]d\xi dt = \int_0^\infty\int_{\partial\Omega_0}\beta\,v\,\varphi\,dH^{n-1}(x)dt \quad \forall\varphi\in C^\infty\cap W^{1,2}(\overline{Q_0})\ , \tag{1.2}$$

where

$$J_t(x) = |\det DT_t(x)|, \quad (t,x)\in\overline{Q_0}, \tag{1.3}$$

$$A_{hk}(t,x) = J_t(x)\left(DT_t(x)^{-1}\right)_{hi}a_{ij}(t,T_t(x))\left[\left(DT_t(x)^{-1}\right)^*\right]_{jk}, \quad (t,x)\in\overline{Q_0}, \tag{1.4}$$

$$C_h(t,x) = J_t(x)\left(DT_t(x)^{-1}\right)_{hj}c_j(t,T_t(x)), \quad (t,x)\in\overline{Q_0}, \tag{1.5}$$

$$B_h(t,x) = \left(DT_t(x)^{-1}\right)_{hj}\left[b_j(t,T_t(x)) - V_j(t,T_t(x))\right], \quad (t,x)\in\overline{Q_0}, \tag{1.6}$$

$$H(t,x) = h(t,T_t(x)), \quad (t,x)\in\overline{Q_0}, \tag{1.7}$$

$$\beta(t,x) = J_t(x)\left|\left[\left(DT_t(x)^{-1}\right)^*\right]\cdot\nu(0,x)\right|, \quad (t,x)\in\overline{\Sigma_0}; \tag{1.8}$$

here we have set $\Sigma_0=]0,\infty[\times\partial\Omega_0$.

Thus the function z solves the following non-autonomous problem:

$$\begin{cases} z_t - J_t^{-1}D_h\left(A_{hk}(t,\cdot)D_kz\right) + B_h(t,\cdot)D_hz - J_t^{-1}D_h\left(C_h(t,\cdot)z\right) + H(t,\cdot)z = 0 \quad \text{in } \Omega_0,\ t>0, \\ A_{hk}(t,\cdot)D_kz\,\nu_h(0,\cdot) + C_h(t,\cdot)\nu_h(0,\cdot)z = \beta(t,\cdot)v(t,\cdot) \quad \text{on } \partial\Omega_0,\ t>0, \\ z(0,\cdot) = y_0(\cdot) \quad \text{in } \Omega_0. \end{cases} \tag{1.9}$$

We recall that the "area" element $d\Sigma(t,\xi)$ can be written as

$$d\Sigma(t,\xi) = \sqrt{1+\left(V(t,\xi)\cdot\nu(t,\xi)\right)^2}\,dH^{n-1}(\xi)dt = \beta(t,x)\sqrt{1+\left(V(t,T_t(x))\cdot\nu(t,T_t(x))\right)^2}\,dH^{n-1}(x)dt; \tag{1.10}$$

consequently the cost functional I(u) transforms into

$$I_0(v) = \int_0^\infty\int_{\Omega_0} M(t,\xi)|z(t,x)|^2dxdt + \int_0^\infty\int_{\partial\Omega_0} N(t,x)|v(t,x)|^2dH^{n-1}(x)dt, \tag{1.11}$$

where

$$M(t,x) = J_t(x)m(t,T_t(x)), \quad (t,x)\in\overline{Q_0}, \tag{1.12}$$

$$N(t,x) = n(t,T_t(x))\beta(t,x)\sqrt{1+\left(V(t,T_t(x))\cdot\nu(t,T_t(x))\right)^2}\ , \quad (t,x)\in\overline{\Sigma_0}. \tag{1.13}$$

The problem is now to minimize (1.11) among all $v\in L^2(\Sigma_0)$ subject to the state equation (1.9) (or (1.2)).

