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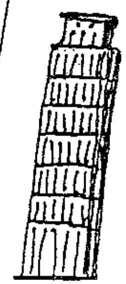

ABSTRACT LINEAR NON-AUTONOMOUS PARABOLIC EQUATIONS: A SURVEY

Paolo Acquistapace

0. Introduction.

Let E be a Banach space. Consider the abstract Cauchy problem

$$\begin{cases} u'(t)-A(t)u(t)=f(t), \quad t\in]0,T], & (0.1)\\ u(0)=x, & (0.2)\end{cases}$$

where $T>0$, $x\in E$, $f\in C([0,T],E)$ and $\{A(t), t\in[0,T]\}$ is a family of generators of analytic semigroups in E; more precisely we assume for each $t\in[0,T]$:

$$\begin{cases} A(t):D_{A(t)}\subseteq E\longrightarrow E \text{ is a closed linear operator,} \\ \rho(A(t))\supseteq S(\vartheta_0):=\{0\}\cup\{z\in\mathbb{C}: |\arg z|<\vartheta_0\} \quad \text{with } \vartheta_0>\pi/2, \\ \|[\lambda-A(t)]^{-1}\|_{\mathcal{L}(E)}\le \frac{M}{1+|\lambda|} \quad \forall\lambda\in S(\vartheta_0). \end{cases} \tag{0.3}$$

The domains $D_{A(t)}$ may depend on t and be not dense in E (however they are necessarily dense if E is reflexive, see [15]). Further regularity assumptions on the map $t\longrightarrow A(t)$ will be introduced later; there is a lot of different hypotheses of this kind in the literature, generally independent of one another, which lead to existence and regularity results for strict and classical solutions of problem (0.1)-(0.2).

The aim of this paper is to describe the main known results, improve some of them and, above all, show how slight modifications of a unique approach naturally lead to consider such different kinds of assumptions.

1. Notations.

In the sequel we write Au instead of $A(\cdot)u(\cdot)$, and use the spaces

$$C([0,T],D_A):=\{v\in C([0,T],E):\ v(t)\in D_{A(t)}\ \forall t\in[0,T] \text{ and } Av\in C([0,T],E)\}$$

and $C(]0,T],D_A)$, whose definition is similar.

A strict solution of (0.1) is a function $u\in C^1([0,T],E)\cap C([0,T],D_A)$ which satisfies (0.1) in $[0,T]$; a classical solution is a function $u\in C([0,T],E)\cap C^1(]0,T],E)\cap C(]0,T],D_A)$ which satisfies (0.1) in $]0,T]$.

We will deal with the real interpolation spaces

$$D_{A(t)}(\vartheta,\infty):=\big(D_{A(t)},E\big)_{1-\vartheta,\infty}$$

whose use is crucial in regularity questions; such spaces are exactly characterized in several concrete cases (for $E=L^p(\Omega)$ see [14], for $E=C(\overline{\Omega})$ see [21,10,1]). We will also use the space

$$B(0,T;D_A(\vartheta,\infty)):=$$

$$\Big\{v:[0,T]\longrightarrow E:\ v(t)\in D_{A(t)}(\vartheta,\infty)\ \forall t\in[0,T] \text{ and } \sup_{t\in[0,T]}\|v(t)\|_{D_{A(t)}(\vartheta,\infty)}<\infty\Big\}.$$

Remark 1.1 The notation $L^\infty(0,T;D_A(\vartheta,\infty))$ has no meaning in general. But even in the case $D_{A(t)}(\vartheta,\infty)\equiv D_\vartheta$ (independent of t), an element of $B(0,T;D_\vartheta)$ needs not be Bochner measurable with values in D_ϑ, i.e. $L^\infty(0,T;D_\vartheta)$ is strictly contained in $B(0,T;D_\vartheta)$, as the following example shows.

Example 1.2 This example is due to T. Zabczyk (unpublished). Set $T:=1$, $E:=C([0,1])$, $A:=d^2/dx^2$ with domain $D_A:=\{u\in C^2([0,1]):u'(0)=u'(1)=0\}$; then it is known that $D_A(\alpha/2,\infty)=C^\alpha([0,1])$ for each $\alpha\in]0,1[$ (see [10]). We sketch the construction of a function $f\in C\big([0,1],C([0,1])\big)\cap B\big(0,1;C^\alpha([0,1])\big)$ which is not Bochner measurable as a $C^\alpha([0,1])$-valued function.

Firstly, we remark that there exists a sequence $\{\varphi_n\}\subseteq C([0,1])$ such that $0\le\varphi_n(t)\le1$ for each $t\in[0,1]$, and the map $\Phi:[0,1]\longrightarrow[0,1]^{\mathbb{N}}$, defined by $\Phi(t):=\{\varphi_n(t)\}_{n\in\mathbb{N}}$, is onto (Peano's map). Next, we fix a strictly increasing sequence $\{r_n\}\subseteq[0,1[$ such that $r_0=0$, $r_n\longrightarrow1$ as $n\longrightarrow\infty$, and $\sum_{n=0}^{\infty}\big(r_{n+1}-r_n\big)^\alpha<\infty$; now we define

$$\psi_n(x):=\begin{cases} c_n(x-r_n)^\alpha & \text{if } x\in[r_n,(r_n+r_{n+1})/2],\\ c_n(r_{n+1}-x)^\alpha & \text{if } x\in[(r_n+r_{n+1})/2,r_{n+1}],\\ 0 & \text{otherwise,}\end{cases}$$

where c_n is such that $[\psi_n]_{C^\alpha([0,1])}=1$ for each $n\in\mathbb{N}$. Finally, we set

$$\big[f(t)\big](x):=\sum_{n=0}^{\infty}\varphi_n(t)\psi_n(x)\qquad(t,x\in[0,1]).$$

