Differential and Integral Equations, Volume 5, Number 5, September 1992, pp. 1151–1184.

REGULARITY PROPERTIES OF THE EVOLUTION OPERATOR FOR ABSTRACT LINEAR PARABOLIC EQUATIONS

PAOLO ACQUISTAPACE
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Università di Roma "La Sapienza", 00161 Roma, Italy

BRUNELLO TERRENI
Dipartimento di Matematica "F. Enriques", Università di Milano, 20133 Milano, Italy

(Submitted by: G. Da Prato)

0. Introduction. Consider the parabolic Cauchy problem

$$u'(t) - A(t)u(t) = f(t), \quad t \in [s,T], \quad u(s) = x \tag{0.1}$$

in a general Banach space E, where the operators $A(t)$ generate analytic semigroups in E, with possibly non-dense domains. The existence of the evolution operator $U(t,s)$ for problem (0.1) is guaranteed provided the family $\{A(t)\}_{t\in[0,T]}$ enjoys some regularity with respect to t. It is shown in [5, §7] that all kinds of hypotheses used in the literature can be essentially reduced to two independent sets of assumptions which are weaker than any other: namely, the classical ones by Kato and Tanabe [10], revisited in [3, 4], and those introduced in [5] and used in [6, 1].

The properties of $U(t,s)$ and its regularity with respect to t are very well known now. Much less information is available on the regularity of $s \to U(t,s)$; the only classical result is that

$$\exists \left[\tfrac{d}{d\sigma}U(t,\sigma)x\right]_{\sigma=s} = -U(t,s)A(s)x, \quad \forall x \in D_{A(s)}, \quad \forall 0 \le s \le t \le T, \tag{0.2}$$

provided all domains $D_{A(t)}$ are dense in E (see e.g., [17, Theorems 5.2.1 and 5.3.3]); some improvements of (0.2) can be found in [9, Theorem 1], [16, §1.11]. In addition, under the assumptions of [10], it is known that the operator-valued function $s \to U(t,s)$ is differentiable in $\mathcal{L}(E)$ for $s < t$, and $dU(t,s)/ds$ is a bounded extension of the closed operator $-U(t,s)A(s)$ (see [17, Theorem 5.3.3]). In a recent paper by Lunardi [13], the latter property, with several related results, has been shown to be true if the $A(t)$'s have a (possibly non-dense) common domain D and satisfy strong regularity assumptions.

The goal of this paper is a systematic study of the properties of $s \to U(t,s)$ under the assumptions of [3]; in this case, our results extend those of [13] and seem to be optimal. We also study the same problem under the assumptions of [5], but the situation here is more complicated and requires additional assumptions.

Received for publication December 3, 1990.
AMS Subject Classifications: 34G10, 47D05, 58D25.

Here is the plan of the paper. Section 1 is devoted to the list of assumptions and results, with some related remarks; Section 2 contains a collection of preparatory lemmas; in Sections 3, 4 and 5 we prove our main Theorems 1.6, 1.8 and 1.9, respectively; Section 6 contains statements and proofs of the analogous results under the hypotheses of [5]. Finally, there is an appendix containing the proof of some further regularity results for $U(t,s)$ which were only stated in [7]. Due to the length of the paper, examples and applications will appear elsewhere.

1. Assumptions and main results. Let us list the hypothesis of [3].

$$\begin{cases} \text{For each } t \in [0,T],\ A(t) : D_{A(t)} \subseteq E \to E \text{ is a closed linear} \\ \text{operator with possibly non-dense domain;} \end{cases} \tag{1.1}$$

$$\begin{cases} \text{there exist } \vartheta_0 \in]\pi/2, \pi[\text{ and } M > 0 \text{ such that} \\ \rho(A(t)) \supseteq S(\vartheta_0) := \{z \in \mathbb{C} : |\arg z| \le \vartheta_0\} \cup \{0\} \text{ and} \\ \quad \|[\lambda - A(t)]^{-1}\|_{\mathcal{L}(E)} \le \frac{M}{1+|\lambda|}, \quad \forall t \in [0,T], \quad \forall \lambda \in S(\vartheta_0); \end{cases} \tag{1.2}$$

$$\begin{cases} t \to [\lambda - A(t)]^{-1} \in C^1([0,T], \mathcal{L}(E)) \text{ for each } \lambda \in S(\vartheta_0), \\ \text{and there exist } \alpha \in]0,1],\ L > 0 \text{ such that} \\ \quad \|\frac{d}{dt}[\lambda - A(t)^{-1}\|_{\mathcal{L}(E)} \le \frac{L}{1+|\lambda|^{\alpha}}, \quad \forall t \in [0,T], \quad \forall \lambda \in S(\vartheta_0); \end{cases} \tag{1.3}$$

$$\begin{cases} \text{there exist } \eta \in]0,1],\ N > 0 \text{ such that} \\ \quad \|\frac{d}{dt}A(t)^{-1} - \frac{d}{ds}A(s)^{-1}\|_{\mathcal{L}(E)} \le N|t-s|^{\eta}, \quad \forall t,s \in [0,T]. \end{cases} \tag{1.4}$$

Remark 1.1. Hypothesis (1.3) can be relaxed somewhat; compare with [4, Hypothesis II(iii) and Remark 1.1].

Remark 1.2. We will often use the well known representation of the semigroups $\{e^{\xi A(t)}\}$ by Dunford integrals; in fact, we have

$$A(t)^h e^{\xi A(t)} = \fint_\gamma \lambda^h e^{\xi\lambda}[\lambda - A(t)]^{-1} d\lambda, \quad \forall t \in [0,T], \ \forall \xi > 0, \ \forall h \in \mathbb{N}, \tag{1.5}$$

where $\gamma \subset S(\vartheta_0)$ joins $+\infty e^{-i\vartheta}$ to $+\infty e^{i\vartheta}$, $\pi/2 < \vartheta < \vartheta_0$, and the symbol $\fint_\gamma$ means $(1/2\pi i)\int_\gamma$.

An easy consequence of the results of [3] is the following statement (compare also with [17, §5.3]):

Proposition 1.3. *Under assumptions* (1.1)–(1.4), *problem* (0.1) *possesses a unique evolution operator* $U(t,s) : \bar{\Delta} \to \mathcal{L}(E)$, *where* $\Delta : \{(t,s) \in [0,T]^2 : s < t\}$, *enjoying the following properties:*

(i) $U(t,s) = U(t,r)U(r,s), \forall 0 \le s < r < t \le T$; $U(t,t) = 1, \forall t \in [0,T]$;
(ii) $\frac{d}{dt}U(t,s) = A(t)U(t,s), \forall 0 \le s < t \le T$;
(iii) $\|U(t,s)\|_{\mathcal{L}(E)} + (t-s)\|A(t)U(t,s)\|_{\mathcal{L}(E)} \le c(M, L, N, \alpha, \eta, \vartheta_0, T), \forall (t,s) \in \Delta$.

Remark 1.4. There is a representation formula for $U(t,s)$, namely

$$U(t,s) = [(1-R_s)^{-1}(e^{(\cdot - s)A(s)})](t), \quad 0 \le s \le t \le T, \tag{1.6}$$

where R_s is the integral operator

$$[R_s g](t) := \int_s^t R(t,r) g(r)\, dr, \quad t \in [s,T], \quad g \in L^1(s,T;E), \tag{1.7}$$

whose kernel $R(t,r)$ is given by the Dunford integral

$$R(t,r) := \int_\gamma e^{(t-r)\lambda} \frac{d}{dr}[\lambda - A(r)]^{-1} d\lambda, \quad 0 \le r < t \le T; \tag{1.8}$$

here γ is the same curve appearing in (1.5). Formula (1.6) follows easily by [3, Proposition 1.9] (the operator R_s was named $\tilde{P}$ there).

Remark 1.5. By (1.6) and the results of [3], it follows that under suitable assumptions on the data x, f, the function

$$u(t) := U(t,s)x + \int_s^t U(t,r) f(r)\, dr, \quad t \in [s,T]$$

is the unique solution of problem (0.1).

Let us state now our first main result.

Theorem 1.6. *Under assumptions* (1.1)–(1.4), *there exists an operator* $V(t,s) \in \mathcal{L}(E)$ *satisfying for each* $(t,s) \in \Delta$:

(i) $\frac{d}{ds}U(t,s) = V(t,s)$, *and* $V(t,s)x = -U(t,s)A(s)x$, $\forall x \in D_{A(s)}$;

(ii) $\|V(t,s)\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,T)(t-s)^{-1}$;

(iii) $\|V(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,T)(t-s)^{\delta-1}$, *where* $\delta := \alpha \wedge \eta$.

Proof. See Section 3.

