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0. INTRODUCTION

Let $\{A(t),\ t\in[0,T]\}$ be a family of generators of analytic semigroups in a complex Hilbert space H, and suppose that both $\{A(t)\}$ and $\{A(t)^*\}$ fulfil the assumptions of (Acquistapace and Terreni, 1987) in a somewhat strengthened form, i.e. assume that:

for each $t\in[0,T]$, $A(t):D_{A(t)}\subseteq H\longrightarrow H$ is a closed linear operator; in addition there exist $\vartheta\in]\pi/2,\pi[$ and $M>0$ such that $\rho(A(t))\supseteq\overline{S(\vartheta)}$, where $S(\vartheta):=\{z\in\mathbb{C}: |\arg z|<\vartheta\}$, and

$$\|[\lambda-A(t)]^{-1}\|_{\mathcal{L}(H)}\le M[1+|\lambda|]^{-1}\quad \forall\lambda\in\overline{S(\vartheta)},\ \forall t\in[0,T]; \tag{0.1}$$

there exist $N>0$ and $\alpha,\rho\in]0,1]$ with $\alpha+\rho>1$, such that

$$\|A(t)[\lambda-A(t)]^{-1}[A(t)^{-1}-A(s)^{-1}]\|_{\mathcal{L}(H)}\le N|t-s|^{\alpha}[1+|\lambda|]^{-\rho}\quad \forall\lambda\in\overline{S(\vartheta)},\ \forall t,s\in[0,T]; \tag{0.2}$$

the operators $\{A(t)^*, t\in[0,T]\}$ satisfy (0.1) and (0.2) with the same constants $\vartheta, M, N, \alpha, \rho$. (0.3)

REMARK 0.1 By (0.1), the domains $D_{A(t)}$ are necessarily dense in H, so that $A(t)^*$ is well defined (and densely defined too).□

Denote by $\Sigma(H)$ the set of self-adjoint bounded linear operators on H: $\Sigma(H)$ is a Banach space with the $\mathscr{L}(H)$ norm. Consider for each $t\in[0,T]$ the operator

$$\Lambda(t)P = A(t)^{*}P + PA(t), \quad P\in\Sigma(H), \tag{0.4}$$

whose precise definition will be given in Section 1. It is known (see Sections 6.1, 6.2 in (Da Prato, 1973)) that for each $t\in[0,T]$, $\Lambda(t)$ generates an analytic semigroup in $\Sigma(H)$, and in addition $\Lambda(t)$ preserves positivity, i.e. if $P\in D_{\Lambda(t)}$ and $P\geq 0$, then $\Lambda(t)P\geq 0$.

Our goal is to show that under the above assumptions the family $\{\Lambda(t),\ t\in[0,T]\}$ fulfils the assumptions of (Acquistapace and Terreni, 1987), or, more precisely, satisfies (0.1) and (0.2), with ρ replaced by any smaller number, in the Banach space $\Sigma(H)$.

As an application of this result, we are able to show existence of classical solutions for an abstract non-autonomous Riccati equation arising in the study of the Linear Quadratic Regulator Problem for parabolic systems with boundary control. Due to lack of space, this application will appear in a forthcoming paper (Acquistapace and Terreni, in preparation).

REMARK 0.2 We may replace (0.2) by the slightly weaker condition

$$\|A(t)[\lambda-A(t)]^{-1}[A(t)^{-1}-A(s)^{-1}]\|_{\mathscr{L}(H)}\leq N\sum_{i=1}^{k}|t-s|^{\alpha_i}[1+|\lambda|]^{-\rho_i}$$
$$\forall\lambda\in\overline{S(\vartheta)},\ \forall t,s\in[0,T],$$

where $\alpha_i,\rho_i\in]0,1]$ and $\alpha_i+\rho_i>1$ for $i=1,\dots,k$; what is crucial here is that $\rho_i>0$, and this requirement makes such assumption stronger than that of (Acquistapace and Terreni, 1987), where on the contrary the ρ_i's are allowed to be possibly 0.

1. THE OPERATOR $\Lambda(t)$ FOR FIXED t.

A precise definition of the operator (0.4), for fixed $t\in[0,T]$, can be given in the following way (compare with (Da Prato, 1973)). Fix $P\in\Sigma(H)$ and consider the sesquilinear form defined on $D_{A(t)}\times D_{A(t)}$ by:

$$\Phi_P(t;x,y):=(A(t)x,Py)_H+(Px,A(t)y)_H, \quad x,y\in D_{A(t)}. \tag{1.1}$$

We set

$$D_{\Lambda(t)}:=\{P\in\Sigma(H):\ \exists\ c(t;P)>0 \text{ such that}$$
$$|\Phi_P(t;x,y)|\leq c(t;P)\|x\|_H\|y\|_H\ \ \forall x,y\in D_{A(t)}\}. \tag{1.2}$$