§2. The abstract formulation

We set $H=L^2(\Omega_0)$, $U=L^2(\partial\Omega_0)$ and consider the abstract linear operator $A(t):D_{A(t)}\subseteq H \longrightarrow H$ defined by:

$$\begin{cases} D_{A(t)} = \{z\in W^{2,2}(\Omega_0):\ \mathcal{B}(t,\cdot,D)z = 0 \text{ on } \partial\Omega_0\} \\ A(t)z = \mathcal{A}(t,\cdot,D)z, \end{cases} \tag{2.1}$$

where

$$\begin{cases} \mathcal{A}(t,\cdot,D)z = J_t^{-1}D_h\big(A_{hk}(t,\cdot)D_k z\big) - B_h(t,\cdot)D_h z + J_t^{-1}D_h\big(C_h(t,\cdot)z\big) - H(t,\cdot)z, \\ \mathcal{B}(t,\cdot,D)z = A_{hk}(t,\cdot)D_h z\, \nu_k(0,\cdot) + C_h(t,\cdot)\nu_h(0,\cdot)z \end{cases} \tag{2.2}$$

We want to show that, under suitable assumptions, $\{A(t), t\geq 0\}$ fulfills the abstract hypotheses of [5].

LEMMA 2.1 Under the assumptions of §§0,1, suppose in addition that

$$\nu(t)^{-1}\Big[|c(t,\xi)|^2+|b(t,\xi)-V(t,\xi))|^2\Big] \leq K \quad \forall(t,\xi)\in\overline{Q}; \tag{2.3}$$

then

$$\big(A(t)z|J_t z\big)_{L^2(\Omega_0)} \leq K\int_{\Omega_0} J_t(x)|z(x)|^2dx \quad \forall z\in D_{A(t)}. \tag{2.4}$$

Proof Straightforward. □

As a consequence of Lemma 2.1 we have:

PROPOSITION 2.2 Under the assumptions of §§0,1, suppose in addition that (2.3) holds. Then the control problem (1.9)-(1.11) fulfills Hypotheses 1.1, 1.2, 1.3 and 1.4 of [5].

Proof By Lemma 2.1 we see at once that if $\lambda\in\mathbb{R}$ and $\lambda>K$, then λ cannot be an eigenvalue of $A(t)$. On the other hand, in particular, each $A(t)$ is the infinitesimal generator of an analytic semigroup in H, whose resolvent operators $[\lambda-A(t)]^{-1}$ are compact; thus, clearly, there exists $\lambda_0\in\mathbb{R}$ such that $\lambda_0\in\rho(A(t))$ and the operators $[\lambda_0-A(t)]^{\alpha}$ are well defined for each $t\geq 0$ and $\alpha>0$. This shows that Hypothesis 1.1 of [5] holds true.

It is also clear that we can apply to both $\{A(t)\}$ and $\{A(t)^*\}$ the abstract results of [4,3,1], so that there exists the evolution operator

$U(t,s)$ associated to $\{A(t)\}$, with its usual properties. Moreover by [2,§2.3] we deduce that Hypothesis 1.2 of [5] holds; namely, for each $\beta,\mu\in[-1,1]$, $[a,b]\subset[0,\infty[$ and $a\leq s<t\leq b$ the operator $[\lambda_0-A(t)]^{\beta}U(t,s)[\lambda_0-A(s)]^{-\mu}$ can be continuously extended to H, it is strongly continuous as a function of (t,s) and there exists $M_{\beta,\mu}>0$ such that

$$\|[\lambda_0-A(t)]^{\beta}U(t,s)[\lambda_0-A(s)]^{-\mu}\|_{\mathcal{L}(H)} \leq M_{\beta,\mu}\big[1+(t-s)^{\mu-\beta}\big] \quad \forall a\leq s<t\leq b. \tag{2.5}$$

Next, for each $t>0$ we define the "Green map" $G(t)$ by:

$$w=G(t)g \iff \begin{cases} \lambda_0 w - \mathcal{A}(t,\cdot,D)w = 0 & \text{in } \Omega_0 \\ \mathcal{B}(t,\cdot,D)w = g & \text{on } \partial\Omega_0 . \end{cases} \tag{2.6}$$