We are able to give a representation formula for $V(t,s)$ (see (1.10) below). A heuristic derivation of such a formula can be obtained in the following way: by (0.2) and (1.6), we formally get

$$\frac{d}{ds}U(t,s)x = -U(t,s)A(s)x = -[(1-R_s)^{-1}(A(s)e^{(\cdot - s)A(s)}x)](t).$$

Now, $(1-R_s)^{-1} = \sum_{n=0}^{\infty}(R_s)^n$, and the powers of R_s are integral operators with kernels inductively defined by

$$R_1(t,s) := R(t,s), \quad R_n(t,s) := \int_s^t R_{n-1}(t,r)R(r,s)\, dr, \quad \forall n > 1; \tag{1.9}$$

hence,

$$\begin{aligned}\frac{d}{ds}U(t,s)x &= -A(s)e^{(t-s)A(s)}x - \sum_{n=1}^{\infty}\int_s^t R_n(t,r)A(s)e^{(r-s)A(s)}x\, dr \\ &= -A(s)e^{(t-s)A(s)}x - \sum_{n=1}^{\infty}\int_s^t [R_n(t,r) - R_n(t,s)]A(s)e^{(r-s)A(s)}x\, dr \\ &\quad - \sum_{n=1}^{\infty} R_n(t,s)[e^{(t-s)A(s)} - 1]x.\end{aligned}$$

The last expression is meaningful for each $x \in E$ in view of Lemma 2.6 below. Thus, if we set

$$\begin{aligned}V(t,s)x := &-A(s)e^{(t-s)A(s)}x \\ &- \sum_{n=1}^{\infty}\int_s^t [R_n(t,r) - R_n(t,s)]A(s)e^{(r-s)A(s)}x\, dr \\ &- \sum_{n=1}^{\infty} R_n(t,s)[e^{(t-s)A(s)} - 1]x,\end{aligned} \tag{1.10}$$

the above argument "shows" that $dU(t,s)x/ds = V(t,s)x$, $\forall (t,s) \in \Delta$, $\forall x \in E$. In fact, in Section 3 we will prove more, i.e., as in [17, Theorem 5.3.3], that

$$\frac{d}{ds}U(t,s) = V(t,s), \quad \forall (t,s) \in \Delta \text{ (in the } \mathcal{L}(E) \text{ norm)}. \tag{1.11}$$

The operator $V(t,s)$ has some further regularity properties, whose description needs some notations. We denote by $D_{A(s)}(\vartheta,\infty)$ the real interpolation space $(D_{A(s)},E)_{1-\vartheta,\infty}$, $\vartheta \in]0,1[$; for brevity we agree that $D_{A(s)}(1,\infty)$ means $D_{A(s)}$. We also need to recall the definition of the following function spaces:

Definition 1.7. (i) If $\mu \ge 0$, $B_\mu([a,b[,E)$ (resp. $B_\mu(]a,b],E)$) is the Banach space of continuous functions $u : [a,b[\to E$ (resp. $u :]a,b] \to E$) such that $\|u\|_\mu < \infty$, where

$$\|u\|_\mu := \sup_{s\in[a,b[} (b-s)^\mu \|u(s)\|_E \quad \left(\text{resp. } \sup_{s\in]a,b]} (s-a)^\mu \|u(s)\|_E\right).$$

(ii) If $\mu \ge 0$ and $\beta \in]0,1]$, $Z_{\mu,\beta}([a,b[,E)$ (resp. $Z_{\mu,\beta}(]a,b],E)$) is the space of functions $u \in B_\mu([a,b[,E)$ (resp. $B_\mu(]a,b],E)$) such that $[u]_{\mu,\beta} < \infty$, where

$$[u]_{\mu,\beta} := \sup_{s\in[a,b[}\left\{(b-s)^{\mu+\beta} \sup_{s\le p<q\le \frac{s+b}{2}} (q-p)^{-\beta}\|u(q)-u(p)\|_E\right\}$$

(resp.

$$\left.\sup_{s\in]a,b]}\left\{(s-a)^{\mu+\beta} \sup_{\frac{s+a}{2}\le p<q\le s} (q-p)^{-\beta}\|u(q)-u(p)\|_E\right\}\right).$$

The spaces $Z_{\mu,\beta}$ are Banach spaces with their obvious norms; they are useful in treating Hölder continuous functions which blow up at an endpoint of their interval of definition. These spaces were introduced in [5] (with blow up at $a = 0$) and used in various situations [11, 12, 13, 8], but an earlier use of them can be found in [15].

Here is our result concerning the regularity of $s \to V(t,s)$.

Theorem 1.8. *Under assumptions* (1.1)-(1.4), *the operator* $V(t,s)$, *given by* (1.10), *satisfies for each* $(t,s) \in \Delta$ *and* $\vartheta \in]0,1]$:

(i) $\|V(t,s)\|_{\mathcal{L}(D_{A(s)}(\vartheta,\infty),E)} \le c(\vartheta, M, L, N, \alpha, \eta, \vartheta_0, T)(t-s)^{\vartheta-1}$;

(ii) $\|V(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(D_{A(s)}(\vartheta,\infty),E)}$
$\le c(\vartheta, M, L, N, \alpha, \eta, \vartheta_0, T)(t-s)^{\delta+\vartheta-1}$, *where* $\delta = \alpha \wedge \eta$;

(iii) $s \to V(t,s) \in Z_{1,\delta-\epsilon}([0,t[, \mathcal{L}(E))$, $\forall \epsilon \in]0,\delta[$;

(iv) $s \to V(t,s) \in Z_{1-\vartheta,\delta-\epsilon}([0,t[, \mathcal{L}(D_{A(t)}(\vartheta,\infty),E))$, $\forall \epsilon \in]0,\delta[$;

(v) $s \to V(t,s)x \in C([0,t],E)$ *if and only if* $x \in D_{A(t)}$ *and* $A(t)x$, $dA(t)^{-1}/dt \cdot A(t)x$ *both belong to* $\overline{D_{A(t)}}$; *in this case*

$$V(t,t)x = -[\tfrac{d}{ds}U(t,s)x]_{s=t} = -A(t)x;$$

(vi) *if* $x \in D_{A(t)}$ *and* $A(t)x$, $dA(t)^{-1}/dt \cdot A(t)x \in D_{A(t)}(\beta,\infty)$, *with* $\beta \in]0,\delta[$, *then* $s \to V(t,s)x \in C^\beta([0,t],E)$.

Proof. See Section 4.

Concerning the regularity of $t \to V(t,s)$, we have the following result:

Theorem 1.9. *Under assumptions* (1.1)-(1.4), *the operator* $V(t,s)$, *given by* (1.10), *satisfies:*

(i) $V(t,s) = U(t,r)V(r,s)$, $\forall 0 \le s < r < t \le T$;

(ii) $\frac{d}{dt}V(t,s) = A(t)V(t,s)$, $\forall 0 \le s < t \le T$;

(iii) $t \to V(t,s) \in Z_{1,1}(]s,T], \mathcal{L}(E))$, $\forall s \in [0,T[$;

(iv) $t \to V(t,s) \in Z_{1-\vartheta,1}(]s,T], L(D_{A(s)}(\vartheta,\infty),E))$, $\forall \vartheta \in]0,1]$;

(v) $t \to V(t,s)x \in C([s,T],E)$ *if and only if* $x \in D_{A(s)}$ *and* $A(s)x \in \overline{D_{A(s)}}$;

(vi) $t \to V(t,s)x \in C^\beta([s,T],E)$, *with* $\beta \in]0,\alpha]$, *if and only if* $x \in D_{A(s)}$ *and* $A(s)x \in D_{A(s)}(\beta,\infty)$.

Proof. See Section 5.

Remark 1.10. We recall that, by (1.2), the fractional powers $[-A(s)]^\vartheta$ of $-A(s)$ are well defined for each $\vartheta \in [0,1]$ (see e.g., [17, §2.3]. Then, due to the continuous inclusion

$$D_{[-A(s)]^\vartheta} \subseteq D_{A(s)}(\vartheta,\infty), \quad \forall \vartheta \in]0,1],$$

by Theorem 1.8(i)-(ii) we immediately get

$$\|V(t,s)\|_{\mathcal{L}(D_{[-A(s)]^\vartheta},E)} \le c(\vartheta, M, L, N, \alpha, \eta, \vartheta_0, T)(t-s)^{\vartheta-1}, \tag{1.12}$$

$$\|V(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(D_{[-A(s)]^\vartheta},E)} \le c(\vartheta, M, L, N, \alpha, \eta, \vartheta_0, T)(t-s)^{\delta+\vartheta-1}. \tag{1.13}$$

This proves [7, Proposition 3.1(ii)] which was stated without proof there (the other parts of that Proposition are proved in the Appendix below).

We end this section with some remarks concerning the Yosida approximations of $A(t)$, defined by

$$A_k(t) := kA(t)[k - A(t)]^{-1}, \quad k \in \mathrm{N}^+. \tag{1.14}$$

We will use such operators in the proof of our main Theorem 1.6. It is well known and easily seen that for each $k \in \mathrm{N}^+$ and $t \in [0,T]$ we have $\rho(A_k(t)) \supseteq S(\vartheta_0)$ and the following properties are true for each $\lambda \in S(\vartheta_0)$, $t,s \in [0,T]$ and $k \in \mathrm{N}^+$:

$$\|[\lambda - A_k(t)]^{-1}\|_{\mathcal{L}(E)} \le \frac{M'}{1+|\lambda|}, \tag{1.15}$$

$$\|\frac{d}{dt}[\lambda - A_k(t)]^{-1}\|_{\mathcal{L}(E)} \le \frac{L'}{1+|\lambda|^\alpha}, \tag{1.16}$$

$$\|\frac{d}{dt}A_k(t)^{-1} - \frac{d}{ds}A_k(s)^{-1}\|_{\mathcal{L}(E)} \le N|t-s|^\eta. \tag{1.17}$$

where M' depends on M, ϑ_0 and L' depends on L, ϑ_0. In particular, $\{A_k(t)\} \subset \mathcal{L}(E)$ and fulfills (1.1)–(1.4) uniformly with respect to $k \in \mathrm{N}^+$. Thus, we can construct the corresponding evolution operators $\{U_k(t,s)\}$, given (as in (1.6)), by

$$U_k(t,s) = [(1 - R_{s,k})^{-1}(e^{(\cdot-s)A_k(s)})](t), \quad 0 \le s \le t \le T, \ k \in \mathrm{N}^+, \tag{1.18}$$

where

$$[R_{s,k}g](t) := \int_s^t R_{1,k}(t,r)g(r)\,dr, \quad t \in [s,T], \ k \in \mathrm{N}^+, \tag{1.19}$$

with

$$R_{1,k}(t,r) := \int_\gamma e^{(t-r)\lambda}\frac{d}{dr}[\lambda - A_k(r)]^{-1}d\lambda, \quad 0 \le r < t \le T. \ k \in \mathrm{N}^+. \tag{1.20}$$

We can also construct the operator (analogous to (1.10))

$$\begin{aligned} V_k(t,s) := &- A_k(s)e^{(t-s)A_k(s)} \\ &- \sum_{n=1}^\infty \int_s^t [R_{n,k}(t,r) - R_{n,k}(t,s)]A_k(s)e^{(r-s)A_k(s)}dr \\ &- \sum_{n=1}^\infty R_{n,k}(t,s)[e^{(t-s)A_k(s)} - 1], \quad 0 \le s < t \le T. \ k,n \in \mathrm{N}^+, \end{aligned} \tag{1.21}$$

where, as in (1.9),

$$R_{n,k}(t,s) := \int_s^t R_{n-1,k}(t,r)R_{1,k}(r,s)\,dr, \quad n,k > 1. \tag{1.22}$$

We remark that all operators constructed above obviously belong to $\mathcal{L}(E)$; we will show in the next section that their norms are, in fact, bounded uniformly with respect to $k \in \mathrm{N}^+$ (see Lemmas 2.7, 2.8 and 2.9 below).