If $P\in D_{\Lambda(t)}$, then $\Phi_P(t;\cdot,\cdot)$ has a unique extension $\hat{\Phi}_P(t;\cdot,\cdot)$ to $H\times H$ such that

$$\left.\begin{aligned}&\hat{\Phi}_P(t;x,y)=\Phi_P(t;x,y)\ \ \forall x,y\in D_{A(t)},\\&|\hat{\Phi}_P(t;x,y)|\leq c(t;P)\|x\|_H\|y\|_H\ \ \forall x,y\in H;\end{aligned}\right\} \tag{1.3}$$

hence by Riesz' Representation Theorem there exists an operator $Q_P(t)\in\mathscr{L}(H)$ such that

$$\hat{\Phi}_P(t;x,y)=(Q_P(t)x,y)_H\ \ \forall x,y\in H. \tag{1.4}$$

Now we define

$$\Lambda(t)P:=Q_P(t)\ \ \forall P\in D_{\Lambda(t)}, \tag{1.5}$$

i.e.

$$(\Lambda(t)Px,y)_H=\hat{\Phi}_P(t;x,y)\ \ \forall x,y\in H. \tag{1.6}$$

We remark that if $P\in D_{\Lambda(t)}$ and $x\in D_{A(t)}$ then in particular

$$|(Px,A(t)y)_H|=|\hat{\Phi}_P(t;x,y)-(A(t)x,Py)_H|\leq$$
$$\leq[c(t;P)\|x\|_H+\|A(t)x\|_H]\|y\|_H;$$

this means $Px\in D_{A(t)^*}$ and

$$\Lambda(t)Px=A(t)^{*}Px+PA(t)x\ \ \forall x\in D_{A(t)},\ \forall P\in D_{\Lambda(t)}, \tag{1.7}$$

i.e. (0.4) holds when evaluated at any $x\in D_{A(t)}$. In particular, by (1.4), (1.3), (1.1) and (1.7) it follows easily that

$$(Q_P(t)x,y)_H=(x,Q_P(t)y)_H\ \ \forall x,y\in D_{A(t)},$$

and therefore $\Lambda(t)P\equiv Q_P(t)\in\Sigma(H)$ for each $P\in D_{\Lambda(t)}$.

The operator $\Lambda(t)$ generates the semigroup $\{e^{\xi\Lambda(t)},\xi\geq 0\}\subseteq\mathscr{L}(\Sigma(H))$, defined by

$$e^{\xi\Lambda(t)}P := e^{\xi A(t)^*} P e^{\xi A(t)}, \quad P \in \Sigma(H); \tag{1.8}$$

indeed, we have:

PROPOSITION 1.1 Denote by 1 the identity operator on $\Sigma(H)$. We have:

(i) $D_{\Lambda(t)} = \{P \in \Sigma(H) : \exists \lim_{\xi \to 0} \left(\frac{e^{\xi\Lambda(t)} - 1}{\xi} Px, y \right)_H = (\Lambda(t)Px, y)_H \ \forall x, y \in H\}$;

(ii) $\overline{D_{\Lambda(t)}} = \{P \in \Sigma(H) : \exists \lim_{\xi \to 0} \| (e^{\xi\Lambda(t)} - 1)P \|_{\Sigma(H)} = 0\}$;

(iii) $\{P \in D_{\Lambda(t)} : \Lambda(t)P \in \overline{D_{\Lambda(t)}}\} =$

$$= \{P \in D_{\Lambda(t)} : \exists \lim_{\xi \to 0} \left\| \frac{e^{\xi\Lambda(t)} - 1}{\xi} P - \Lambda(t)P \right\|_{\Sigma(H)} = 0\}.$$

Proof. (i) By (1.8) and (0.1)-(0.3) it follows that

$$\| e^{\xi\Lambda(t)} \|_{\mathcal{L}(\Sigma(H))} \le c(\vartheta, M) \quad \forall \xi > 0, \ \forall t \in [0,T];$$

hence the argument of Chapter 9, Remark 1.5 of (Kato, 1966) shows that if $P, Q \in \Sigma(H)$ and

$$\lim_{\xi \to 0} \left(\frac{e^{\xi\Lambda(t)} - 1}{\xi} Px, y \right)_H = (Qx, y)_H \quad \forall x, y \in H,$$

then $P \in D_{\Lambda(t)}$ and $\Lambda(t)P = Q$. Suppose conversely that $P \in D_{\Lambda(t)}$: then by (1.7) it is easy to get for each $x \in D_{A(t)}$ and $y \in H$:

$$\lim_{\xi \to 0} \left(\frac{e^{\xi\Lambda(t)} - 1}{\xi} Px, y \right)_H = \lim_{\xi \to 0} \Big((e^{\xi A(t)^*} - 1_H) P [\xi^{-1} (e^{\xi A(t)} - 1_H)] x +$$

$$+ \xi^{-1} (e^{\xi A(t)^*} - 1_H) Px + P[\xi^{-1} (e^{\xi A(t)} - 1_H)] x, y \Big)_H = (\Lambda(t)Px, y)_H;$$

hence by (1.6) we get the result since $D_{A(t)}$ is dense in H.