By the results of [2,§2.5] and [6] we easily get Hypothesis 1.3 of [5], i.e.

$$t\longrightarrow[\lambda_0-A(t)]^{\alpha}G(t) \in L^{\infty}_{loc}([0,\infty[,\mathcal{L}(U,H)) \quad \forall\alpha\in]0,3/4[. \tag{2.7}$$

Hence, following [5], we can now rewrite the state equation (1.9) in abstract form:

$$z(t) = U(t,0)y_0 + \int_0^t\Big[[\lambda_0-A(s)^*]^{1-\alpha}U(t,s)^*\Big]^*[\lambda_0-A(s)^{\alpha}]G(s)[\beta(s)v(s)]ds; \tag{2.8}$$

(writing $z(t),\beta(s),v(s)$ instead of $z(t,\cdot),\beta(s,\cdot),v(s,\cdot)$).

Similarly, introducing the operators $C(t)\in\mathcal{L}(H)$, $N(t)\in\mathcal{L}(U)$ given by

$$C(t)f = M(t,\cdot)^{1/2}f(\cdot), \qquad N(t)g = N(t,\cdot)g(\cdot), \tag{2.9}$$

the cost functional $I_0(v)$ can be rewritten as

$$I_0(v) = \int_0^{\infty}\Big[\|C(t)z(t)\|_H^2 + \big(N(t)v(t)|v(t)\big)_U\Big]dt, \tag{2.10}$$

and by (0.10) it is clear that Hypothesis 1.4 of [5] is fulfilled, i.e.

$$C(\cdot)\in L^{\infty}_{loc}([0,\infty[,\Sigma^+(H)), \quad N(\cdot)\in L^{\infty}_{loc}([0,\infty[,\Sigma^+(U)), \text{ and } N(t)>0\ \forall t\geq 0. \tag{2.11}$$

This concludes the proof of Proposition 2.2. □

Thus the control problem (2.8)-(2.10) is covered by the theory of [5]. In particular, by [5,§2] we can solve the associated Riccati equation - and, consequently, get the synthesis of the control problem - if and only if the following finite cost condition holds true:

$$\begin{cases} \text{for each } t_0 \geq 0 \text{ there exists } c(t_0) \geq 0 \text{ such that to each } y_0 \in H \text{ there} \\ \text{corresponds a control } v \in L^2([t_0,\infty[,U) \text{ for which} \\ \qquad I_{t_0}(v) \leq c(t_0)\|y_0\|_H^2 , \end{cases} \tag{2.12}$$

where

$$I_{t_0}(v) = \int_{t_0}^{\infty} \left[\|C(t)y(t)\|_H^2 + \big(N(t)v(t)|v(t)\big)_U\right]dt, \tag{2.13}$$

with z given by (2.8).

§3. The finite cost condition

We want to prove:

PROPOSITION 3.1 Under the hypotheses of §§0,1,2, assume in addition that:
(i) there exists $\delta>0$ such that

$$2h(t,\xi) \geq \nu(t)^{-1}|c(t,\xi)+b(t,\xi)|^2 + \delta \quad \forall (t,\xi)\in\overline{Q}, \tag{3.1}$$

(ii) the norm $\gamma(t)$ of the trace operator $\Omega_t \longrightarrow \partial\Omega_t$ satisfies (3.15) below (this is a condition on Ω_0 and T_t, see (3.12) below).
Then there exists $\overline{c}>0$ such that to each $y_0\in H$ there corresponds a control $v\in L^2([0,\infty[,U)$ for which

$$I_0(v) \leq \overline{c}\ \|y_0\|_H^2 . \tag{3.2}$$

<u>Proof</u> Firstly we remark that, due to the well known property

$$\frac{d}{dt}J_t(x) = J_t(x) \operatorname{div} V(t,T_t(x)) \quad \forall t>0,\ \forall x\in\Omega_t, \tag{3.3}$$

for each smooth $y:\overline{Q}\longrightarrow\mathbb{R}$ we have the formula

$$\frac{d}{dt}\int_{\Omega_t} |y(t,\xi)|^2 d\xi = 2\int_{\Omega_t} y(t,\xi)y_t(t,\xi)d\xi + \int_{\partial\Omega_t} |y(t,\xi)|^2 V(t,\xi)\cdot\nu(t,\xi)dH^{n-1}(\xi). \tag{3.4}$$