2. Technicalities.

We collect here all preparatory lemmas which are needed in proving Theorems 1.6, 1.8 and 1.9. The statements are grouped according to their subject.

A) Lemmas on resolvents.

Lemma 2.1. *Let γ be the curve appearing in* (1.5). *Under assumptions* (1.1), (1.2), (1.3) *we have for each $\lambda \in S(\vartheta_0)$ and $t,s,r \in [0,T]$:*

(i) $\|A(s)[\lambda - A(s)]^{-1}\|_{\mathcal{L}(E)} \le 1 + M$;

(ii) $\|A(t)[\lambda - A(t)]^{-1} - A(s)[\lambda - A(s)]^{-1}\|_{\mathcal{L}(E)}$
$= \|\lambda([\lambda - A(t)]^{-1} - [\lambda - A(s)]^{-1})\|_{\mathcal{L}(E)} \le L|\lambda|^{1-\alpha}|t-s|$;

(iii) $\|A(s)[\lambda - A(s)]^{-1}A(t)^{-1}\|_{\mathcal{L}(E)} \le c(M,L)(|t-s| + [1+|\lambda|]^{-1})$;

(iv) $\|(A(r)[\lambda - A(r)]^{-1} - A(s)[\lambda - A(s)]^{-1})A(t)^{-1}\|_{\mathcal{L}(E)}$
$\le c(M,L)|r-s|(|\lambda|\,|t-r| + 1)$.

Proof. (i) is a trivial consequence of (1.2). (ii) follows by (1.3) and the identity

$$A(t)[\lambda - A(t)]^{-1} - A(s)[\lambda - A(s)]^{-1} = \lambda([\lambda - A(t)]^{-1} - [\lambda - A(s)]^{-1}) = \lambda \int_s^t \frac{d}{dr}[\lambda - A(r)]^{-1}dr. \tag{2.1}$$

(iv) follows by (i), (1.2), (1.3) and the identity

$$(A(r)[\lambda - A(r)]^{-1} - A(s)[\lambda - A(s)]^{-1})A(t)^{-1} = \lambda A(s)[\lambda - A(s)]^{-1} \times [A(s)^{-1} - A(r)^{-1}](\lambda A(r)[\lambda - A(r)]^{-1}[A(t)^{-1} - A(r)^{-1}] + 1)[\lambda - A(t)]^{-1}. \tag{2.2}$$

(iii) is an easy consequence of (iv) (with $r = t$) and (1.2).

Lemma 2.2. *Under assumptions* (1.1)–(1.4), *we have for each $\lambda \in S(\vartheta_0)$ and $r,s,t \in [0,T]$:*

(i) $\|\frac{d}{dt}[\lambda - A(t)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1}\|_{\mathcal{L}(E)} \le c(M,L,N)(|t-s|^\eta + |\lambda|^{1-\alpha}|t-s|)$;

(ii) $\|A(t)^{-1}\left(\frac{d}{dt}[\lambda - A(t)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1}\right)\|_{\mathcal{L}(E)} \le$
$c(M,L,N)([1+|\lambda|]^{-1}|t-s|^\eta + [1+|\lambda|]^{-\alpha}|t-s|)$;

(iii) $\|\left(\frac{d}{dt}[\lambda - A(t)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1}\right)A(r)^{-1}\|_{\mathcal{L}(E)} \le$
$c(M,L,N)(|t-s|^\eta + |\lambda|^{1-\alpha}|t-s|)((s-r) + [1+|\lambda|]^{-1})$.

Proof. Arguing as in [3, Lemma 3.1], all estimates follow easily by Lemma 2.1(i)-(ii)-(iii), (1.2), (1.4) and the identity

$$\frac{d}{dt}[\lambda - A(t)]^{-1} = -A(t)[\lambda - A(t)]^{-1}\frac{d}{dt}A(t)^{-1} \cdot A(t)[\lambda - A(t)]^{-1}. \tag{2.3}$$

Remark 2.3. Due to (1.15), (1.16) and (1.17), Lemma 2.2 holds unchanged with $A(t)$, $A(s)$ replaced by $A_k(t)$, $A_k(s)$, with bounds independent of $k \in \mathbf{N}^+$.

Lemma 2.4. *Under assumptions* (1.1)–(1.4), *we have for each $t \in [0,T]$, $\xi > 0$ and $h \in \mathbf{N}$ (δ_{h0} being the Kronecker symbol) :*

(i) $$\int_\gamma \lambda^h e^{\xi\lambda} A(t)^{-1}\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda = \int_\gamma \lambda^{h-1}e^{\xi\lambda}\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda + \frac{d}{dt}A(t)^{-1}[\delta_{h0} - A(t)^h e^{\xi A(t)}];$$

(ii) $$\int_\gamma \lambda^h e^{\xi\lambda}\frac{d}{dt}[\lambda - A(t)]^{-1}\cdot A(t)^{-1}d\lambda = \int_\gamma \lambda^{h-1}e^{\xi\lambda}\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda + [\delta_{h0} - A(t)^h e^{\xi A(t)}]\frac{d}{dt}A(t)^{-1}.$$

Proof. Both equalities follow easily by the resolvent identity and (1.5).

B) Lemmas on semigroups.

Lemma 2.5. *Under assumptions* (1.1), (1.2), (1.3), *we have for each $r,s,t \in [0,T]$, $\xi > 0$ and $h \in \mathbf{N}$:*

(i) $\|A(s)^h e^{\xi A(s)}\|_{\mathcal{L}(E)} \le c(h,M,\vartheta_0)\xi^{-h}$;

(ii) $\|A(s)^h e^{\xi A(s)} - A(r)^h e^{\xi A(r)}\|_{\mathcal{L}(E)} \le c(h,L,\alpha,\vartheta_0)\xi^{\alpha-1-h}|s-r|$;

(iii) $\|A(s)^h e^{\xi A(s)}A(t)^{-1} - \delta_{h0}A(t)^{-1}\|_{\mathcal{L}(E)} \le c(h,M,L,\alpha,\vartheta_0)\xi^{1-h}[1+\xi^{-1}|t-s|]$;

(iv) $\|[A(s)^h e^{\xi A(s)} - A(r)^h e^{\xi A(r)}]A(t)^{-1}\|_{\mathcal{L}(E)} \le$
$c(h,M,L,\alpha,\vartheta_0)\xi^{-h}|r-s|[1+\xi^{-1}|t-r|]$.

Moreover, under assumptions (1.1)–(1.4), *we have for each $r,s,t \in [0,T]$, $\xi > 0$ and $h \in \mathbf{N}$:*

(v) $\|A(t)^{-1}[A(s)^h e^{\xi A(s)} - A(r)^h e^{\xi A(r)}] - (s-r)\frac{d}{dt}A(t)^{-1}[\delta_{h0} - A(t)^h e^{\xi A(t)}]\|_{\mathcal{L}(E)}$
$\le c(h,M,L,N,\alpha,\eta,\vartheta_0,T)\xi^{\delta-h}|r-s|[1+\xi^{-1}\max(|t-r|,|t-s|,|r-s|)]$;

(vi) $\|[A(s)^h e^{\xi A(s)} - A(r)^h e^{\xi A(r)}]A(t)^{-1} - (s-r)[\delta_{h0} - A(t)^h e^{\xi A(t)}]\frac{d}{dt}A(t)^{-1}\|_{\mathcal{L}(E)}$
$\le c(h,M,L,N,\alpha,\eta,\vartheta_0,T)\xi^{\delta-h}|r-s|[1+\xi^{-1}\max(|t-r|,|t-s|,|r-s|)]$.

Proof. (i) is a standard consequence of (1.5). (ii) follows by (1.5) and (1.3). (iv) If $h > 0$, (iv) follows by (1.5) and Lemma 2.1(iv); if $h = 0$, we can write (by (2.2))

$$[e^{\xi A(s)} - e^{\xi A(r)}]A(t)^{-1} = -\int_\gamma e^{\xi\lambda}A(s)[\lambda - A(s)]^{-1}[A(s)^{-1} - A(r)^{-1}]A(r)[\lambda - A(r)]^{-1}A(t)^{-1}d\lambda,$$

and the result follows by Lemma 2.1(i)-(iii) and (1.3).

(iii) If $h > 0$, (iii) follows by (1.5) and Lemma 2.1(iii); if $h = 0$, we can write

$$e^{\xi A(s)}A(t)^{-1} - A(t)^{-1} = [e^{\xi A(s)} - e^{\xi A(t)}]A(t)^{-1} + \int_0^\xi e^{\zeta A(t)}d\zeta,$$

and the result follows by (iv) and (i).