It is a straightforward task to show that this function belongs to $C\big([0,1],C([0,1])\big)\cap B\big(0,1;C^\alpha([0,1])\big)$. If f were also Bochner measurable with values in $C^\alpha([0,1])$, then it is well known that its range $f([0,1])$ would be a separable subset of $C^\alpha([0,1])$. But this is not the case: indeed, fix two

distinct subsets $A,B\subseteq\mathbb{N}$ and let $e_A,e_B\in[0,1]^{\mathbb{N}}$ be such that

$$(e_A)_n=\begin{cases}0 & \text{if } n\in\mathbb{N}-A\\ 1 & \text{if } n\in A\end{cases},\qquad (e_B)_n=\begin{cases}0 & \text{if } n\in\mathbb{N}-B\\ 1 & \text{if } n\in B\end{cases};$$

as Φ is onto, there exist $t_A,t_B\in[0,1]$ such that $\Phi(t_A)=e_A$, $\Phi(t_B)=e_B$. Hence it is easy to deduce that

$$\|f(t_A)-f(t_B)\|_{C^\alpha([0,1])}\ge1,$$

and since $\{f(t_A),A\subseteq\mathbb{N}\}$ is uncountable, $f([0,1])$ cannot be separable. □

2. The autonomous case.

Assume $A(t)\equiv A$, with obvious modifications in (0.3). Then the following facts are well known (see [22, Prop.2.4 and Theorems 4.4, 4.5, 5.4, 5.5]):

Theorem 2.1 (i) If a solution of (0.1)-(0.2) exists, then it has the form

$$u(t)=e^{tA}x+\int_0^t e^{(t-s)A}f(s)ds,\qquad t\in[0,T],\tag{2.1}$$

where the semigroup $\{e^{sA}\}$ is expressed by the Dunford integral

$$e^{sA}=\frac{1}{2\pi i}\int_\gamma e^{s\lambda}[\lambda-A]^{-1}d\lambda\qquad\forall s>0,\tag{2.2}$$

γ being a smooth path contained in $S(\vartheta_0)$ and joining $+\infty e^{-i\vartheta}$ to $+\infty e^{i\vartheta}$ $(\pi/2<\vartheta<\vartheta_0)$.

(ii) Assume either $f\in C^\varepsilon([0,T],E)$ or $f\in C([0,T],E)\cap B(0,T;D_A(\varepsilon,\infty))$ $(0<\varepsilon<1)$; then:

- (a) the solution is classical provided $x\in\overline{D_A}$;
- (b) the solution is strict provided $x\in D_A$ and $Ax+f(0)\in\overline{D_A}$;
- (c) The solution has the maximal regularity property, i.e. u' and Au belong to the same space as f, provided $x\in D_A$ and $Ax+f(0)\in D_A(\varepsilon,\infty)$. □

We now wish to generalize Theorem 2.1 to the non-autonomous case, by a suitable perturbation argument.

3. The approach to the non-autonomous case.

We proceed formally: let u be any solution of the non-autonomous problem (0.1)-(0.2). For fixed $t\in]0,T]$ introduce the auxiliary function

$$v(s):=e^{(t-s)B(t,s)}u(s), \quad s\in[0,t], \tag{3.1}$$

where $B(t,s)$ is some operator to be chosen later, defined for $0\le s\le t$ and satisfying (0.3). Differentiating (3.1) with respect to s we get (using (0.3)):

$$v'(s)=$$
$$=\left[-B(t,s)e^{(t-s)B(t,s)}+\left[\frac{\partial}{\partial s}e^{rB(t,s)}\right]_{r=t-s}\right]u(s) + e^{(t-s)B(t,s)}\left[A(s)u(s)+f(s)\right];$$

taking into account (0.2), an integration from 0 to t yields

$$\begin{aligned} u(t)-e^{tB(t,0)}x &= \int_0^t e^{(t-s)B(t,s)}f(s)ds + \\ &+\int_0^t\left[e^{(t-s)B(t,s)}\left[A(s)-B(t,s)\right]+\left[\frac{\partial}{\partial s}e^{rB(t,s)}\right]_{r=t-s}\right]u(s)ds. \end{aligned} \tag{3.2}$$

This is a Volterra integral equation for the solution u, with kernel

$$K(t,s):=e^{(t-s)B(t,s)}\left[A(s)-B(t,s)\right]+\left[\frac{\partial}{\partial s}e^{rB(t,s)}\right]_{r=t-s} \tag{3.3}$$

belonging to $\mathcal{L}(E)$. Thus our strategy will be the following:

Step 1 find some assumption on $t\longrightarrow A(t)$, such that (3.2) is solvable for suitable data x,f;

Step 2 show that the solution of (3.2) is in fact the solution of problem (0.1)-(0.2).

As a consequence of Step 1 we will get a representation formula, and hence uniqueness, for the solution of (0.1)-(0.2); from Step 2 we will deduce existence of (strict and classical) solutions and their maximal regularity. According to different possible choices of $B(t,s)$ we will need different kinds of assumptions on the map $t\longrightarrow A(t)$, corresponding to the various, independent ones appeared in the literature.