(ii)-(iii) See Proposition 1.2(i)-(iii) of (Sinestrari, 1985). □

EXAMPLE 1.2 $D_{\Lambda(t)}$ is not dense in $\Sigma(H)$ in general (unless, of course, the $A(t)$'s are bounded. Indeed, set $H := L^2(0,\pi)$, and $A(t) \equiv A := d^2/dx^2$, with $D_A := W^{2,2}(0,\pi) \cap W_0^{1,2}(0,\pi)$; then we have

$$e^{\xi A} f = e^{\xi A^*} f = \sum_{n=1}^{\infty} \exp(-n^2 \xi) f_n e_n \quad \forall \xi > 0, \ \forall f \in H,$$

where $e_n(x) := (2/\pi)^{1/2} \sin(nx)$, $f_n := (f, e_n)_H$. Now if D_Λ were dense in $\Sigma(H)$, then we should have, choosing $P := 1_H$:

$$\lim_{\xi \to 0} \| (e^{\xi\Lambda} - 1_{\Sigma(H)}) 1_H \|_{\Sigma(H)} = 0,$$

i.e. for each $\varepsilon > 0$ there should exist $\delta_\varepsilon > 0$ such that

$$\sup \left\{ \| (e^{2\xi A} - 1_H) f \|_H : \ \|f\|_H = 1 \right\} < \varepsilon \quad \forall \xi \in]0, \delta_\varepsilon[;$$

hence by taking $f := e_n$, $n \in \mathbb{N}^+$, we would get

$$\| (e^{2\xi A} - 1_H) e_n \|_H = 1 - \exp(-2n^2 \xi) < \varepsilon \quad \forall n \in \mathbb{N}^+, \ \forall \xi \in]0, \delta_\varepsilon[,$$

which is impossible. □

REMARK 1.3 Despite of Example 1.2, we obviously have

$$\lim_{\xi \to 0} \| (e^{\xi\Lambda(t)} - 1_{\Sigma(H)}) Px \|_H = 0 \quad \forall P \in \Sigma(H), \ \forall x \in H, \ \forall t \in [0,T]. \ \square \tag{1.9}$$

2. MAIN RESULT

By (0.1)-(0.3) and the results of (Acquistapace and Terreni, 1986), (Acquistapace, 1988), (Acquistapace, Flandoli and Terreni, 1990, in press), (Acquistapace and Terreni, 1990) we can construct the evolution operator $U(t,s)$ associated to $\{A(t)\}$, and the following properties hold true:

PROPOSITION 2.1 For $0 \le s < t \le T$ we have:

(i) $U(t,s) = U(t,r)U(r,s) \quad \forall r \in [s,t], \quad U(t,t) = 1_H$;

(ii) $U(t,s) \in \mathcal{L}(H, D_{A(t)})$ and $\exists \ dU(t,s)/dt = A(t)U(t,s)$;

(iii) $U(t,s)^* \in \mathcal{L}(H, D_{A(s)^*})$ and $\exists \ dU(t,s)^*/ds = -A(s)^* U(t,s)^*$;

(iv) $\exists \ dU(t,s)/ds = -[A(s)^* U(t,s)^*]^*$;

(v) $\|U(t,s)\|_{\mathcal{L}(H)} + \|U(t,s)^*\|_{\mathcal{L}(H)} + (t-s)\|dU(t,s)/dt\|_{\mathcal{L}(H)} +$
$+ (t-s)\|dU(t,s)/ds\|_{\mathcal{L}(H)} \le c(\vartheta, M, N, \alpha, \rho, T)$.

Proof. (i)-(ii) See Theorem 2.3 of (Acquistapace, 1988).
(iii) See (6.11) of (Acquistapace and Terreni, 1990).
(iv) See Theorem 6.4 of (Acquistapace and Terreni, 1990).

(v) See Theorem 2.3 of (Acquistapace, 1988) and Theorem 6.4 of (Acquistapace and Terreni, 1990). □

Consider now the operator $E(\cdot,\cdot):\Sigma(H)\longrightarrow\Sigma(H)$ defined by

$$E(t,s)P:=U(T-s,T-t)^{*}PU(T-s,T-t),\quad 0\le s\le t\le T,\ P\in\Sigma(H). \tag{2.1}$$

A straightforward computation shows that $E(t,s)$ is strongly continuous in $\Sigma(H)$, and in addition if $0\le s<t\le T$

$$\left.\begin{aligned} &E(t,s)=E(t,r)E(r,s)\quad \forall r\in[s,t],\quad E(t,t)=1_{\Sigma(H)};\\ &\frac{d}{dt}E(t,s)P=\Lambda(T-t)E(t,s)P\quad \forall P\in\Sigma(H),\\ &\frac{d}{ds}E(t,s)P=-E(t,s)\Lambda(T-s)P\quad \forall P\in D_{\Lambda(T-s)}; \end{aligned}\right\} \tag{2.2}$$

hence $E(t,s)$ is the (necessarily unique) evolution operator associated to $\{\Lambda(T-t),t\in[0,T]\}$. We will show in our main Theorem 2.3 below that the family $\{\Lambda(T-t)\}$ satisfies (0.1) and (0.2) (with ρ replaced by any smaller number) in the space $\Sigma(H)$. As a consequence of Theorem 2.3, the results of (Acquistapace and Terreni, 1987), (Acquistapace and Terreni, 1986) and (Acquistapace, 1988) immediately imply several regularity properties for the evolution operator $E(t,s)$.