(v) We can write, using (2.2) and (2.3),

$$A(t)^{-1}[A(s)^h e^{\xi A(s)} - A(r)^h e^{\xi A(r)}] - (s-r)\frac{d}{dt}A(t)^{-1}[\delta_{h0} - A(t)^h e^{\xi A(t)}]$$

$$= \oint_\gamma \lambda^h e^{\xi\lambda}\Big\{[\lambda - A(t)]^{-1}[A(t)^{-1} - A(r)^{-1}]\lambda([\lambda - A(s)]^{-1} - [\lambda - A(r)]^{-1})$$

$$+ [\lambda - A(t)]^{-1}[A(r)^{-1} - A(s)^{-1} - (r-s)\frac{d}{ds}A(s)^{-1}]A(s)[\lambda - A(s)]^{-1}$$

$$+ (r-s)[\lambda - A(t)]^{-1}\Big[\frac{d}{ds}A(s)^{-1} - \frac{d}{dt}A(t)^{-1}\Big]A(s)[\lambda - A(s)]^{-1}$$

$$+ (r-s)[\lambda - A(t)]^{-1}\frac{d}{dt}A(t)^{-1}\lambda([\lambda - A(s)]^{-1} - [\lambda - A(t)]^{-1})$$

$$- (r-s)A(t)^{-1}\frac{d}{dt}[\lambda - A(t)]^{-1}\Big\}d\lambda - (s-r)\frac{d}{dt}A(t)^{-1}[\delta_{h0} - A(t)^h e^{\xi A(t)}].$$

Now, by Lemma 2.4(i), the last term under the integral sign may be rewritten as

$$\oint_\gamma \lambda^h e^{\xi\lambda}(s-r)A(t)^{-1}\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda$$

$$= \oint_\gamma \lambda^{h-1}e^{\xi\lambda}(s-r)\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda + (s-r)\frac{d}{dt}A(t)^{-1}[\delta_{h0} - A(t)^h e^{\xi A(t)}],$$

so that the result follows easily by Lemma 2.1(i)-(ii) and (1.2), (1.3), (1.4).

(vi) is quite similar to the proof of (v). We can write, using (2.2) and (2.3),

$$[A(s)^h e^{\xi A(s)} - A(r)^h e^{\xi A(r)}]A(t)^{-1} - (s-r)[\delta_{h0} - A(t)^h e^{\xi A(t)}]\frac{d}{dt}A(t)^{-1}$$

$$= \oint_\gamma \lambda^h e^{\xi\lambda}\Big\{\lambda([\lambda - A(s)]^{-1} - [\lambda - A(r)]^{-1})[A(r)^{-1} - A(t)^{-1}][\lambda - A(t)]^{-1}$$

$$- A(s)[\lambda - A(s)]^{-1}[A(s)^{-1} - A(r)^{-1} - (s-r)\frac{d}{dr}A(r)^{-1}][\lambda - A(t)]^{-1}$$

$$- (s-r)A(s)[\lambda - A(s)]^{-1}\Big[\frac{d}{dr}A(r)^{-1} - \frac{d}{dt}A(t)^{-1}\Big][\lambda - A(t)]^{-1}$$

$$- (s-r)\lambda([\lambda - A(s)]^{-1} - [\lambda - A(t)]^{-1})\frac{d}{dt}A(t)^{-1}[\lambda - A(t)]^{-1}$$

$$+ (s-r)\frac{d}{dt}[\lambda - A(t)]^{-1}A(t)^{-1}\Big\}d\lambda - (s-r)[\delta_{h0} - A(t)^h e^{\xi A(t)}]\frac{d}{dt}A(t)^{-1}.$$

Now, by Lemma 2.4(ii), the last term under the integral sign may be rewritten as

$$\oint_\gamma \lambda^h e^{\xi\lambda}(s-r)\frac{d}{dt}[\lambda - A(t)]^{-1}\cdot A(t)^{-1}d\lambda$$

$$= \oint_\gamma \lambda^{h-1}e^{\xi\lambda}(s-r)\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda + (s-r)[\delta_{h0} - A(t)^h e^{\xi A(t)}]\frac{d}{dt}A(t)^{-1},$$

so that the result follows easily by Lemma 2.1(i)-(ii) and (1.2), (1.3), (1.4).

C) Lemma on the integral operator.

Lemma 2.6. *Under assumptions* (1.1), (1.2), (1.3), *we have, for each* $(t,s) \in \Delta$:

(i) $\displaystyle\sum_{n=1}^\infty \|R_n(t,s)\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0,T)(t-s)^{\alpha-1};$

(ii) $\displaystyle\Big\|\sum_{n=1}^\infty A(t)^{-1}R_n(t,s) + \frac{d}{ds}A(s)^{-1}[e^{(t-s)A(s)} - 1]\Big\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta_0,T)(t-s)^\alpha.$

Moreover, under assumptions (1.1)–(1.4), *we have, for* $0 \le s \le r < t \le T$ *and* $\epsilon \in]0,\delta[$:

(iii) $$\sum_{n-1}^\infty \|R_n(t,r) - R_n(t,s)\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,\epsilon,T)(r-s)^{\delta-\epsilon}(t-r)^{\epsilon-1};$$

(iv) $$\sum_{n=1}^\infty \int_s^t (r-s)^{-1}\|R_n(t,r) - R_n(t,s)\|_{\mathcal{L}(E)}dr \le c(M,L,N,\alpha,\eta,\vartheta_0,T)(t-s)^{\delta-1};$$

(v) $$\Big\|\sum_{n=1}^\infty A(t)^{-1}[R_n(t,r) - R_n(t,s)] - \frac{d}{dr}A(r)^{-1}[e^{(t-s)A(r)} - e^{(t-r)A(r)}]\Big\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,\epsilon,T)(r-s)^{\delta-\epsilon}.$$

Proof. (i) By induction we easily have

$$\|R_n(t,s)\|_{\mathcal{L}(E)} \le K^n\Gamma(\alpha)^n\Gamma(n\alpha)^{-1}(t-s)^{n\alpha-1}, \quad \forall n \in \mathbb{N}^+, \ \forall (t,s) \in \Delta, \quad (2.4)$$

where $K := (L/2\pi)\int_\gamma e^{\mathrm{Re}\, z}|z|^{-\alpha}|dz|$; the result follows immediately.

(ii) By Lemma 2.4(i),

$$A(t)^{-1}R_1(t,s) = [A(t)^{-1} - A(s)^{-1}]R(t,s) + \oint_\gamma \lambda^{-1}e^{(t-s)\lambda}\frac{d}{ds}[\lambda - A(s)]^{-1}d\lambda + \frac{d}{ds}A(s)^{-1}[1 - e^{(t-s)A(s)}],$$

so that (1.3) yields

$$\|A(t)^{-1}R_1(t,s) + \frac{d}{ds}A(s)^{-1}[e^{(t-s)A(s)} - 1]\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0)(t-s)^\alpha;$$

and, in particular, by Lemma 2.5(i),

$$\|A(t)^{-1}R_1(t,s)\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta_0)[1 + (t-s)^\alpha];$$

hence, by (i),

$$\Big\| \sum_{n=1}^{\infty} A(t)^{-1} R_n(t,s) + \frac{d}{ds} A(s)^{-1}[e^{(t-s)A(s)} - 1] \Big\|_{\mathcal{L}(E)}$$
$$\le \Big\| A(t)^{-1} R_1(t,s) + \frac{d}{ds} A(s)^{-1}[e^{(t-s)A(s)} - 1] \Big\|_{\mathcal{L}(E)}$$
$$+ \sum_{n=2}^{\infty} \Big\| \int_s^t A(t)^{-1} R(t,r) R_{n-1}(r,s)\, dr \Big\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta,T)(t-s)^{\alpha},$$

which proves (ii).

(iii)-(iv) If $0 \le s \le r < t \le T$, we have, by (1.8),

$$\begin{aligned} R_1(t,r) - R_1(t,s) = &- \int_{t-r}^{t-s} \oint_{\gamma} \lambda e^{\xi\lambda} \frac{d}{dr}[\lambda - A(r)]^{-1} d\lambda\, d\xi \\ &+ \oint_{\gamma} e^{(t-s)\lambda} \left(\frac{d}{dr}[\lambda - A(r)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1} \right) d\lambda, \end{aligned} \tag{2.5}$$

so that, recalling Lemma 2.2(i), it is easy to get, for $\epsilon \in]0,\delta[$,

$$\begin{aligned} \|R_1(t,r) - R_1(t,s)\|_{\mathcal{L}(E)} &\le c(M,L,N,\alpha,\eta,\vartheta_0) \Big\{ \int_{t-r}^{t-s} \frac{d\xi}{\xi^{2-\delta}} + \frac{(r-s)^{\delta}}{t-s} \Big\} \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0)(r-s)^{\delta-\epsilon}(t-r)^{\epsilon-1}; \end{aligned} \tag{2.6}$$

which implies in addition

$$\int_s^t (r-s)^{-1} \|R_1(t,r) - R_1(t,s)\|_{\mathcal{L}(E)} dr \le c(M,L,N,\alpha,\eta,\vartheta_0)(t-s)^{\delta-1}. \tag{2.7}$$

Next, if $n \ge 2$ we write for $0 \le s \le r < t \le T$:

$$\begin{aligned} R_n(t,r) - R_n(t,s) = &\int_r^t R_{n-1}(t,q)[R_1(q,r) - R_1(q,s)]dq \\ &- \int_s^r R_{n-1}(t,q) R_1(q,s) dq =: T_1 + T_2. \end{aligned} \tag{2.8}$$