We remark that using (3.3) and (2.2) we can rewrite the integral term of (3.2) in the following way:

$$\begin{aligned} &\int_0^t K(t,s)u(s)ds = \\ &=\frac{1}{2\pi i}\int_0^t\int_\gamma e^{(t-s)\lambda}[\lambda-B(t,s)]^{-1}\left[A(s)-B(t,s)+\frac{\partial}{\partial s}B(t,s)[\lambda-B(t,s)]^{-1}\right]u(s)d\lambda ds \end{aligned} \tag{3.4}$$

or, after an integration by parts,

$$\begin{aligned} \int_0^t K(t,s)u(s)ds &= \frac{1}{2\pi i}\int_0^t\int_\gamma e^{(t-s)\lambda}\Big[[\lambda-B(t,s)]^{-1}\cdot \\ &\cdot\left[A(s)-B(t,s)+\left[\frac{\partial}{\partial s}B(t,s),[\lambda-B(t,s)]^{-1}\right]+(t-s)\frac{\partial}{\partial s}B(t,s)\right]u(s)d\lambda ds, \end{aligned} \tag{3.5}$$

where $[U,V]$ stands for the commutator $UV-VU$.

We now have to choose $B(t,s)$. Reasonable choices are the following:

I. $B(t,s):=A(s)$, II. $B(t,s):=A(t)$,

III. $B(t,s):=A(0)$, IV. $B(t,s):=\frac{1}{t-s}\int_s^t A(\sigma)d\sigma$.

In the next section we will perform Step 1 with each one of the above choices of $B(t,s)$.

4. Solving the integral equation (3.2).

4.1. The first choice, $B(t,s)=A(s)$.

By (3.3)

$$\int_0^t K(t,s)u(s)ds = \frac{1}{2\pi i}\int_0^t\int_\gamma e^{(t-s)\lambda}\frac{d}{ds}[\lambda-A(s)]^{-1}u(s)d\lambda ds.$$

Hence in order to solve (3.2) one is lead to assume, for some $\alpha\in]0,1[$ and $N,\eta>0$,

$$\left\|\frac{d}{ds}[\lambda-A(s)]^{-1}\right\|_{\mathcal{L}(E)} \le \frac{N}{1+|\lambda|^\alpha} \quad \forall s\in[0,T], \tag{4.1}$$

$$\left\|\frac{d}{dt}A(t)^{-1}-\frac{d}{ds}A(s)^{-1}\right\|_{\mathcal{L}(E)} \le N|t-s|^\eta \quad \forall t,s\in[0,T]. \tag{4.2}$$

Conditions (4.1)-(4.2) were introduced in [18] and revisited in [3,11]; they - in fact (4.1) alone: (4.2) is needed only in Step 2 - allow to solve the integral equation (3.2) in the space $C([0,T],E)$, provided $x\in\overline{D_A}$ and $f\in C([0,T],E)$: see [3, Proposition 1.9].

4.2. The second choice, $B(t,s)=A(t)$.

The integral term (3.4) becomes

$$\int_0^t K(t,s)u(s)ds = \int_0^t A(t)e^{(t-s)A(t)}\left[A(t)^{-1}-A(s)^{-1}\right]A(s)u(s)ds.$$

Applying $A(t)$ to both sides of (3.2) we get

$$\begin{aligned} A(t)u(t)-A(t)e^{tA(t)}x &= A(t)\int_0^t e^{(t-s)A(t)}f(s)ds + \\ &+ \int_0^t A(t)^2 e^{(t-s)A(t)}\left[A(t)^{-1}-A(s)^{-1}\right]A(s)u(s)ds, \end{aligned} \tag{4.3}$$

and this is an integral equation in the unknown Au.

If we assume

$$\begin{cases} \|A(t)[\lambda-A(t)]^{-1}\left[A(t)^{-1}-A(s)^{-1}\right]\|_{\mathcal{L}(E)} \le N \dfrac{|t-s|^{\alpha}}{1+|\lambda|^{\rho}} \ \forall t,s\in[0,T], \ \forall\lambda\in S(\vartheta_0) \\ \delta := \alpha+\rho-1>0, \end{cases} \tag{4.4}$$

then it is easy to check that

$$\|A(t)^2 e^{(t-s)A(t)}\left[A(t)^{-1}-A(s)^{-1}\right]\|_{\mathcal{L}(E)} \le \frac{C}{|t-s|^{1-\delta}} \quad \forall t,s\in[0,T], \tag{4.5}$$

so that (4.3) can be solved in $C([0,T],E)$ for suitable data x,f: see [9, Proposition 3.1]. Of course, Step 2 will consist in showing that $A^{-1}v$, where v is the solution of (4.3), is in fact a strict solution of (0.1) - (0.2).

Condition (4.4), in a somewhat weaker form, was introduced in [9] (see also [8,2]) and used in [30,31].

4.3. The third choice, $B(t,s)=A(0)$.

The integral equation (3.2) just reduces to the Korn device applied to the usual variation of parameters formula (compare with (2.1)):

$$u(t) = e^{tA(0)}x + \int_0^t e^{(t-s)A(0)}\left[f(s)+[A(s)-A(0)]u(s)\right]ds. \tag{4.6}$$

Assume:

$$\begin{cases} D_{A(t)} \equiv D \ ; \quad \exists\vartheta\in]0,1[: \ D_{A(t)}(\sigma,\infty)\equiv D_{\sigma} \ \forall\sigma\in]0,\vartheta+1]-\{1\} \\ A\in C([0,T],\mathcal{L}(D,E))\cap C([0,T],\mathcal{L}(D_{\vartheta+1},D_{\vartheta})); \end{cases} \tag{4.7}$$

this allows to solve (4.6), by a fixed point argument, in the space

$$X := C_{1-\vartheta}(]0,T],D)\cap B_1(]0,T],D_{\vartheta+1})$$

consisting of functions $u\in C(]0,T],D)\cap B(0,T,D_{\vartheta+1})$ such that

$$\|u\|_X := \sup_{t>0} t^{1-\vartheta}\|u(t)\|_D + \sup_{t>0} t\|u(t)\|_{D_{\vartheta+1}} < \infty.$$

This is proved in [12], generalizing previous results of [13].