REMARK 2.2 Of course, many smoothness properties for $E(t,s)$ and $E(t,s)^{*}$ may also be directly derived by (2.1), using the regularity results for $U(t,s)$ and $U(t,s)^{*}$ proved in (Acquistapace, 1988), (Acquistapace, Flandoli and Terreni, 1990, in press), (Acquistapace and Terreni, 1990). However we believe that Theorem 2.3 has some interest in itself, since it provides a new class of generators of analytic semigroups having a good dependence on t (i.e. satisfying (0.1) and (0.2)); this class is not the "usual" abstract version of some elliptic operator with time-dependent coefficients and homogeneous boundary conditions, acting on some concrete function space, although its construction in fact starts from an operator of that kind. □

THEOREM 2.3 Under assumptions (0.1)-(0.3) the operators $\Lambda(t)$, defined by (1.2), (1.6), enjoy the following properties:

(i) $\Lambda(t):D_{\Lambda(t)}\subseteq\Sigma(H)\longrightarrow\Sigma(H)$ is a closed linear operator; in addition there exist $\vartheta_0\in]\pi/2,\vartheta[$ and $M_0>0$, depending on ϑ,M, such that

$$\|[\lambda-\Lambda(t)]^{-1}\|_{\mathcal{L}(\Sigma(H))}\le M_0[1+|\lambda|]^{-1}\quad \forall\lambda\in\overline{S(\vartheta_0)},\ \forall t\in[0,T];$$

(ii) for each $\varepsilon\in]0,1[$ there exists $N_0>0$, depending on $\vartheta,M,N,\alpha,\rho,\varepsilon$, such that

$$\|\Lambda(t)[\lambda-\Lambda(t)]^{-1}[\Lambda(t)^{-1}-\Lambda(s)^{-1}]\|_{\mathcal{L}(\Sigma(H))}\le$$
$$\le N_0|t-s|^{\alpha}[1+|\lambda|]^{-\rho(1-\varepsilon)}\quad \forall\lambda\in\overline{S(\vartheta_0)},\ \forall t,s\in[0,T].$$

Proof. See Section 3.

3. PROOF OF THEOREM 2.3

Assume (0.1)-(0.3) and let $\Lambda(t)$ be the operator defined in $\Sigma(H)$ by (1.2), (1.6). First of all we need a representation of the resolvent operator $[\lambda-\Lambda(t)]^{-1}$.

PROPOSITION 3.1 Part (i) of Theorem 2.3 holds true and, in addition, we have

$$[\lambda-\Lambda(t)]^{-1}P=⨍_{\gamma}[\mu-A(t)^{*}]^{-1}P[\lambda-\mu-A(t)]^{-1}d\mu \qquad \forall P\in\Sigma(H),\ \forall\lambda\in\overline{S(\vartheta_0)}, \tag{3.1}$$

ϑ_0 being defined in Theorem 2.1(i); here γ is any curve lying in $\rho(A(t))\cap\rho(A(t)^{*})$ and joining $+\infty e^{-i\eta}$ to $+\infty e^{i\eta}$ for some $\eta\in]\pi/2,\vartheta[$, and the symbol $⨍_{\gamma}$ means $(2\pi i)^{-1}\int_{\gamma}$.

Proof. Clearly, if $\xi>0$ we have $e^{\xi\Lambda(t)}P\in D_{\Lambda(t)}$ for each $P\in\Sigma(H)$, and

$$\Lambda(t)e^{\xi\Lambda(t)}P=A(t)^{*}e^{\xi A(t)^{*}}Pe^{\xi A(t)}+e^{\xi A(t)^{*}}PA(t)e^{\xi A(t)}\quad \forall\xi>0,$$

so that

$$\|\Lambda(t)e^{\xi\Lambda(t)}P\|_{\mathcal{L}(\Sigma(H))}\le c(\vartheta_0,M)\ \xi^{-1}\quad \forall\xi>0;$$

part (i) of Theorem 2.3 then follows by standard arguments.