Now by (2.6) and (2.4) (with α replaced by δ, which is obviously allowed), we have for $\epsilon \in]0,\delta[$:

$$\begin{aligned} &\|T_1\|_{\mathcal{L}(E)} \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0) \frac{K^{n-1}\Gamma(\delta)^{n-1}}{\Gamma(n\delta-\delta)} \int_r^t (t-q)^{(n-1)\delta-1} \frac{(r-s)^{\delta-\epsilon}}{(q-r)^{1-\epsilon}} dq \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0) \frac{K^{n-1}\Gamma(\delta)^{n-1}\Gamma(\epsilon)}{\Gamma(n\delta-\delta+\epsilon)} (t-r)^{(n-1)\delta+\epsilon-1}(r-s)^{\delta-\epsilon}, \end{aligned} \tag{2.9}$$

$$\|T_2\|_{\mathcal{L}(E)} \le \frac{K^{n-1}\Gamma(\delta)^{n-1}}{\Gamma(n\delta-\delta)} \int_s^r (t-q)^{(n-1)\delta-1}(q-s)^{\delta-1} dq; \tag{2.10}$$

by (2.6), (2.8), (2.9) and (2.10), we easily get (iii). In addition, choosing $\epsilon = \delta/2$, we obtain

$$\begin{aligned} &\int_s^t (r-s)^{-1} \|R_n(t,r) - R_n(t,s)\|_{\mathcal{L}(E)} dr \\ &\le c(M,L,N,\alpha,\eta,\vartheta_0) \frac{K^{n-1}\Gamma(\delta)^{n-1}\Gamma(\delta/2)}{\Gamma(n\delta-\delta/2)} \int_s^t (t-r)^{(n-1/2)\delta-1}(r-s)^{\delta/2-1} dr \\ &+ \frac{K^{n-1}\Gamma(\delta)^{n-1}}{\Gamma(n\delta-\delta)} \int_s^t (r-s)^{\delta/2-1} \int_s^r (t-q)^{(n-1)\delta-1}(q-s)^{\delta/2-1} dq\, dr \\ &\le c(M,L,N,\alpha,\eta,\vartheta_0) \frac{4^{n\delta}K^n\Gamma(\delta)^n}{\Gamma(n\delta)} (t-s)^{n\delta-1}. \end{aligned} \tag{2.11}$$

In the last estimate, we have used the easy inequality

$$\Gamma(n\delta - \delta/2) \ge c(\delta) 2^{-n\delta}\Gamma(n\delta).$$

By (2.7) and (2.11) we get (iv).

(v) By (2.6) and Lemma 2.4(i) (with $h = 1$ and $t = r$) we can write:

$$\begin{aligned} A(t)^{-1}[R(t,r) - R(t,s)] = &- \int_{t-r}^{t-s} [A(t)^{-1} - A(r)^{-1}] \oint_{\gamma} \lambda e^{\xi\lambda} \frac{d}{dr}[\lambda - A(r)]^{-1} d\lambda\, d\xi \\ &- \int_{t-r}^{t-s} \oint_{\gamma} e^{\xi\lambda} \frac{d}{dr}[\lambda - A(r)]^{-1} d\lambda\, d\xi + \frac{d}{dr} A(r)^{-1}[e^{(t-s)A(r)} - e^{(t-r)A(r)}] \\ &+ \oint_{\gamma} e^{(t-s)\lambda}[A(t)^{-1} - A(r)^{-1}] \left[\frac{d}{dr}[\lambda - A(r)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1} \right] d\lambda \\ &+ \oint_{\gamma} e^{(t-s)\lambda} A(r)^{-1} \left[\frac{d}{dr}[\lambda - A(r)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1} \right] d\lambda; \end{aligned}$$

thus, by (1.3) and Lemma 2.2, we obtain

$$A(t)^{-1}[R(t,r) - R(t,s)] = \frac{d}{dr} A(r)^{-1}[e^{(t-s)A(r)} - e^{(t-r)A(r)}] + O((r-s)^{\delta})$$

as $r - s \searrow 0$. On the other hand, by (ii), (iii) and (i) we get for each $\epsilon \in]0,\delta[$

$$\begin{aligned} \sum_{n=2}^{\infty} A(t)^{-1}[R_n(t,r) - R_n(t,s)] = &\sum_{n=2}^{\infty} \int_r^t A(t)^{-1} R(t,q)[R_{n-1}(q,r) - R_{n-1}(q,s)]dq \\ &- \sum_{n=2}^{\infty} \int_s^r A(t)^{-1} R(t,q) R_{n-1}(q,s)\, dq = O((r-s)^{\delta-\epsilon}) \end{aligned}$$

as $r - s \searrow 0$, and (v) is proved.

D) Lemmas on approximation.

Lemma 2.7. *Let the operators $A_k(t)$ be defined by (1.14). Under assumptions (1.1)–(1.4), we have for each $\lambda \in S(\vartheta_0)$, $k \in \mathbb{N}^+$, $h \in \mathbb{N}$, $\xi > 0$ and $s \in [0,T]$*

(i) $\|[\lambda - A_k(s)]^{-1} - [\lambda - A(s)]^{-1}\|_{\mathcal{L}(E)} \le c(M,L,\vartheta_0)k^{-\alpha/2}[1+|\lambda|]^{\alpha/2-1}$;

(ii) $\|\frac{d}{ds}[\lambda - A_k(s)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1}\|_{\mathcal{L}(E)} \le c(M,L,\vartheta_0)k^{-\alpha/2}[1+|\lambda|]^{-\alpha/2}$;

(iii) $\|A_k(s)^h e^{\xi A_k(s)}\|_{\mathcal{L}(E)} \le c(M,\vartheta_0)\xi^{-h}$;

(iv) $\|A_k(s)^h e^{\xi A_k(s)} - A(s)^h e^{\xi A(s)}\|_{\mathcal{L}(E)} \le c(M,\alpha,\eta,\vartheta_0)k^{-\alpha/2}\xi^{-h-\alpha/2}$.

Proof. (i) We have

$$[\lambda - A_k(s)]^{-1} - [\lambda - A(s)]^{-1} = \frac{1}{\lambda+k}A(s)\left[\frac{\lambda k}{\lambda+k} - A(s)\right]^{-1}A(s)[\lambda - A(s)]^{-1}, \tag{2.12}$$

which implies, by (1.2),

$$\|[\lambda - A_k(s)]^{-1} - [\lambda - A(s)]^{-1}\|_{\mathcal{L}(E)} \le c(M)|\lambda+k|^{-1};$$

the result follows since

$$|\lambda + k| \ge c(\vartheta_0)\max\{|\lambda|, k\} \quad \forall\lambda \in S(\vartheta_0), \quad ,\forall k \in \mathbb{N}^+. \tag{2.13}$$

(ii) We have

$$\begin{aligned}
&\frac{d}{ds}[\lambda - A_k(s)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1} \\
&= \frac{k^2}{(\lambda+k)^2}\frac{d}{ds}\left[\frac{\lambda k}{\lambda+k} - A(s)\right]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1} \\
&= \frac{k^2}{(\lambda+k)^2}\left[\frac{d}{ds}\left[\frac{\lambda k}{\lambda+k} - A(s)\right]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1}\right] \\
&\quad + \left[\frac{k^2}{(\lambda+k)^2} - 1\right]\frac{d}{ds}[\lambda - A(s)]^{-1} \\
&= \frac{k^2}{(\lambda+k)^2}\frac{\lambda^2}{\lambda+k}\frac{d}{ds}\left(\left[\frac{\lambda k}{\lambda+k} - A(s)\right]^{-1}[\lambda - A(s)]^{-1}\right) \\
&\quad - \frac{\lambda(\lambda+2k)}{(\lambda+k)^2}\frac{d}{ds}[\lambda - A(s)]^{-1},
\end{aligned}$$

and by (2.13), (1.2) and (1.3) the result follows easily.

(iii) is an easy consequence of (1.5) and (1.15).

(iv) is an easy consequence of (1.5) and (i).

Lemma 2.8. *Let the operators $R_n(t,s)$, $R_{n,k}(t,s)$ be defined by (1.9), (1.20)–(1.22). Under assumptions (1.1)–(1.4), we have for each $0 \le s \le r < t \le T$ and $k, n \in \mathbb{N}^+$:*

(i) $\sum_{n=-1}^{\infty}\|R_{n,k}(t,s)\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0,T)(t-s)^{\alpha-1}$;

(ii) $\sum_{n=1}^{\infty}\|R_{n,k}(t,r) - R_{n,k}(t,s)\|_{\mathcal{L}(E)}$
$\le c(M,L,N,\alpha,\eta,\vartheta_0,T)(r-s)^{\delta/2}(t-r)^{-1+\delta/2}$;

(iii) $\sum_{n-1}^{\infty}\|R_{n,k}(t,s) - R_n(t,s)\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\vartheta_0,T)k^{-\alpha/2}(t-s)^{-1+\alpha/2}$;

(iv) $\sum_{n=1}^{\infty}\|R_{n,k}(t,r) - R_{n,k}(t,s) - R_n(t,r) + R_n(t,s)\|_{\mathcal{L}(E)}$
$\le c(M,L,N,\alpha,\eta,\vartheta_0,T)k^{-\delta/4}(r-s)^{\delta/4}(t-r)^{-1+\delta/2}$.

Proof. (i) As in the proof of Lemma 2.6(i), we get, by induction,

$$\|R_{n,k}(t,s)\|_{\mathcal{L}(E)} \le (K')^n\frac{\Gamma(\alpha)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}, \quad \forall (t,s)\in\Delta,\ \forall n,k \in \mathbb{N}^+, \tag{2.14}$$

where $K' := (L'/2\pi)\int_\gamma e^{Re z}|z|^{-\alpha}|dz|$ depends on L,α,ϑ_0; the result follows easily by this estimate.

(ii) Quite similar to the proof of Lemma 2.6 (iii) (using Remark 2.3 instead of Lemma 2.2).