4.4. The fourth choice, $B(t,s)=\frac{1}{t-s}\int_s^t A(\sigma)d\sigma$.

Firstly we remark that

$$(t-s)\frac{\partial}{\partial s}B(t,s)=B(t,s)-A(s),$$

so that the integral term (3.5) becomes

$$\begin{aligned} &\int_0^t K(t,s)u(s)ds = \\ &= \frac{1}{2\pi i}\int_0^t\int_{\gamma} e^{(t-s)\lambda}[\lambda-B(t,s)]^{-1}\frac{1}{t-s}\left[[\lambda-B(t,s)]^{-1},A(s)\right]u(s)d\lambda ds. \end{aligned} \tag{4.8}$$

We are then led to the following assumption:

$$\begin{cases} D_{A(t)} \equiv D, \quad D \text{ dense in } E, \\ A\in C([0,T],\mathcal{L}(D,E)), \\ \|[\lambda-B(t,s)]^{-1}\|_{\mathcal{L}(E)} \le \dfrac{N}{1+|\lambda|}, \quad \|[\lambda-B(t,s)]^{-1}\|_{\mathcal{L}(E,D)} \le N, \\ \|\left[[\lambda-B(t,s)]^{-1},A(s)\right]\|_{\mathcal{L}(D,E)} \le \dfrac{N}{1+|\lambda|^{\beta+1}} (0<\beta<1). \end{cases} \tag{4.9}$$

Furthermore, in Step 2 we will also need:

$$\begin{cases} \|[\lambda-B(t,s)]^{-1}\|_{\mathcal{L}(D_A(\eta,\infty),D)} \le \dfrac{N}{1+|\lambda|^{\eta}}, \\ \|\left[[\lambda-B(t,s)]^{-1},A(s)\right]\|_{\mathcal{L}(D,D_A(\eta,\infty))} \le \dfrac{N}{1+|\lambda|^{1+\beta-\eta}} \end{cases} \quad \forall\eta\in]0,\beta]. \tag{4.10}$$

This set of assumptions was introduced by Sobolevskii [25]. By (4.8) and (4.9) it is easy to deduce that

$$\|K(t,s)\|_{\mathcal{L}(D)} \le \frac{C}{|t-s|^{1-\beta}},$$

so that (3.2) can be solved in the space $C([0,T],D)$.

5. Existence of solutions of (0.1)-(0.2).

For suitable data x,f we now have a solution v of the integral equation (3.2) (cases I,III,IV) or (4.3) (case II), which can be written as the sum of a uniformly convergent Neumann series; the goal of Step 2 is to show that v coincides with u (cases I,III,IV) or Au (case II), u being the strict solution of (0.1)-(0.2). In fact, this goal is achieved in a somewhat indirect method, since some different representation formulas are used.

Theorem 5.1 (case I) Assume (0.3) and (4.1)-(4.2). Suppose that $x\in\overline{D_{A(0)}}$ and either $f\in C^{\varepsilon}([0,T],E)$ or $f\in C([0,T],E)\cap B(0,T;D_A(\varepsilon,\infty))$; then the solution v of (3.2) is the unique classical solution of problem (0.1)-(0.2). If in addition $x\in D_{A(0)}$ and $A(0)x+f(0)-[dA(t)^{-1}/dt]_{t=0}A(0)x \in \overline{D_{A(0)}}$, then v is strict.

Proof In [3] it is shown that any strict solution of (0.1)-(0.2) can be represented as

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}\left[(1+P)^{-1}(f-P(\cdot,0)x)\right](s)ds, \tag{5.1}$$

where P is the integral operator

$$(Pg)(t):=\int_0^t P(t,s)g(s)ds,$$

whose kernel $P(t,s)$ is

$$P(t,s):=\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}\frac{d}{dt}[\lambda-A(t)]^{-1}d\lambda.$$

Then if $f\in C^{\varepsilon}([0,T],E)$ all statements follow by direct computation (see [3, Theorems 4.1-5.1]. On the other hand if $f\in C([0,T],E)\cap B(0,T;D_A(\varepsilon,\infty))$ we can rewrite (5.1) as

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}\left[(1+P)^{-1}(-Pf-P(\cdot,0)x)\right](s)ds + \int_0^t e^{(t-s)A(t)}f(s)ds; \tag{5.2}$$

now the first two terms represent the solution of problem (0.1)-(0.2) with right member $-Pf\in C^{\varepsilon}([0,T],E)$ (by [3, Proposition 3.5(v)]) and initial datum x, so that the previous argument applies. The third term is in $C^{\varepsilon}([0,T],D_A) \cap C^{1+\varepsilon}([0,T],E)$ by direct computation, as it is easily seen using [11,Lemma 2.5(ii)]. This implies the result. □

Theorem 5.2 (case II) Assume (0.3) and (4.4). Suppose that $x\in\overline{D_{A(0)}}$ and either $f\in C^{\varepsilon}([0,T],E)$ or $f\in C([0,T],E)\cap B(0,T;D_A(\varepsilon,\infty))$; then the function $A^{-1}v$, where v is the solution of (4.3), is the unique classical solution of (0.1)-(0.2). If in addition $x\in D_{A(0)}$ and $A(0)x+f(0)\in\overline{D_{A(0)}}$, then $A^{-1}v$ is strict.