Fix now $P\in\Sigma(H)$ and $\lambda\in\overline{S(\vartheta_0)}$. Then by the Laplace transform formula we get

$$[\lambda-A(t)]^{-1}P = \int_0^\infty e^{-\lambda\xi} e^{\xi A(t)^*} P e^{\xi A(t)} d\xi.$$

On the other hand, we have

$$e^{\xi A(t)} = \int_{\gamma_1} e^{\xi\nu}[\nu-A(t)]^{-1}d\nu, \quad e^{\xi A(t)^*} = \int_{\gamma_2} e^{\xi\mu}[\mu-A(t)^*]^{-1}d\mu,$$

where γ_1, γ_2 obey the requirements listed above; hence by Fubini's Theorem and the resolvent identity we get

$$[\lambda-A(t)]^{-1}P = \int_0^\infty \int_{\gamma_1}\int_{\gamma_2} e^{-\lambda\xi} e^{\xi\mu} e^{\xi\nu}[\mu-A(t)^*]^{-1}P[\nu-A(t)]^{-1}d\mu d\nu =$$

$$= -\int_{\gamma_1}\int_{\gamma_2} (\nu+\mu-\lambda)^{-1}[\mu-A(t)^*]^{-1}P[\nu-A(t)]^{-1}d\mu d\nu.$$

We can select the curves γ_1, γ_2 in such a way that: (a) for each $\lambda\in\overline{S(\vartheta_0)}$ and $\mu\in\gamma_2$ the point $\lambda-\mu$ lies on the right-hand side of γ_1, and similarly (b) for each $\lambda\in\overline{S(\vartheta_0)}$ and $\nu\in\gamma_1$ the point $\lambda-\nu$ lies on the right-hand side of γ_2. This can be achieved by choosing, for instance,

$$\gamma_1 := \left\{z=r\exp(i\vartheta_1),\ r\ge\tfrac{1}{2}r_0|\cos\vartheta_1|^{-1}\right\} \cup \left\{z=-\tfrac{1}{2}r_0+is,\ |s|\le\tfrac{1}{2}r_0|\operatorname{tg}\vartheta_1|\right\},$$

$$\gamma_2 := \left\{z=r\exp(i\vartheta_2),\ r\ge r_0|\cos\vartheta_2|^{-1}\right\} \cup \left\{z=-r_0+is,\ |s|\le r_0|\operatorname{tg}\vartheta_2|\right\},$$

(oriented from $+\infty\exp(-i\vartheta_j)$ to $+\infty\exp(i\vartheta_j)$, $j=1,2$), where $\vartheta_0<\vartheta_1<\vartheta_2<\vartheta$ and $r_0\in]0,M^{-1}|\operatorname{tg}\vartheta|^{-1}[$, so that by (0.1) both γ_1 and γ_2 are contained in $\rho(A(t))\cap\rho(A(t)^*)$ for each $t\in[0,T]$.

Now if $\lambda\in\overline{S(\vartheta_0)}$ we may "close the curve γ_1 on the right", and evaluate the integral over γ_1 by means of residues' theorem, obtaining (3.1). The proof is complete. □

REMARK 3.2 Of course we might also "close the curve γ_2 on the right" (instead of γ_1), obtaining similarly

$$[\lambda-A(t)]^{-1}P = \int_\gamma [\lambda-\nu-A(t)^*]^{-1}P[\nu-A(t)]^{-1}d\mu \qquad (3.2)$$

$$\forall P\in\Sigma(H),\ \forall\lambda\in\overline{S(\vartheta_0)},$$

where γ satisfies the requirements listed in Proposition 3.1. □

Fix now $P\in\Sigma(H)$, $s,t\in[0,T]$ and $\lambda\in\overline{S(\vartheta_0)}$; consider the operator

$$Z:=[\lambda-A(t)]^{-1}[A(t)^{-1}-A(s)^{-1}]P. \qquad (3.3)$$

Obviously $Z\in D_{A(t)}$; we have to show that

$$\|A(t)Z\|_{\mathcal{L}(H)} \le c(\vartheta,M,N,\alpha,\rho,\varepsilon)|t-s|^\alpha[1+|\lambda|]^{-\rho(1-\varepsilon)} \quad \forall\varepsilon\in]0,1[, \qquad (3.4)$$

and this will prove part (ii) of Theorem 2.3.

We remark that if (3.4) holds with $\lambda=1$, then for each $\lambda\in\overline{S(\vartheta_0)}$ with $|\lambda|<1$ we have

$$\|A(t)Z\|_{\mathcal{L}(H)} =$$

$$= \|[1-A(t)][\lambda-A(t)]^{-1}A(t)[1-A(t)]^{-1}[A(t)^{-1}-A(s)^{-1}]P\|_{\mathcal{L}(H)} \le$$

$$\le c(\vartheta,M,N,\alpha,\rho,\varepsilon)|t-s|^\alpha \le c(\vartheta,M,N,\alpha,\rho,\varepsilon)|t-s|^\alpha[1+|\lambda|]^{-\rho(1-\varepsilon)},$$

i.e. (3.4) holds for each $\lambda\in\overline{S(\vartheta_0)}$ with $|\lambda|<1$ as well. Hence it is sufficient to prove (3.4) for each $\lambda\in\overline{S(\vartheta_0)}$ with $|\lambda|\ge1$.