(iii) By Lemma 2.7(ii), we readily get for each $(t,s)\in\Delta$ and $k \in \mathbb{N}^+$

$$\|R_{1,k}(t,s) - R_1(t,s)\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta_0,T)k^{-\alpha/2}(t-s)^{-1+\alpha/2}. \tag{2.15}$$

Next, assume that for each $(t,s)\in\Delta$ and $k \in \mathbb{N}^+$

$$\|R_{n,k}(t,s) - R_n(t,s)\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta_0)\frac{n(K')^{n-1}\Gamma(\alpha)^{n-1}}{k^{\alpha/2}\Gamma((n-1/2)\alpha)}(t-s)^{(n-1/2)\alpha-1}; \tag{2.16}$$

then writing

$$R_{n+1,k}(t,s) - R_{n+1}(t,s) =$$
$$\int_s^t R_{n,k}(t,r)[R_{1,k}(r,s) - R_1(r,s)]dr + \int_s^t [R_{n,k}(t,r) - R_n(t,r)]R_1(r,s)dr,$$

we get by (2.14), (2.15) and (2.16):

$$\begin{aligned}
&\|R_{n+1,k}(t,s) - R_{n+1}(t,s)\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta_0) \\
&\times\left[(K')^n\frac{\Gamma(\alpha)^n\Gamma(\alpha/2)}{\Gamma((n+1/2)\alpha)} + n(K')^{n-1}\frac{k\Gamma(\alpha)^n}{\Gamma((n+1/2)\alpha)}\right]k^{-\alpha/2}(t-s)^{(n+1/2)\alpha-1},
\end{aligned}$$

which implies (2.16) with n replaced by $n+1$. Thus (2.16) holds for each $n \in \mathbf{N}^+$.
(iv) We write

$$\sum_{n=1}^{\infty} \|R_{n,k}(t,r) - R_{n,k}(t,s) - R_n(t,r) + R_n(t,s)\|_{\mathcal{L}(E)}$$
$$\le \sum_{n=1}^{\infty} \Big[\|R_{n,k}(t,r) - R_n(t,r)\|_{\mathcal{L}(E)} + \|R_{n,k}(t,s) - R_n(t,s)\|_{\mathcal{L}(E)}\Big]^{1/2}$$
$$\times \Big[\|R_{n,k}(t,r) - R_{n,k}(t,s)\|_{\mathcal{L}(E)} + \|R_n(t,r) - R_n(t,s)\|_{\mathcal{L}(E)}\Big]^{1/2}.$$

and the result follows easily by (iii) (with α replaced by δ), (ii) and Lemma 2.6(iii).

Lemma 2.9. *Let R_s, $R_{s,k}$ be the operators defined by* (1.7) *and* (1.19). *Under assumptions* (1.1)–(1.4), *we have for each* $(t,s) \in \Delta$ *and* $k \in \mathbf{N}^+$:

(i) $\|R_{s,k}\|_{\mathcal{L}(B_{\alpha/2}(]s,t],E),B_0(]s,t],E))} \le c(L,\alpha,\vartheta_0)(t-s)^{\alpha/2}$;

(ii) $\|R_{s,k} - R_s\|_{\mathcal{L}(B_{\alpha/2}(]s,t],E),B_0(]s,t],E))} \le c(M,L,\alpha,\vartheta_0)k^{-\alpha/2}$;

(iii) $\|[1-R_{s,k}]^{-1}\|_{\mathcal{L}(B_{\alpha/2}(]s,t],E))} \le c(L,\alpha,\vartheta_0,T)$.

Proof. (i) Easy consequence of (1.19) and (2.14) (with $n=1$).
(ii) Easy consequence of Lemma 2.7(ii).
(iii) If $g \in B_{\alpha/2}(]s,t],E)$, by Lemma 2.8(i), we get for each $r \in]s,t]$:

$$(r-s)^{\alpha/2}\|[1-R_{s,k}]^{-1}g(r)\|_E$$
$$\le (r-s)^{\alpha/2}\Big[\|g(r)\|_E + \int_s^r \sum_{n=1}^{\infty} \|R_{n,k}(r,q)\|_{\mathcal{L}(E)}\|g(q)\|_E\, dq\Big]$$
$$\le c(L,\alpha,\vartheta_0)[1+(r-s)^{\alpha}]\|g\|_{B_{\alpha/2}(]s,t],E)}.$$

3. Proof of Theorem 1.6. Our main task in the proof of Theorem 1.6 is the verification of the first part of (i), i.e., of (1.10) and (1.11), since the other properties will then follow easily.

First of all, as a consequence of Lemmas 2.6(i)-(iii) and 2.5(i), we see that formula (1.10) defines $V(t,s)$ as an element of $\mathcal{L}(E)$ for each $(t,s) \in \Delta$; moreover, it is quite easy to check that if $x \in D_{A(s)}$ then $V(t,s)x$ reduces to $-U(t,s)A(s)x$, with $U(t,s)$ given by (1.6). Parts (ii)-(iii) of Theorem 1.6 follow also immediately: thus we just have to verify (1.11).

We will use the Yosida operators $\{A_k(t)\}$ defined by (1.14). It is clear, since $A_k(t)$ is a bounded opertor, that

$$\frac{d}{ds}U_k(t,s) = -U_k(t,s)A_k(s) = V_k(t,s), \quad \forall (t,s) \in \Delta, \forall k \in \mathbf{N}^+, \tag{3.1}$$

where the operators $U_k(t,s)$ and $V_k(t,s)$ were defined in (1.18) and (1.21). Our goal now is to show that

$$\lim_{k\to\infty} \|U_k(t,s) - U(t,s)\|_{\mathcal{L}(E)} = 0, \quad \forall (t,s) \in \Delta, \tag{3.2}$$
$$\lim_{k\to\infty} \|V_k(t,s) - V(t,s)\|_{\mathcal{L}(E)} = 0, \quad \forall (t,s) \in \Delta: \tag{3.3}$$

taking into account (3.1), by (3.2) and (3.3), we obtain (1.11) and the proof of Theorem 1.6 will be complete.

Now, recalling (1.6) and (1.18), we can write for $0 \le s \le t \le T$ and $k \in \mathbf{N}^+$,

$$U_k(t,s) - U(t,s) =$$
$$\Big[(1-R_{s,k})^{-1}[(R_{s,k}-R_s)U(\cdot,s) + (e^{(\cdot-s)A_k(s)} - e^{(\cdot-s)A(s)})]\Big](t);$$

on the other hand, (1.6) implies that $U(\cdot,s)$ is uniformly bounded, so that by Lemmas 2.9(iii)-(ii) and 2.7(iv) we get (3.2).

Concerning (3.3), by (1.10) and (1.21), we can write for each $(t,s) \in \Delta$ and $k \in \mathbf{N}^+$

$$\|V_k(t,s) - V(t,s)\|_{\mathcal{L}(E)} \le \|A_k(s)e^{(t-s)A_k(s)} - A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(E)}$$
$$+ \sum_{n=1}^{\infty} \int_s^t \|R_{n,k}(t,r) - R_{n,k}(t,s) - R_n(t,r) + R_n(t,s)\|_{\mathcal{L}(E)}$$
$$\times \|A_k(s)e^{(t-s)A_k(s)}\|_{\mathcal{L}(E)}dr$$
$$+ \sum_{n=1}^{\infty} \int_s^t \|R_n(t,r) - R_n(t,s)\|_{\mathcal{L}(E)}\|A_k(s)e^{(r-s)A_k(s)} - A(s)e^{(r-s)A(s)}\|_{\mathcal{L}(E)}dr$$
$$+ \sum_{n=1}^{\infty} \|R_{n,k}(t,s) - R_n(t,s)\|_{\mathcal{L}(E)}\|e^{(t-s)A_k(s)} - 1\|_{\mathcal{L}(E)}$$
$$+ \sum_{n=1}^{\infty} \|R_n(t,s)\|_{\mathcal{L}(E)}\|e^{(t-s)A_k(s)} - e^{(t-s)A(s)}\|_{\mathcal{L}(E)};$$

thus using Lemmas 2.7(iv)-(iii), 2.6(iii)-(i) and 2.8(iii), a straightforward calculation yields

$$\|V_k(t,s) - V(t,s)\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,T)k^{-\delta/4}(t-s)^{-1-\delta/4},$$

which proves (3.3). The proof of Theorem 1.6 is complete.

4. Proof of Theorem 1.8. The proof of parts (i)-(ii) is very simple: recalling that (see, e.g., [14, Definition 1.4 and Proposition 1.12])

$$\begin{aligned} x \in D_{A(s)}(\vartheta,\infty) &\iff \sup_{\xi>0} \|\xi^{h-\vartheta}A(s)^h e^{\xi A(s)}x\|_E < \infty, \ \forall h \in \mathbf{N}^+ \\ &\iff \sup_{\xi>0} \xi^{-\vartheta}\|(e^{\xi A(s)}-1)x\|_E < \infty, \end{aligned} \tag{4.1}$$

both results follow easily by (1.10) and Lemma 2.6(i)-(iii).

In order to prove (iii) and (iv), fix $\epsilon \in]0,\delta[$. For $0 \le \tau \le s \le \sigma \le (\tau+t)/2 < t \le T$, according to (1.10) we split $V(t,s) - V(t,\sigma)$ as follows:

$$\begin{aligned} V(t,s) - V(t,\sigma) &= [A(\sigma)e^{(t-\sigma)A(\sigma)} - A(s)e^{(t-s)A(s)}] \\ &+ \sum_{n=1}^{\infty}\Big[\int_s^t [R_n(t,r) - R_n(t,s)]A(s)e^{(r-s)A(s)}dr \\ &- \int_\sigma^t [R_n(t,r) - R_n(t,\sigma)]A(\sigma)e^{(r-\sigma)A(\sigma)}dr\Big] \qquad (4.2) \\ &+ \sum_{n=1}^{\infty}\Big[R_n(t,s)[e^{(t-s)A(s)} - 1] - R_n(t,\sigma)[e^{(t-\sigma)A(\sigma)} - 1]\Big] \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