Proof (See [9].) Consider the Yosida approximations $A_n(t)$ of $A(t)$, i.e.

$$A_n(t)=nA(t)[n-A(t)]^{-1}, \quad n\in\mathbb{N}^+. \tag{5.3}$$

Let u_n be the solution of the Cauchy problem

$$\begin{cases} u_n'(t)-A_n(t)u_n(t)=f(t), & t\in[0,T], \\ u_n(0)=x \end{cases} \tag{5.4}$$

(obviously existing, since $A_n\in C([0,T],\mathcal{L}(E))$!). Then A_nu_n is the solution of an integral equation whose kernel is (compare with (4.5))

$$A_n(t)^2e^{(t-s)A_n(t)}[A_n(t)^{-1}-A_n(s)^{-1}],$$

and converges in $\mathcal{L}(E)$, as $n\to\infty$, to the kernel of (4.3). Then one sees that u_n converges to some u in $C([0,T],E)$ and $A_nu_n\to v$ in $C([0,T],E)$; hence $v\equiv Au$ and $u_n'\equiv A_nu_n+f\to Au+f$ in $C([0,T],E)$, so that $u'\equiv(A^{-1}v)'$ exists and is equal to $Au+f$, i.e. $A^{-1}v$ solves (0.1)-(0.2). □

Theorem 5.3 (case III) Assume (0.3) and (4.7). Suppose that $x\in D_\varepsilon$ and $f\in C([0,T],E)\cap B(0,T;D_\varepsilon))$; then the solution v of (3.2) (i.e. of (4.6)) is the unique classical solution of problem (0.1)-(0.2). If in addition $x\in D$ and $A(0)x\in\overline{D}$, then v is strict.

Proof The first assertion follows by direct computation (see [12, Theorem 2.2]). Let us prove the second one: fix $x\in D$ such that $A(0)x\in\bar{D}$, and select a sequence $\{x_n\}\subseteq D_{\varepsilon+1}$ such that $x_n\longrightarrow x$ in D. For each n let v_n be the strict solution of (0.1)-(0.2) with data x_n,f; then, by an argument used in [5], we can write a different representation formula for v_n:

$$v_n(t) = A(t)^{-1}\Big[(1-H)^{-1}\Big(L(\cdot,0)x_n + Lf\Big)\Big](t)$$

where

$$L(t,s):=A(t)e^{(t-s)A(t)}, \qquad \Big(Lf\Big)(t):=\int_0^t L(t,s)f(s)ds,$$

$$\Big(Hg\Big)(t):=\int_0^t A(t)e^{(t-s)A(t)}(1-A(t)A(s)^{-1})g(s).$$

Now it can be seen that $L(\cdot,0)\in\mathcal{L}(D,C([0,T],E))$ and $(1-H)^{-1}\in\mathcal{L}(C([0,T],E))$, so that $v_n\longrightarrow u$ in $C([0,T],D)$, where

$$u(t) = A(t)^{-1}\Big[(1-H)^{-1}\Big(L(\cdot,0)x + Lf\Big)\Big](t). \tag{5.5}$$

Moreover $v_n'=Av_n+f \longrightarrow Au+f$ in $C([0,T],E)$, which implies that u is a strict solution of (0.1)-(0.2). □

Theorem 5.4 (case IV) Assume (0.3) and (4.9)-(4.10). Suppose that $x\in E$ and $f\in C([0,T],E)\cap B(0,T;D_\varepsilon))$; then the solution v of (3.2) is the unique classical solution of problem (0.1)-(0.2). If in addition $x\in D$, then v is strict.

Proof It is similar to the case II: let u_n be the solution of problem (5.4), with $A_n(t)$ given by (5.3). If $x\in D$, then u_n is the solution, in the space $C([0,T],D)$, of an integral equation whose kernel is (compare with (4.8)):

$$\frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}[\lambda-B_n(t,s)]^{-1}\frac{1}{t-s}\Big[[\lambda-B_n(t,s)]^{-1},A_n(s)\Big]d\lambda,$$

where

$$B_n(t,s):=\frac{1}{t-s}\int_s^t A_n(\sigma)d\sigma;$$

hence the above kernel converges as $n\longrightarrow\infty$ to the kernel of (4.8). As a consequence we get $u_n\longrightarrow v$ in $C([0,T],D)$ and, in particular, $A_nu_n\longrightarrow Av$ in

$C([0,T],E)$. But then $u_n'\equiv A_nu_n+f\longrightarrow Av+f$, so that $v\in C^1([0,T],E)$ and $v'\equiv Av+f$. If $x\in E$ only, the above argument works, with some more technicalities, in the space $X:=C_{1-\eta}(]0,T],D_\eta)\cap C_1(]0,T],D)$. □

Remark 5.5 In cases III and IV we are not able to find classical or strict solutions of problem (0.1)-(0.2) with $f\in C^\varepsilon([0,T],E)$, because the $A(t)$'s are just continuous in t.

6. Maximal regularity.

There are two kinds of maximal regularity, namely with respect to time and with respect to space: that is, there are certain subspaces M of $C([0,T],E)$, consisting either of E-valued Hölder continuous functions, or of functions which are bounded with values in some interpolation space, which have the following property: if the right member of (0.1)-(0.2) belongs to M, then the solution u is such that both u' and Au belong to M, provided suitable compatibility conditions hold. Such conditions turn out to be necessary and sufficient for maximal regularity.

Theorem 6.1 (case I) Assume (0.3) and (4.1)-(4.2), and fix $\varepsilon\in]0,\alpha\wedge\eta]$.