To this purpose using (3.2) we split $[A(t)^{-1}-A(s)^{-1}]P$ in the following manner:

$$[A(t)^{-1}-A(s)^{-1}]P =$$

$$= \int_\gamma \left\{[-\nu-A(t)^*]^{-1}P[\nu-A(t)]^{-1} - [-\nu-A(s)^*]^{-1}P[\nu-A(s)]^{-1}\right\}d\nu =$$

$$= \int_\gamma \left\{\left[[-\nu-A(t)^*]^{-1}-[-\nu-A(s)^*]^{-1}\right]P[\nu-A(t)]^{-1} + \right. \qquad (3.5)$$

$$\left. + [-\nu-A(s)^*]^{-1}P\left[[\nu-A(t)]^{-1}-[\nu-A(s)]^{-1}\right]\right\}d\nu.$$

Clearly, the curve γ here must be contained in $\rho(A(t))\cap\rho(A(t)^*)$ and in $\rho(A(s))\cap\rho(A(s)^*)$, and in addition $-\gamma:=\{z\in\mathbb{C}:\ -z\in\gamma\}$ must have the same property; for instance we may take $\gamma:=\partial S(\vartheta_0)$, oriented from $+\infty\exp(-i\vartheta_0)$ to $+\infty\exp(i\vartheta_0)$.

Now let us fix $\lambda\in\overline{S(\vartheta_0)}$ with $|\lambda|\ge1$. Using (3.5), (3.3) and taking into account (3.1) and (3.2) we split Z as follows:

$$Z = \int_{\gamma_1}\int_{\gamma_2} [\mu-A(t)^*]^{-1}\Big[[-\nu-A(t)^*]^{-1}-[-\nu-A(s)^*]^{-1}\Big]P\,\cdot$$
$$\cdot\ [\nu-A(t)]^{-1}[\lambda-\mu-A(t)]^{-1}d\mu d\nu\ + \tag{3.6}$$
$$+\int_{\gamma_1}\int_{\gamma_2} [\lambda-\mu-A(t)^*]^{-1}[-\nu-A(s)^*]^{-1}P\,\cdot$$
$$\cdot\ \Big[[\nu-A(t)]^{-1}-[\nu-A(s)]^{-1}\Big][\mu-A(t)]^{-1}d\mu d\nu =: Z_1 + Z_2,$$

where we may choose $\gamma_1 := \partial S(\vartheta_1)$ and $\gamma_2 := \partial S(\vartheta_2)$, with $\vartheta_0<\vartheta_1<\vartheta_2<\vartheta$; for instance we may choose $\vartheta_1 := (2\vartheta_0+\vartheta)/3$, $\vartheta_2 := (\vartheta_0+2\vartheta)/3$, so that ϑ_1 and ϑ_2 depend only on ϑ, M.

Next, we rewrite Z_1 using the resolvent identity:

$$Z_1 = \int_{\gamma_1}\int_{\gamma_2} [\mu-A(t)^*]^{-1}\Big[[-\nu-A(t)^*]^{-1}-[-\nu-A(s)^*]^{-1}\Big]P\,\cdot$$
$$\cdot\ (\lambda-\mu-\nu)^{-1}[\nu-A(t)]^{-1}d\mu d\nu\ - \tag{3.7}$$
$$-\int_{\gamma_1}\int_{\gamma_2} [\mu-A(t)^*]^{-1}\Big[[-\nu-A(t)^*]^{-1}-[-\nu-A(s)^*]^{-1}\Big]P\,\cdot$$
$$\cdot\ (\lambda-\mu-\nu)^{-1}[\lambda-\mu-A(t)]^{-1}d\mu d\nu =: Z_{11} + Z_{12};$$

of course both Z_{11} and Z_{12} are absolutely convergent integrals.

In Z_{11} we may evaluate the integral over γ_2 by "closing γ_2 on the right" and using residues' theorem: we find (since the point $\lambda-\nu$ lies on the right-hand side of γ_2)

$$Z_{11} = \int_{\gamma_1} [\lambda-\nu-A(t)^*]^{-1}\Big[[-\nu-A(t)^*]^{-1}-[-\nu-A(s)^*]^{-1}\Big]P[\nu-A(t)]^{-1}d\nu. \tag{3.8}$$