Hence if y is a solution of (0.1) with given control u, we easily get

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} |y|^2 d\xi &= \int_{\partial\Omega_t} y\left[2u + y\ V(t,\xi)\cdot\nu(t,\xi)\right]dH^{n-1}(\xi) - \\ &- 2\int_{\Omega_t} \left[a_{ij}(t,\xi)D_j y\ D_i y + (c_j(t,\xi)+b_j(t,\xi))D_j y\ y + h(t,\xi)\ y^2\right]d\xi.\end{aligned} \tag{3.5}$$

Now we choose the feedback control

$$u(t,\xi) = -\frac{1}{2}\, y(t,\xi)\ V(t,\xi)\cdot\nu(t,\xi), \qquad (t,\xi)\in\overline{\Sigma}; \tag{3.6}$$

inserting (3.6) into (3.5) we easily get:

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} |y|^2 d\xi &\leq -2\nu(t)\int_{\Omega_t} |Dy|^2 d\xi + 2\int_{\Omega_t} \left[|c(t,\xi)+b(t,\xi)|\,|y|\,|Dy| - h(t,\xi)y^2\right]d\xi \\ &\leq -\nu(t)\int_{\Omega_t} |Dy|^2 d\xi + \int_{\Omega_t} \left[\nu(t)^{-1}|c(t,\xi)+b(t,\xi)|^2 - 2h(t,\xi)\right]y^2 d\xi.\end{aligned} \tag{3.7}$$

By (3.1) we deduce

$$\frac{d}{dt}\int_{\Omega_t} |y|^2 d\xi \leq -\delta \int_{\Omega_t} |y|^2 d\xi - \nu(t)\int_{\Omega_t} |Dy|^2 d\xi, \tag{3.8}$$

so that

$$\int_{\Omega_t} |y|^2 d\xi \leq e^{-\delta t}\int_{\Omega_0} |y_0|^2 dx - \int_0^t e^{-\delta(t-s)}\nu(s)\int_{\Omega_s} |Dy|^2 d\xi ds; \tag{3.9}$$

hence inserting (3.9) into (0.9) we easily obtain

$$\begin{aligned} I(u) \leq& \int_0^\infty e^{-\delta t}\|m(t,\cdot)\|_{L^\infty(\Omega_t)} dt\int_{\Omega_0} |y_0|^2 dx - \\ &- \int_0^\infty \nu(s)\int_s^\infty e^{-\delta(t-s)}\|m(t,\cdot)\|_{L^\infty(\Omega_t)} dt\int_{\Omega_s} |Dy|^2 d\xi ds + \\ &+ \frac{1}{4}\int_0^\infty \|n(t,\cdot)\|_{L^\infty(\partial\Omega_t)} \|V(t,\cdot)\|^2_{L^\infty(\partial\Omega_t)}\int_{\partial\Omega_t} y^2 dH^{n-1}(\xi)dt .\end{aligned} \tag{3.10}$$

Now by the inclusion (for instance) $H^{3/4,2}(\Omega_t)\subset L^2(\partial\Omega_t)$ and by interpolation we have, for each $\varepsilon(t)>0$:

$$\int_{\partial\Omega_t} y^2 dH^{n-1}(\xi)dt \leq \gamma(t)\|y\|^2_{H^{3/4,2}(\Omega_t)} \leq \varepsilon(t)\int_{\Omega_t} |Dy|^2 d\xi + \frac{\gamma(t)^3}{\varepsilon(t)^3}\int_{\Omega_t} |y|^2 d\xi \tag{3.11}$$

where $\gamma(t)$ is the norm of the trace operator $\Omega_t \longrightarrow \partial\Omega_t$; taking into account