(i) Suppose $x\in D_{A(0)}$, $f\in C^\varepsilon([0,T],E)$ and $A(0)x+f(0)-\big[dA(t)^{-1}/dt\big]_{t=0}A(0)x \in \overline{D_{A(0)}}$; let u be the strict solution of problem (0.1)-(0.2). Then u', $Au \in C^\varepsilon([0,T],E)$ if and only if $A(0)x+f(0)-\big[dA(t)^{-1}/dt\big]_{t=0}A(0)x \in D_{A(0)}(\varepsilon,\infty)$; in this case, one has also $u'-\big[dA(\cdot)^{-1}/dt\big]Au \in B(0,T;D_A(\varepsilon,\infty))$.

(ii) Suppose that $x\in D_{A(0)}$, $f \in C([0,T],E) \cap B(0,T;D_A(\varepsilon,\infty))$ and $A(0)x-\big[dA(t)^{-1}/dt\big]_{t=0}A(0)x \in \overline{D_{A(0)}}$; let u be the strict solution of problem (0.1)-(0.2). Then u', $Au \in B(0,T;D_A(\varepsilon,\infty))$ if and only if $A(0)x-\big[dA(t)^{-1}/dt\big]_{t=0}A(0)x \in D_{A(0)}(\varepsilon,\infty)$; in this case, one has also $Au-\big[dA(\cdot)^{-1}/dt\big]Au \in C^\varepsilon([0,T],E)$.

Proof The first assertion of (i) is proved in [3, Theorem 5.3] and [6, Appendix]; let us prove the second one. To this purpose we need to rewrite the integral equation (3.2) with the choice II instead of I, i.e. taking $B(t,s)=A(t)$: then we get

$$u(t) = \int_0^t A(t)e^{(t-s)A(t)}\big[A(t)^{-1}-A(s)^{-1}\big]A(s)u(s)ds + \\ + e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}f(s)ds. \tag{6.1}$$

Next, we split it conveniently:

$$u(t) = \int_0^t A(t)e^{(t-s)A(t)}\left[A(t)^{-1}-A(s)^{-1}-(t-s)\frac{d}{dt}A(t)^{-1}\right]A(s)u(s)ds +$$

$$+ \int_0^t A(t)e^{(t-s)A(t)}(t-s)\frac{d}{dt}A(t)^{-1}\left[A(s)u(s)-A(t)u(t)\right]ds +$$

$$+ \left[te^{tA(t)}-A(t)^{-1}\left[e^{tA(t)}-1\right]\right]\frac{d}{dt}A(t)^{-1}\cdot A(t)u(t) + e^{tA(t)}x +$$

$$+ \int_0^t e^{(t-s)A(t)}\left[f(s)-f(t)\right]ds + A(t)^{-1}\left[e^{tA(t)}-1\right]f(t).$$

It is clear that each term belongs to $D_{A(t)}$, so that after a further splitting we easily have:

$$u'(t) - \frac{d}{dt}A(t)^{-1}\cdot A(t)u(t) = A(t)u(t) + f(t) - \frac{d}{dt}A(t)^{-1}\cdot A(t)u(t) =$$

$$= \int_0^t A(t)e^{(t-s)A(t)}\left[A(t)^{-1}-A(s)^{-1}-(t-s)\frac{d}{dt}A(t)^{-1}\right]A(s)u(s)ds +$$

$$+ \int_0^t A(t)^2e^{(t-s)A(t)}(t-s)\frac{d}{dt}A(t)^{-1}\left[A(s)u(s)-A(t)u(t)\right]ds +$$

$$+ A(t)e^{tA(t)}\left[t\frac{d}{dt}A(t)^{-1}\left[A(t)u(t)-A(0)x\right] + \left[t\frac{d}{dt}A(t)^{-1}+A(0)^{-1}-A(t)^{-1}\right]\right]A(0)x +$$

$$+ e^{tA(t)}\left[\left[f(t)-f(0)\right] - \frac{d}{dt}A(t)^{-1}\left[A(t)u(t)-A(0)x\right] -\right.$$

$$\left. - \left[\frac{d}{dt}A(t)^{-1}-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}\right]A(0)x\right] + e^{tA(t)}\left[A(0)x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x\right] -$$

$$+ \int_0^t A(t)e^{(t-s)A(t)}f(s)ds.$$

Now it is a straightforward task to verify that each term in the right-hand side is in $D_{A(t)}(\varepsilon,\infty)$, with bounded norms (using, for the last two, [11, Lemma 2.5(i)-(ii)]). This proves (i).

Let us prove (ii). We start from the last assertion. Using (5.3) we see that

$$A(t)u(t) = A(t)w(t) + A(t)\int_0^t e^{(t-s)A(t)}f(s)ds,$$

where w is the strict solution of problem (0.1)-(0.2) with data -Pf, x. Hence Aw is in $C^{\varepsilon}([0,T],E)$ by (i), whereas the second term is in the same space as remarked in the proof of Theorem 5.1. This proves the last assertion of (ii). Finally, concerning the first one, starting from (6.1)

we can repeat the argument used in the proof of the last part of (i), and the result follows. □

Theorem 6.2 (case II) Assume (0.3) and (4.4), and fix $\varepsilon\in]0,\delta]$.

(i) Suppose $x\in D_{A(0)}$, $f\in C^{\varepsilon}([0,T],E)$ and $A(0)x+f(0)\in\overline{D_{A(0)}}$; let u be the strict solution of problem (0.1)-(0.2). Then u', Au $\in C^{\varepsilon}([0,T],E)$ if and only if $A(0)x+f(0)\in D_{A(0)}(\varepsilon,\infty)$; in this case, one has also $u'\in B(0,T;D_A(\varepsilon,\infty))$.