Similarly, in Z_{12} we evaluate the integral over γ_1 by "closing γ_1 on the left", finding (since $\lambda-\mu$ lies on the right-hand side of γ_1):

$$Z_{12} = 0. \tag{3.9}$$

Consider now Z_2. By the change of variable $\nu=-z$, we have

$$Z_2 = \int_{\gamma_2}\int_{-\gamma_1} [\lambda-\mu-A(t)^*]^{-1}[z-A(s)^*]^{-1}P\,\cdot$$
$$\cdot\ \Big[[-z-A(t)]^{-1}-[-z-A(s)]^{-1}\Big][\mu-A(t)]^{-1}dz d\mu, \tag{3.10}$$

where $-\gamma_1 := \{z\in\mathbb{C}:\ -z\in\gamma_1\}$, oriented from $+\infty\exp(i(\pi-\vartheta_1))$ to $+\infty\exp(-i(\pi-\vartheta_1))$. But the function

$$z \longmapsto [z-A(s)^*]^{-1}P\Big[[-z-A(t)]^{-1}-[-z-A(s)]^{-1}\Big]$$

is absolutely integrable and holomorphic in the region

$$\{z\in\mathbb{C}:\ |\arg z| \in [\pi-\vartheta_1, \vartheta_1]\},$$

so that in (3.10) we can replace $-\gamma_1$ by γ_1; thus, writing again ν in place of z,

$$Z_2 = \int_{\gamma_2}\int_{\gamma_1} [\lambda-\mu-A(t)^*]^{-1}[\nu-A(s)^*]^{-1}P\,\cdot$$
$$\cdot\ \Big[[-\nu-A(t)]^{-1}-[-\nu-A(s)]^{-1}\Big][\mu-A(t)]^{-1}d\nu d\mu. \tag{3.11}$$

Next, using the resolvent identity we rewrite Z_2 as the sum of three absolutely convergent integrals:

$$Z_2 = \int_{\gamma_2}\int_{\gamma_1} [\lambda-\mu-A(t)^*]^{-1}\Big[[\nu-A(s)^*]^{-1}-[\nu-A(t)^*]^{-1}\Big]P\,\cdot$$
$$\cdot\ \Big[[-\nu-A(t)]^{-1}-[-\nu-A(s)]^{-1}\Big][\mu-A(t)]^{-1}d\nu d\mu\ -$$
$$-\int_{\gamma_2}\int_{\gamma_1} (\lambda-\mu-\nu)^{-1}[\lambda-\mu-A(t)^*]^{-1}P\,\cdot \tag{3.12}$$
$$\cdot\ \Big[[-\nu-A(t)]^{-1}-[-\nu-A(s)]^{-1}\Big][\mu-A(t)]^{-1}d\nu d\mu\ +$$
$$+\int_{\gamma_2}\int_{\gamma_1} (\lambda-\mu-\nu)^{-1}[\nu-A(t)^*]^{-1}P\,\cdot$$
$$\cdot\ \Big[[-\nu-A(t)]^{-1}-[-\nu-A(s)]^{-1}\Big][\mu-A(t)]^{-1}d\nu d\mu =: Z_0 + Z_{21} + Z_{22},$$

and as before we can evaluate in Z_{21} the integral over γ_1 and in Z_{22} the integral over γ_2, obtaining

$$Z_{21} = 0, \tag{3.13}$$

$$Z_{22} = \int_{\gamma_1} [\nu - A(t)^*]^{-1} P \Big[[-\nu - A(t)]^{-1} - [-\nu - A(s)]^{-1} \Big] [\lambda - \nu - A(t)]^{-1} d\nu. \tag{3.14}$$

By (3.6)-(3.9) and (3.12)-(3.14) we finally have

$$Z = Z_0 + Z_{11} + Z_{22}, \tag{3.15}$$

where Z_0 is defined in (3.12) and Z_{11}, Z_{22} are given by (3.8), (3.14).

Let us compute now, according to (1.1), the quantity $\Phi_Z(t;x,y)$ for $x,y \in D_{A(t)}$ and $t \in [0,T]$:

$$\Phi_Z(t;x,y) = (A(t)x, Zy)_H + (Zx, A(t)y)_H = \tag{3.16}$$

$$= (x, A(t)^* Zy)_H + (A(t)^* Zx, y)_H ,$$

so that by (3.15) we have to estimate $A(t)^* Z_0$, $A(t)^* Z_{11}$ and $A(t)^* Z_{22}$ in the $\mathcal{L}(H)$ norm.

To this purpose we need two lemmas.

LEMMA 3.3 If $\lambda \in \overline{S(\vartheta_0)}$ with $|\lambda| \geq 1$ and $\mu \in \gamma_1 \cup \gamma_2$, then

$$|\lambda - \mu| \geq [|\lambda| \vee |\mu|] \sin(\vartheta_1 - \vartheta_0).$$

Proof. Quite easy. □

LEMMA 3.4 If $\lambda \in \overline{S(\vartheta_0)}$ with $|\lambda| \geq 1$, $\mu \in \gamma_1 \cup \gamma_2$ and $\nu \in \gamma_1$, then for each $\varepsilon \in]0,1[$ we have

$$\Big\| A(t)^* [\lambda - \mu - A(t)^*]^{-1} \Big[[\nu - A(t)^*]^{-1} - [\nu - A(s)^*]^{-1} \Big] \Big\|_{\mathcal{L}(H)} \leq$$

$$\leq c(\vartheta, M, N, \alpha, \rho) |t-s|^{\alpha} [1+|\lambda|]^{-\rho(1-\varepsilon)} [1+|\mu|]^{-\rho\varepsilon/2} [1+|\nu|]^{-\rho\varepsilon/2}.$$