(0.2), (0.3) and (0.4) it is easy to show that

$$\gamma(t) \le c(\Omega_0)\|J_t\|_{L^\infty(\Omega_0)}\|J_t^{-1}\|_{L^\infty(\Omega_0)}\|DT_t\|_{L^\infty(\Omega_0)}\|(DT_t)^{-1}\|_{L^\infty(\Omega_0)}. \tag{3.12}$$

By (3.10) and (3.11) we easily deduce

$$\begin{aligned} I(u) \le & \int_0^\infty e^{-\delta t}\Big[\|m(t,\cdot)\|_{L^\infty(\Omega_t)} + \\ & + \tfrac{1}{4}\|n(t,\cdot)\|_{L^\infty(\partial\Omega_t)}\|V(t,\cdot)\|^2_{L^\infty(\partial\Omega_t)}\frac{\gamma(t)^3}{\varepsilon(t)^3}\Big]dt\int_{\Omega_0}|y_0|^2dx - \\ & - \int_0^\infty\Big\{\nu(s)\int_s^\infty e^{-\delta(t-s)}\Big[\|m(t,\cdot)\|_{L^\infty(\Omega_t)} + \\ & + \tfrac{1}{4}\|n(t,\cdot)\|_{L^\infty(\partial\Omega_t)}\|V(t,\cdot)\|^2_{L^\infty(\partial\Omega_t)}\frac{\gamma(t)^3}{\varepsilon(t)^3}\Big]dt - \\ & - \tfrac{1}{4}\|n(s,\cdot)\|_{L^\infty(\partial\Omega_s)}\|V(s,\cdot)\|^2_{L^\infty(\partial\Omega_s)}\varepsilon(s)\Big\}\int_{\Omega_s}|Dy|^2d\xi ds. \end{aligned} \tag{3.13}$$

Now we take $\varepsilon(t)$ sufficiently small, in such a way that

$$\begin{aligned} & \tfrac{1}{4}\|n(s,\cdot)\|_{L^\infty(\partial\Omega_s)}\|V(s,\cdot)\|^2_{L^\infty(\partial\Omega_s)}\varepsilon(s) \le \\ & \le \nu(s)\int_s^\infty e^{-\delta(t-s)}\Big[\|m(t,\cdot)\|_{L^\infty(\Omega_t)} + \tfrac{1}{4}\|n(t,\cdot)\|_{L^\infty(\partial\Omega_t)}\|V(t,\cdot)\|^2_{L^\infty(\partial\Omega_t)}\frac{\gamma(t)^3}{\varepsilon(t)^3}\Big]dt \end{aligned} \tag{3.14}$$

(which is always possible); thus if we assume that

$$\int_0^\infty e^{-\delta t}\Big[\|m(t,\cdot)\|_{L^\infty(\Omega_t)} + \tfrac{1}{4}\|n(t,\cdot)\|_{L^\infty(\partial\Omega_t)}\|V(t,\cdot)\|^2_{L^\infty(\partial\Omega_t)}\frac{\gamma(t)^3}{\varepsilon(t)^3}\Big]dt < \infty, \tag{3.15}$$

then by (3.13) we conclude that there exists $c>0$ such that for each $y_0\in H$ we can find a control $u\in L^2(Q)$ for which $I(u)\le c\|y_0\|_H^2$; hence the same is true for $I_0(v)$, i.e. (3.2) holds. The proof is complete. □

In particular, Proposition 3.1 shows that the finite cost condition (2.12) holds true for $I(u)$ and hence for $I_0(v)$; in fact it holds uniformly with respect to t, i.e. Hypothesis 3.1 of [5] is fulfilled too. Thus by [5, Proposition 3.2], the Riccati equation associated to the control problem (2.8)-(2.10) has a unique minimal solution $P(\cdot)\in L^\infty([0,\infty[,\Sigma^+(H))$; hence, following [5, §2.5], we get a unique optimal pair $(\hat{z},\hat{v})$ for problem (2.8)-

-(2.10). Going back to the corresponding $(\hat{y},\hat{u})$ we obtain a unique optimal pair for problem (0.1)-(0.9).