(ii) Suppose that $x\in D_{A(0)}$, $f\in C([0,T],E)\cap B(0,T;D_A(\varepsilon,\infty))$ and $A(0)x\in\overline{D_{A(0)}}$; let u be the strict solution of problem (0.1)-(0.2). Then u', Au $\in B(0,T;D_A(\varepsilon,\infty))$ if and only if $A(0)x\in D_{A(0)}(\varepsilon,\infty)$; in this case, one has also Au $\in C^{\varepsilon}([0,T],E)$.

Proof See [9, Theorems 6.1(iii)-6.2(iii)]. □

Theorem 6.3 (case III) Assume (0.3) and (4.7), and fix $\varepsilon\in]0,\vartheta]$. Suppose $x\in D$, $A(0)x\in\overline{D}$ and $f\in C([0,T],E)\cap B(0,T;D_{\varepsilon})$), and let u be the strict solution of problem (0.1)-(0.2). Then u', Au $\in B(0,T;D_{\varepsilon})$ if and only if $A(0)x\in D_{\varepsilon}$.

Proof By direct computation, starting from (5.5) (the original proof is different: see [12, Corollary 2.3]). □

Theorem 6.4 (case IV) Assume (0.3) and (4.9)-(4.10), and fix $\varepsilon\in]0,\beta[$. Suppose $x\in D$ and $f\in C([0,T],E)\cap B(0,T;D_{\varepsilon})$), and let u be the strict solution of (0.1)-(0.2). Then u', Au $\in B(0,T;D_{\varepsilon})$ if and only if $A(0)x\in D_{\varepsilon}$.

Proof The solution u is in $C([0,T],D)$ and solves the integral equation

$$u(t) - \int_0^t K(t,s)u(s)ds = e^{tB(t,0)}x + \int_0^t e^{(t-s)B(t,s)}f(s)ds, \qquad (6.2)$$

with K(t,s) given by (4.8). Now it is easily seen that, since $u\in C([0,T],D)$,

$$t\longrightarrow\int_0^t K(t,s)u(s)ds \in B(0,T;D_{1+\eta}) \quad \forall\eta\in]0,\beta[,$$

whereas it is not difficult to verify that the right member of (6.2) belongs to $B(0,T;D_{1+\varepsilon})$ if and only if $A(0)x\in D_{\varepsilon}$. Thus the result follows at once. □

Remark 6.5 In cases III and IV one cannot expect maximal regularity in time i.e. that $f\in C^{\varepsilon}([0,T],E)$ implies u', $Au\in C^{\varepsilon}([0,T],E)$: this is false even in the scalar case $E=\mathbb{R}$.

7. The evolution operator.

In all cases I-...-IV one can construct the evolution operator associated to problem (0.1), i.e. an operator $U(t,s)\in\mathcal{L}(E)$ defined for $0\le s\le t\le T$, such that

$$U(t,s) = U(t,\tau)U(\tau,s) \quad \forall\tau\in[s,t], \qquad U(t,t) = I \quad \forall t\in[0,T], \tag{7.1}$$

$$\frac{d}{dt}U(t,s) = A(t)U(t,s) \quad \forall t\in]s,T], \tag{7.2}$$

$$\frac{d}{ds}U(t,s) = -U(t,s)A(s) \quad \forall s\in[0,t[. \tag{7.3}$$

More precisely:

Theorem 7.1 (case I) Assume (0.3) and (4.1)-(4.2). Then there exists a unique operator $U(t,s)$ satisfying (7.1)-(7.2); moreover $dU(t,s)/ds$ exists in $\mathcal{L}(E)$ for each $s\in[0,t[$ and satisfies (7.3) pointwise for each $x\in D_{A(s)}$.

Proof The result is classical [18] in the case of dense domains; for the general case see [11]. □

Theorem 7.2 (case II) Assume (0.3) and (4.4). Then there exists a unique operator $U(t,s)$ satisfying (7.1)-(7.2); if in addition the domains $D_{A(t)}$ are dense in E and the operators $\{A(t)^*\}$ satisfy (0.3) and (4.4) in the space E^*, then $dU(t,s)/ds$ exists in $\mathcal{L}(E)$ for each $s\in[0,t[$ and satisfies (7.3) pointwise for each $x\in D_{A(s)}$.

Proof See [2] and [11]. □

Theorem 7.3 (case III) Assume (0.3) and (4.7). Then there exists a unique operator $U(t,s)$ satisfying (7.1)-(7.2); moreover, setting $X:=\{x\in D: A(0)x\in\overline{D}\}$, $dU(t,s)/ds$ exists in $\mathcal{L}(X,E)$ for each $s\in[0,t[$ and satisfies (7.3).

Proof First of all, we remark that if $x\in X$, then $A(s)x\in\overline{D}$ for each $s\in[0,T]$. Now, the first assertion is essentially proved in [12]. Let us show the second one. Fix $x\in D_{1+\varepsilon}$: by (5.5) we can write

$$U(t,s)x = A(t)^{-1}\left[(1-H)^{-1}\left(L(\cdot,0)x\right)\right](t) =$$

$$= e^{(t-s)A(t)}x + \sum_{n=1}^{\infty} A(t)^{-1}\left[H^n\left(L(\cdot,0)x\right)\right](t) =$$

$$= e^{(t-s)A(t)}x + \sum_{n=1}^{\infty} A(t)^{-1}\int_s^t H_n(t,\sigma)A(\sigma)e^{(\sigma-s)A(\sigma)}x\,d\sigma ,$$

where $H_n(t,\sigma)$ is the iterated kernel of the integral operator H^n, i.e.

$$H_1(t,\sigma):=H(t,\sigma); \qquad H_{n+1}(t,\sigma):=\int_\sigma^t H_n(t,r)H(r,\sigma)dr \quad \forall n\in\mathbb{N}.$$