Proof. We write

$$\Big\| A(t)^* [\lambda - \mu - A(t)^*]^{-1} \Big[[\nu - A(t)^*]^{-1} - [\nu - A(s)^*]^{-1} \Big] \Big\|_{\mathcal{L}(H)} =$$

$$= \Big\| A(t)^* [\lambda - \mu - A(t)^*]^{-1} A(t)^* [\nu - A(t)^*]^{-1} \Big[[A(t)^*]^{-1} - [A(s)^*]^{-1} \Big] \cdot$$

$$\cdot A(s)^* [\nu - A(s)^*]^{-1} \Big\|_{\mathcal{L}(H)}^{(1-\varepsilon/2)+\varepsilon/2},$$

and using hypothesis (0.2) for $\{A(t)^*\}$ we get

$$\Big\| A(t)^* [\lambda - \mu - A(t)^*]^{-1} \Big[[\nu - A(t)^*]^{-1} - [\nu - A(s)^*]^{-1} \Big] \Big\|_{\mathcal{L}(H)} \leq$$

$$\leq c(M, N, \alpha, \rho) |t-s|^{\alpha} [1+|\lambda-\mu|]^{-\rho(1-\varepsilon/2)} [1+|\nu|]^{-\rho\varepsilon/2};$$

by Lemma 3.3 we get the result. □

Let us now estimate $A(t)^* Z_0$. By (3.12) we have

$$A(t)^* Z_0 = \int_{\gamma_2} \int_{\gamma_1} A(t)^* [\lambda - \mu - A(t)^*]^{-1} \Big[[\nu - A(t)^*]^{-1} - [\nu - A(s)^*]^{-1} \Big] P \cdot$$

$$\cdot \Big[[-\nu - A(t)]^{-1} - [-\nu - A(s)]^{-1} \Big] [\mu - A(t)]^{-1} d\nu d\mu,$$

so that by Lemma 3.4 and (0.1) we get for each $\varepsilon \in]0,1[$

$$\|A(t)^* Z_0\|_{\mathcal{L}(H)} \leq c(\vartheta, M, N, \alpha, \rho) \|P\|_{\Sigma(H)} |t-s|^{\alpha} [1+|\lambda|]^{-\rho(1-\varepsilon)} \cdot$$

$$\cdot \int_{\gamma_2} \int_{\gamma_1} [1+|\mu|]^{-\rho\varepsilon/2} [1+|\nu|]^{-\rho\varepsilon/2} |d\nu| |d\mu| \leq \tag{3.17}$$

$$\leq c(\vartheta, M, N, \alpha, \rho) \|P\|_{\Sigma(H)} |t-s|^{\alpha} [1+|\lambda|]^{-\rho(1-\varepsilon)}.$$

Concerning $A(t)^* Z_{11}$, by (3.8) we have

$$A(t)^* Z_{11} = \int_{\gamma_1} A(t)^* [\lambda - \nu - A(t)^*]^{-1} A(t)^* [-\nu - A(t)^*]^{-1} \cdot$$

$$\cdot \Big[[A(t)^*]^{-1} - [A(s)^*]^{-1} \Big] A(s)^* [-\nu - A(s)^*]^{-1} P [\nu - A(t)]^{-1} d\nu,$$

and by Lemma 3.4 we easily get for each $\varepsilon \in]0,1[$

$$\|A(t)^* Z_{11}\|_{\mathcal{L}(H)} \leq c(\vartheta, M, N, \alpha, \rho) \cdot$$

$$\cdot \|P\|_{\Sigma(H)} |t-s|^{\alpha} [1+|\lambda|]^{-\rho(1-\varepsilon)} \int_{\gamma_1} [1+|\nu|]^{-\rho\varepsilon-1} |d\nu| \leq \tag{3.18}$$

$$\leq c(\vartheta, M, N, \alpha, \rho, \varepsilon) \|P\|_{\Sigma(H)} |t-s|^{\alpha} [1+|\lambda|]^{-\rho(1-\varepsilon)}.$$

The estimate for $A(t)^* Z_{22}$ is quite similar: by (3.14) we have analogously

$$\|A(t)^* Z_{22}\|_{\mathcal{L}(H)} \le c(\vartheta,M,N,\alpha,\rho)\|P\|_{\Sigma(H)}|t-s|^{\alpha} \cdot \int_{\gamma_1} [1+|\nu|]^{-\rho}[1+|\lambda-\nu|]^{-1}|d\nu| \le \le c(\vartheta,M,N,\alpha,\rho,\varepsilon)\|P\|_{\Sigma(H)}|t-s|^{\alpha}[1+|\lambda|]^{-\rho(1-\varepsilon)}. \tag{3.19}$$

Estimates (3.17)-(3.19) show that (3.4) holds true: this concludes the proof of Theorem 2.3. □

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