Next we have

$$A_1 = -\int_{t-\sigma}^{t-s} A(\sigma)^2 e^{\xi A(\sigma)}d\xi + [A(\sigma)e^{(t-s)A(\sigma)} - A(s)e^{(t-s)A(s)}] =: A_{11} + A_{12},$$

$$\begin{aligned} A_2 &= \sum_{n=1}^{\infty}\int_s^\sigma [R_n(t,r) - R_n(t,s)]A(s)e^{(r-s)A(s)}dr \\ &+ \sum_{n=1}^{\infty}\int_\sigma^t [R_n(t,r) - R_n(t,s)][A(s)e^{(r-s)A(s)} - A(\sigma)e^{(r-s)A(\sigma)}]dr \\ &+ \sum_{n=1}^{\infty}[R_n(t,\sigma) - R_n(t,s)][e^{(t-s)A(\sigma)} - e^{(\sigma-s)A(\sigma)}] \\ &+ \sum_{n=1}^{\infty}\int_\sigma^t [R_n(t,r) - R_n(t,\sigma)]\int_{r-\sigma}^{r-s} A(\sigma)^2 e^{\xi A(\sigma)}d\xi\, dr \\ &=: A_{21} + A_{22} + A_{23} + A_{24}, \end{aligned}$$

$$\begin{aligned} A_3 &= \sum_{n=1}^{\infty} R_n(t,s)[e^{(t-s)A(s)} - e^{(t-s)A(\sigma)}] + \sum_{n=1}^{\infty} R_n(t,s)\int_{t-\sigma}^{t-s} A(\sigma)e^{\xi A(\sigma)}d\xi \\ &+ \sum_{n=1}^{\infty}[R_n(t,s) - R_n(t,\sigma)][e^{(t-\sigma)A(\sigma)} - 1] =: A_{31} + A_{32} + A_{33}. \end{aligned}$$

Now, using Lemma 2.5(i)-(ii), we easily get

$$\|A_{11}\|_{\mathcal{L}(E)} \le c(M,\vartheta_0)(\sigma - s)(t-\tau)^{-2},$$
$$\|A_{12}\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0)(\sigma - s)(t-\tau)^{-2+\alpha};$$

next, by Lemmas 2.5(i)-(ii) and 2.6(iii),

$$\begin{aligned} \|A_{21}\|_{\mathcal{L}(E)} &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)\int_s^\sigma (t-s)^{\epsilon-1}(r-s)^{\delta-\epsilon-1}dr \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}(t-\tau)^{\epsilon-1}, \end{aligned}$$

$$\begin{aligned} \|A_{22}\|_{\mathcal{L}(E)} &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)\int_\sigma^t (t-r)^{\epsilon-1}(r-s)^{\delta-\epsilon+\alpha-2}(\sigma-s)dr \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}(t-\tau)^{\epsilon+\delta-1}, \end{aligned}$$

$$\begin{aligned} \|A_{23}\|_{\mathcal{L}(E)} &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}(t-\sigma)^{\epsilon-1} \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}(t-\tau)^{\epsilon-1}, \end{aligned}$$

$$\begin{aligned} \|A_{24}\|_{\mathcal{L}(E)} &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)\int_\sigma^t (t-r)^{\epsilon-1}(r-\sigma)^{\delta-\epsilon}\int_{r-\sigma}^{r-s}\xi^{-2}d\xi\, dr \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-2\epsilon}(t-\tau)^{2\epsilon-1}; \end{aligned}$$

finally, by Lemmas 2.5(ii)-(i) and 2.6(i)-(iii), we similarly obtain

$$\|A_{31}\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0,T)(t-s)^{2\alpha-2}(\sigma-s) \le c(L,\alpha,\vartheta_0,T)(t-\tau)^{2\sigma-2}(\sigma-s),$$

$$\begin{aligned} \|A_{32}\|_{\mathcal{L}(E)} &\le c(M,L,\alpha,\vartheta_0,T)(t-s)^{\alpha-1}\int_{t-\sigma}^{t-s}\xi^{-1}d\xi \\ &\le c(M,L,\alpha,\vartheta_0,T)(\sigma-s)(t-\tau)^{\alpha-2}, \end{aligned}$$

$$\begin{aligned} \|A_{33}\|_{\mathcal{L}(E)} &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}(t-\sigma)^{\epsilon-1} \\ &\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}(t-\tau)^{\epsilon-1}. \end{aligned}$$

Summing up, we get for $0 \le \tau \le s \le \sigma \le (\tau+t)/2 < t \le T$:

$$\|V(t,s) - V(t,\sigma)\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-2\epsilon}(t-\tau)^{2\epsilon-\delta-1};$$

recalling Theorem 1.6(ii) and Definition 1.7, we obtain (iii). As for (iv), we start with the case $\vartheta = 1$; i.e., we evaluate the quantity $\|[V(t,s) - V(t,\sigma)]A(t)^{-1}\|_{\mathcal{L}(E)}$. Using the decomposition (4.2) and Lemmas 2.5(iv)-(iii) and 2.6(iii)-(i), we easily have for all $0 \le \tau \le s \le \sigma \le (\tau+t)/2 < t \le T$

$$\begin{aligned} \|A_{11}A(t)^{-1}\|_{\mathcal{L}(E)} &\le c(M,L,\alpha,\vartheta_0)\int_{t-\sigma}^{t-s}\xi^{-1}[1 + \xi^{-1}(t-s)]d\xi \\ &\le c(M,L,\alpha,\vartheta_0)(\sigma-s)(t-\tau)^{-1}, \end{aligned}$$

$$\|A_{12}A(t)^{-1}\|_{\mathcal{L}(E)} \le c(M,L,\alpha,\vartheta_0)(t-s)^{-1}(\sigma-s)[1+(t-s)^{-1}(t-\sigma)]$$
$$\le c(M,L,\alpha,\vartheta_0)(\sigma-s)(t-\tau)^{-1},$$

$$\|A_{21}A(t)^{-1}\|_{\mathcal{L}(E)}$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)\int_s^\sigma (t-r)^{\epsilon-1}(r-s)^{\delta-\epsilon}[1+(r-s)^{-1}(t-s)]dr$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon},$$

$$\|A_{22}A(t)^{-1}\|_{\mathcal{L}(E)}$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)\int_\sigma^t (t-r)^{\epsilon-1}(r-s)^{\delta-\epsilon-1}(\sigma-s)[1+(r-s)^{-1}(t-\sigma)]dr$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-2\epsilon},$$

$$\|A_{23}A(t)^{-1}\|_{\mathcal{L}(E)}$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(t-\sigma)^{\epsilon-1}(\sigma-s)^{\delta-\epsilon}\int_{\sigma-s}^{t-s}[1+\xi^{-1}(t-\sigma)]d\xi$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-2\epsilon},$$

$$\|A_{24}A(t)^{-1}\|_{\mathcal{L}(E)}$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)\int_\sigma^t (t-r)^{\epsilon-1}(r-\sigma)^{\delta-\epsilon}\int_{r-\sigma}^{r-s}\xi^{-1}[1+\xi^{-1}(t-\sigma)]d\xi\,dr$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-2\epsilon},$$

$$\|A_{31}A(t)^{-1}\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0,T)(t-s)^{\alpha-1}(\sigma-s)[1+(t-s)^{-1}(t-\sigma)]$$
$$\le c(L,\alpha,\vartheta_0,T)(\sigma-s)^{\alpha},$$

$$\|A_{32}A(t)^{-1}\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0,T)(t-s)^{\alpha-1}\int_{t-\sigma}^{t-s}[1+\xi^{-1}(t-\sigma)]d\xi$$
$$\le c(L,\alpha,\vartheta_0,T)(\sigma-s)^{\alpha},$$

$$\|A_{33}A(t)^{-1}\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(t-s)^{\epsilon-1}(\sigma-s)^{\delta-\epsilon}(t-\sigma)$$
$$\le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-\epsilon}.$$

Hence we get for all $0\le\tau\le s\le\sigma\le(\tau+t)/2<t\le T$ and for each $\epsilon\in]0,\delta[$

$$\|V(t,s)-V(t,\sigma)\|_{\mathcal{L}(D_{A(t)},E)} \le c(M,L,N,\alpha,\eta,\epsilon,\vartheta_0,T)(\sigma-s)^{\delta-2\epsilon}(t-\tau)^{2\epsilon-\delta};$$

recalling (i) (with $\vartheta=1$) and Definition 1.7, we get (iv) with $\vartheta=1$. The general case of (iv) follows by a standard interpolation argument between cases (iii) and (iv) (with $\vartheta=1$).

Let us prove (v). Suppose that $x\in D_{A(t)}$ and that both $A(t)x$, $dA(t)^{-1}/dt\cdot A(t)x$ belong to $\overline{D_{A(t)}}$. We shall prove that

$$\lim_{s\nearrow t}\|V(t,s)x+A(t)x\|_E=0; \tag{4.3}$$

by (iii) (with $\vartheta=0$), this will imply that $V(t,\cdot)x\in C([0,t],E)$ and $V(t,t)x=-A(t)x$. Now we have by (1.10)

$$V(t,s)x+A(t)x=[V(t,s)+A(s)e^{(t-s)A(s)}]x$$
$$-[A(s)e^{(t-s)A(s)}-A(t)e^{(t-s)A(t)}]x-[e^{(t-s)A(t)}-1]A(t)x,$$

so that by (ii) (with $\vartheta=1$) and Lemma 2.5(vi), we easily get as $s\nearrow t$.

$$V(t,s)x+A(t)x=$$
$$[1-e^{(t-s)A(t)}]A(t)x-(t-s)A(t)e^{(t-s)A(t)}\frac{d}{dt}A(t)^{-1}\cdot A(t)x+O((t-s)^\delta); \tag{4.4}$$

thus (4.3) follows by recalling that (see [14, Proposition 1.2], [3, Lemma 1.2(v)])

$$y\in\overline{D_{A(t)}}\iff\lim_{\xi\to0}\|[e^{\xi A(t)}-1]y\|_E=0\iff\lim_{\xi\to0}\|\xi A(t)e^{\xi A(t)}y\|_E=0. \tag{4.5}$$