Now differentiating with respect to s we get

$$\frac{d}{ds}U(t,s)x = -A(t)e^{(t-s)A(t)}x - \sum_{n=1}^{\infty} A(t)^{-1}H_n(t,s)A(s)x -$$

$$-\sum_{n=1}^{\infty} A(t)^{-1}\int_s^t H_n(t,\sigma)A(\sigma)^2 e^{(\sigma-s)A(\sigma)}x\,d\sigma =$$

$$= e^{(t-s)A(t)}\left[1-A(t)A(s)^{-1}\right]A(s)x - e^{(t-s)A(t)}A(s)x - \sum_{n=1}^{\infty} A(t)^{-1}H_n(t,s)A(s)x -$$

$$+\sum_{n=1}^{\infty} A(t)^{-1}\int_s^t H_n(t,\sigma)A(\sigma)e^{(\sigma-s)A(\sigma)}\left[1-A(\sigma)A(s)^{-1}\right]A(s)x\,d\sigma -$$

$$-\sum_{n=1}^{\infty} A(t)^{-1}\left[H^n\left(L(\cdot,0)A(s)x\right)\right](t) =$$

$$= A(t)^{-1}H_1(t,s)A(s)x - e^{(t-s)A(t)}A(s)x - \sum_{n=1}^{\infty} A(t)^{-1}H_n(t,s)A(s)x +$$

$$+\sum_{n=1}^{\infty} A(t)^{-1}H_{n+1}(t,s)A(s)x - \sum_{n=1}^{\infty} A(t)^{-1}\left[H^n\left(L(\cdot,0)A(s)x\right)\right](t) =$$

$$= -U(t,s)A(s)x .$$

Thus (7.3) is established when $x\in D_{1+\varepsilon}$. Finally we extend (7.3) to the whole X by approaching any $x\in X$ by a suitable sequence $\{x_n\}\subseteq D_{1+\varepsilon}$. □

Theorem 7.4 (case IV) Assume (0.3) and (4.9)-(4.10). Then there exists a unique operator $U(t,s)$ satisfying (7.1)-(7.2); moreover $dU(t,s)/ds$ exists in $\mathcal{L}(D,E)$ for each $s\in[0,t[$ and satisfies (7.3).

Proof See [25]. □

Remark 7.5 In all cases the operators $U(t,s)$ fulfill further regularity properties with respect to t and s. □

8. Examples and remarks.

Consider a general linear, non-autonomous parabolic initial-boundary value problem. The various assumptions of cases I-...-IV correspond to different concrete situations: case II means moderate regularity in t, with a strong parabolic structure, whereas case I allows a less stringent parabolicity (the boundary operators may reduce their order for some t) provided there is a very good dependence on t; on the other hand cases III and IV concern very restrictive boundary conditions (independent of t), requiring however just continuity with respect to t. Thus it is not surprising that the assumptions of the four cases are independent of one another. To verify this independence it is sufficient to take a one space dimension example. Consider a second order operator

$$A(t,x,D)u := u'' + a(t,x)u' + b(t,x)u, \quad t\in[0,T],\ x\in[0,1],$$

equipped with endpoint conditions

$$u(0) = 0, \quad u(1) + c(t)u'(1) = 0;$$

here $a,b:[0,T]\times[0,1]\to\mathbb{R}$ and $c:[0,T]\to\mathbb{R}$ are continuous functions. Next, set

$$\begin{cases} D_{A(t)} := \{u\in E:\ A(t,\cdot,D)u\in E,\ u(0)=0,\ u(1)+c(t)u'(1)=0\} \\ A(t)u := A(t,\cdot,D)u \end{cases}$$

with $E:=C([0,1])$.

Assume now $a\equiv b\equiv 0$, $c(t)=1+t^{2/3}$; then it is easy to see that assumptions of case II hold and assumptions of case I do not, and conversely, if we take again $a\equiv b\equiv 0$ and $c(t)=t$, then assumptions of case I hold and those of case II do not (see [9, §7] . Moreover in both situations above the assumptions of cases III and IV are not fulfilled (since $D_{A(t)}$ is not constant).

On the other hand, choose $a\equiv 0$, $b(t,x)=\omega(t)\gamma(x)$ and $c\equiv 0$, with $\gamma\in C^{\varepsilon}([0,1])$ $(0<\varepsilon<1)$ and $\omega\in C([0,T])$ (but not Hölder continuous for any $\alpha\in]0,1[$): then an easy inspection shows that assumptions of case III hold but those of case IV do not (since the constant domain is not dense), and assumptions of cases I-II do not too (since ω is not Hölder continuous).

Finally it does not seem easy to construct an example where assumptions of case IV hold but assumptions of case III do not: in fact the hypotheses of case IV, and in particular the existence of the commutator $\left[[\lambda-B(t,s)]^{-1},A(s)\right]$ in $\mathcal{L}(D,E)$, seem to imply that $D_{A(t)}(\vartheta+1,\infty)$ is independent of t in all "reasonable" examples.

The assumptions of cases I-...-IV cover most part of the available literature. Case II generalizes the papers [23,4,5,24,16,17,7]; case I

contains [18,3,26,27,28], case III corresponds to [12,13] and finally case IV is introduced in [25]. It is to be noted that in the paper [29] there is an assumption which is related to and weaker than that of case I: it implies Theorem 7.1 and existence of classical solutions if $f\in C^{\varepsilon}([0,T],E)$, but no maximal regularity results seem to hold in this case; in fact we are not able even to include it in our unitary approach.

We also have to mention the results of [19,20] where assumptions and methods of case II are generalized to the more abstract setting of the sum of two closed linear operators, with applications to elliptic as well as parabolic problems.

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