§4. Further properties

Following [5], we look for further properties of our control problem. We have:

PROPOSITION 4.1 Under the hypotheses of §§0,1,2, assume in addition that

$$|\operatorname{div} V(t,\xi)| \le L \quad \forall (t,\xi)\in\overline{Q}, \tag{4.1}$$

and

$$\max_{\overline{\Omega}_0} J_s \le m_0 \min_{\overline{\Omega}_0} J_t \quad \forall 0\le s<t<\infty. \tag{4.2}$$

Then Hypothesis 3.3 of [5] holds true; in particular the evolution operator $U(t,s)$ introduced in Section 2 decays exponentially as $t-s \to \infty$.

<u>Proof</u> Set $w(s)=U(t,s)y_0$. Then w solves

$$\begin{cases} w'(t)=A(t)w(t), & t>s, \\ w(s)=y_0. \end{cases} \tag{4.3}$$

By (2.4) we have:

$$\big(w'(t)\,|\,J_t w(t)\big)_H = \big(A(t)w(t)\,|\,J_t w(t)\big)_H \le K\int_{\Omega_0} J_t(x)|w(t,x)|^2dx; \tag{4.4}$$

on the other hand

$$\big(w'(t)\,|\,J_t w(t)\big)_H = \frac{1}{2}\frac{d}{dt}\big(w(t)\,|\,J_t w(t)\big)_H - \big(w(t)\,|\,w(t)\tfrac{d}{dt}J_t\big)_H ,$$

so that using (3.4) we get

$$\frac{1}{2}\frac{d}{dt}\big(w(t)\,|\,J_t w(t)\big)_H \le K\int_{\Omega_0} J_t(x)|w(t,x)|^2\big[1 + \operatorname{div} V(t,T_t(x))\big]dx ,$$

and by (4.1) we obtain

$$\frac{d}{dt}\int_{\Omega_0} J_t(x)|w(t,x)|^2dx \le 2KL\int_{\Omega_0} J_t(x)|w(t,x)|^2dx ,$$

which implies

$$\int_{\Omega_0} J_t(x)\,|w(t,x)|^2 dx \le \int_{\Omega_0} J_s(x)\,|y_0(x)|^2 dx\; e^{2KL(t-s)}.$$

Finally by (4.2) we easily deduce for $0\le s<t$

$$\|w(t)\|_H = \|U(t,s)y_0\|_H \le \sqrt{m_0}\;\|y_0\|_H\; e^{KL(t-s)}; \tag{4.5}$$

the other properties of [5, Hypothesis 3.3] are straightforward. □

Let us now try to verify Hypothesis 3.4 of [5], i.e. that

$$[\lambda_0-A(\cdot)]^{\alpha}G(\cdot)\in L^{\infty}([0,\infty[,\mathcal{L}(U,H)) \quad \forall\alpha\in]0,3/4[. \tag{4.6}$$

PROPOSITION 4.2 Under the hypotheses of §§0,1,2, assume in addition that condition (3.1) holds and that

$$\nu(t)\,\min_{\overline{\Omega}_0}\left(J_t(\cdot)\,|DT_t(\cdot)|^{-2}\right) \ge \varepsilon_0>0 \quad \forall t\ge 0, \tag{4.7}$$

$$a_{ij},\,b_j-V_j,\,c_j,\,h\in L^{\infty}(Q), \tag{4.8}$$

$$\sup_{t\ge 0}\left\|D\left(J_t(\cdot)^{-1}\right)\right\|_{L^{\infty}(\Omega_0)},\quad \sup_{t\ge 0}\left\|DT_t(\cdot)^{-1}\right\|_{L^{\infty}(\Omega_0)} < \infty . \tag{4.9}$$

Then [5,Hypothesis 3.4] is fulfilled, i.e.(4.6) holds.

<u>Proof</u> Set $w(t)=G(t)g$, i.e. $w(t)$ is the solution of problem (2.6). If $g\in W^{1/2,2}(\Omega_0)$, then by classical results $w(t)\in W^{2,2}(\Omega_0)$ and

$$\|w(t)\|_{W^{2,2}(\Omega_0)} \le c_1\|g\|_{W^{1/2,2}(\partial\Omega_0)} . \quad \forall t\ge 0. \tag{4.10}$$

On the other hand, multiplying in (2.6) by $J_t w(t)$ and integrating by parts, we easily get, after straightforward calculations:

$$\int_{\Omega_0}\left[h(t,T_t(x)) - \frac{1}{2}\,\nu(t)^{-1}\left|c(t,T_t(x))+b(t,T_t(x))\right|\right]J_t(x)\,|w(t,x)|^2 dx\; +$$

$$+\;\frac{1}{2}\,\nu(t)\int_{\Omega_0} J_t(x)\,|DT_t(x)|^{-2}\,|Dw(t,x)|^2 dx \le$$

$$\le \left|\int_{\partial\Omega_0} g(x)w(t,x)\,dH^{n-1}(x)\right| \le \|g\|_{W^{-1/2,2}(\partial\Omega_0)}\|w(t)\|_{W^{1,2}(\Omega_0)},$$

which implies, taking into account (3.1),

$$\|w(t)\|_{W^{1,2}(\Omega_0)} \le c_2\|g\|_{W^{-1/2,2}(\partial\Omega_0)} \quad \forall t\ge 0. \tag{4.11}$$

The result follows by interpolation between (4.10) and (4.11). □

Finally, Hypotheses 3.5 and 3.8 of [5] are clearly satisfied provided we strengthen assumption (0.10) by requiring

$$m\in C^0\cap L^{\infty}(\overline{Q}),\quad n\in C^0\cap L^{\infty}(\overline{\Sigma}),\quad m(t,\xi)\ge m_0>0 \text{ in } \overline{Q},\quad n(t,\xi)\ge n_0>0 \text{ in } \overline{\Sigma}. \tag{4.12}$$

By [5,Theorems 3.10 and 3.11] we obtain the following result:

PROPOSITION 4.3 Under the hypotheses of §§0,1,2, assume in addition that conditions (3.1), (4.7), (4.8), (4.9) and (4.12) hold. Then the optimal trajectory $\hat{z}$ of the control problem (2.8)-(2.10) is stable, and consequently the associated Riccati equation has a unique bounded solution. □

§5. An example

The list of all assumptions required in the various propositions above is rather long and involved. Here is a very simple example where almost all of the above conditions are fulfilled.

Take as Ω_0 the open unit ball of R^n, and set

$$V(t,\xi) = \frac{\xi}{1+t}\;;$$

then $T_t(x)=(1+t)x$, $J_t(x)=(1+t)^n$ and

$$\Omega_t=\{\xi\in R^n: |\xi|<1+t\}.$$

Next, choose

$$a_{ij}=(1+t)^{2-n}\delta_{ij}\;,\quad b_j=(1+t)^{-1}\xi_j\;,\quad c_j=0,\quad h=(1+t)^{n-3}(1+|\xi|)\;,$$

$$m=t^a,\quad n=(1+t)^b,\quad \text{with } a,b>0.$$

Then it is a straightforward task to verify that the hypotheses of §§0,1 and conditions (2.3), (3.1), (3.15), (4.1), (4.2), (4.7), (4.9) hold true, whereas (4.8) and (4.12) do not. In order to satisfy (4.12) we just need obvious modifications in the choice of m and n; on the contrary, condition (4.8) essentially says that V is bounded as well as b_j, a_{ij} and h: this in particular requires a strong restriction in the choice of Ω_t.

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