Suppose conversely that $V(t,\cdot)x\in C([0,t[,E)$ and define $y:=V(t,t)x$. We have to prove that $x\in D_{A(t)}$, $A(t)x=-y$, and that both y, $dA(t)^{-1}/dt\cdot y$ belong to $\overline{D_{A(t)}}$. In fact, first the identity

$$V(t,s)x=\frac{d}{ds}U(t,s)x=\frac{d}{ds}[U(t,r)U(r,s)x]$$
$$=U(t,r)\frac{d}{ds}U(r,s)x=U(t,r)V(r,s)x,\quad\forall 0\le s<r<t\le T \tag{4.6}$$

shows that $V(t,s)x\in D_{A(t)}$ for each $(t,s)\in\Delta$, so that

$$y=\lim_{s\nearrow t}V(t,s)x\in\overline{D_{A(t)}}.$$

Next, we need a preparatory result; i.e., we show that $x\in\overline{D_{A(t)}}$. Indeed, we have for each $r\in]s,t[$

$$U(t,r)x-U(t,s)x=\int_s^r V(t,\sigma)x\,d\sigma,$$

and since, by assumption, $\sigma\to V(t,\sigma)x$ is continuous at $\sigma=t$, we get

$$\exists\lim_{r\nearrow t}U(t,r)x=U(t,s)x+\int_s^t V(t,\sigma)x\,d\sigma.$$

which implies $x \in \overline{D_{A(t)}}$ and

$$x = U(t,s)x + \int_s^t V(t,\sigma)x\,d\sigma, \quad \forall s \in [0,t].$$

In particular, this gives

$$\exists \left[\frac{d}{ds}U(t,s)x\right]_{s=t} = -V(t,t)x = y. \tag{4.7}$$

Our next step is to show that $x \in D_{A(t)}$ and $A(t)x = -y$. It is enough to verify that

$$A(t)^{-1}V(t,s)x \to -x \quad \text{as } s \nearrow t, \tag{4.8}$$

since this property, coupled with the fact that $V(t,s)x \to y$ as $s \nearrow t$, yields the desired result. To start with, by (1.10) we have

$$\begin{aligned} A(t)^{-1}V(t,s)x + x &= [1 - A(t)^{-1}A(s)e^{(t-s)A(s)}]x \\ &- \sum_{n=1}^{\infty}\int_s^t A(t)^{-1}[R_n(t,r) - R_n(t,s)]A(s)e^{(r-s)A(s)}x\,dr \\ &- \sum_{n=1}^{\infty} A(t)^{-1}R_n(t,s)[e^{(t-s)A(s)} - 1]x =: I_1 + I_2 + I_3. \end{aligned} \tag{4.9}$$

Now we split further I_1 as

$$I_1 = [1 - e^{(t-s)A(t)}]x + A(t)^{-1}[A(t)e^{(t-s)A(t)} - A(s)e^{(t-s)A(s)}]x;$$

hence, by Lemma 2.5(v) (with $r = t$), we have as $s \nearrow t$

$$I_1 = [1 - e^{(t-s)A(t)}]x + (t-s)\frac{d}{dt}A(t)^{-1} \cdot A(t)e^{(t-s)A(t)}x + O((t-s)^\delta),$$

and since $x \in \overline{D_{A(t)}}$, by (4.5) we conclude that

$$I_1 = o(1) \quad \text{as } s \nearrow t. \tag{4.10}$$

Next, we split I_2 in the following way:

$$\begin{aligned} I_2 = &-\int_s^t \left[\sum_{n=1}^{\infty} A(t)^{-1}[R_n(t,r) - R_n(t,s)] - \frac{d}{dr}A(r)^{-1}[e^{(t-s)A(r)} - e^{(t-r)A(r)}]\right] \\ &\quad \times A(s)e^{(r-s)A(s)}x\,dr \\ &- \int_s^t \frac{d}{dr}A(r)^{-1}[e^{(t-s)A(r)} - e^{(t-r)A(r)}][A(s)e^{(r-s)A(s)} - A(r)e^{(r-s)A(r)}]x\,dr \\ &- \int_s^t \frac{d}{dr}A(r)^{-1}\int_{t-s}^{t+r-2s}[A(r)^2e^{\xi A(s)} - A(t)^2e^{\xi A(t)}]x\,d\xi\,dr \\ &- \int_s^t \left[\frac{d}{dr}A(r)^{-1} - \frac{d}{dt}A(t)^{-1}\right][A(t)e^{(t+r-2s)A(t)} - A(t)e^{(t-s)A(t)}]x\,dr \\ &- \frac{d}{dt}A(t)^{-1}[e^{(t-s)A(t)}[e^{(t-s)A(t)} - 1] + (t-s)A(t)e^{(t-s)A(t)}]x; \end{aligned}$$

then using Lemmas 2.6(v) and 2.5(i)-(ii) as well as (1.4) and (4.5), it is easy to see that

$$I_2 = o(1) \quad \text{as } s \nearrow t. \tag{4.11}$$

Finally, we rewrite I_3 as

$$\begin{aligned} I_3 = &-\sum_{n=1}^{\infty} A(t)^{-1}R_n(t,s)[e^{(t-s)A(s)} - e^{(t-s)A(t)}]x \\ &- \sum_{n=1}^{\infty} A(t)^{-1}R_n(t,s)[e^{(t-s)A(t)} - 1]x, \end{aligned}$$

so that by Lemma 2.6(ii) and 2.5(ii) and by (4.5), we obtain

$$I_3 = o(1) \quad \text{as } s \nearrow t. \tag{4.12}$$

Using (4.10), (4.11) and (4.12), by (4.9) we deduce (4.8), which shows that $x \in D_{A(t)}$ and $A(t)x = -y$.

Now since $y \in \overline{D_{A(t)}}$, (4.5) yields

$$[1 - e^{(t-s)A(t)}]A(t)x = o(1) \quad \text{as } s \nearrow t;$$

hence recalling (4.4) and the fact that, by assumption, $V(t,s)x \to y$ as $s \nearrow t$, we find

$$(t-s)A(t)e^{(t-s)A(t)}\frac{d}{dt}A(t)^{-1} \cdot A(t)x = o(1) \quad \text{as } s \nearrow t,$$

so that, again by (4.5), we get $dA(t)^{-1}/dt \cdot A(t)x \in \overline{D_{A(t)}}$. Taking into account (4.7), part (v) is completely proved.

Finally, let us prove (vi). We start from (4.2): with the same calculations made in the proof of (iv), we easily check that if $x \in D_{A(t)}$, $0 \le s \le \sigma < t \le T$ and $\epsilon \in]0,\delta[$ we have

$$V(t,s)x - V(t,\sigma)x = A_{11}x + A_{12}x + O((t-s)^{\delta-\epsilon}) \quad \text{as } \sigma - s \searrow 0. \tag{4.13}$$

Now by Lemma 2.5(vi), we obtain

$$\begin{aligned} A_{11}x &= -\int_{t-\sigma}^{t-s}\left[[A(\sigma)^2e^{\xi A(\sigma)} - A(t)^2e^{\xi A(t)}]A(t)^{-1}\right]A(t)x\,d\xi \\ &\quad - \int_{t-\sigma}^{t-s}[A(t)e^{\xi A(t)}]A(t)x\,d\xi \\ &= -\int_{t-\sigma}^{t-s}(t-\sigma)A(t)^2e^{\xi A(t)}\frac{d}{dt}A(t)^{-1} \cdot A(t)x\,d\xi \\ &\quad + [e^{(t-\sigma)A(t)} - e^{(t-s)A(t)}]A(t)x + O((\sigma-s)^\delta) \quad \text{as } \sigma - s \searrow 0. \end{aligned}$$

$$\begin{aligned} A_{12}x &= \left[[A(\sigma)e^{(t-s)A(\sigma)} - A(s)e^{(t-s)A(s)}]A(t)^{-1}\right]A(t)x \\ &= -(\sigma-s)A(t)e^{(t-s)A(t)}\frac{d}{dt}A(t)^{-1} \cdot A(t)x + O((\sigma-s)^\delta) \quad \text{as } \sigma - s \searrow 0; \end{aligned}$$

hence, by (4.13) we deduce that

$$V(t,s)x - V(t,\sigma)x$$
$$= \Big[(t-\sigma)A(t)e^{(t-\sigma)A(t)} - (t-s)A(t)e^{(t-s)A(t)}\Big]\frac{d}{dt}A(t)^{-1}\cdot A(t)x \quad (4.14)$$
$$+ [e^{(t-\sigma)A(t)} - e^{(t-s)A(t)}]A(t)x + O((\sigma-s)^\delta).$$

Now suppose that both $A(t)x$, $dA(t)^{-1}/dt\cdot A(t)x$ belong to $D_{A(t)}(\beta,\infty)$; then by (4.14) and (4.5), we readily get

$$V(t,s)x - V(t,\sigma)x = -\int_{t-\sigma}^{t-s}[A(t)e^{\xi A(t)} + \xi A(t)^2e^{\xi A(t)}]\frac{d}{dt}A(t)^{-1}\cdot A(t)x\,d\xi$$
$$+ [e^{(t-\sigma)A(t)} - e^{(t-s)A(t)}]A(t)x + O((\sigma-s)^\delta)$$
$$= O((\sigma-s)^{\delta\wedge\beta}) \quad \text{as } \sigma - s \searrow 0,$$

and (vi) is proved. The proof of Theorem 1.8 is complete.

5. Proof of Theorem 1.9. Part (i) was already proved in Section 4 (see (4.6)). Part (ii) follows by (i) and Proposition 1.3(ii): indeed, for $0\le s<t\le T$ we have

$$\frac{d}{dt}V(t,s) = \frac{d}{dt}U(t,r)V(r,s) = A(t)U(t,r)V(r,s) = A(t)V(t,s).$$

Let us prove (iii). If $0\le s<(s+t)/2\le\sigma<\tau\le t\le T$, by (i), Theorem 1.6(ii) and Proposition 1.3(iii), we can write

$$\|V(\tau,s)-V(\sigma,s)\|_{\mathcal{L}(E)} \le \Big\|\int_\sigma^\tau A(r)U(r,\tfrac{3s+t}{4})\,dr\cdot V(\tfrac{3s+t}{4},s)\Big\|_{\mathcal{L}(E)}$$
$$\le c(M,L,N,\alpha,\eta,\vartheta_0,T)(\tau-\sigma)(t-s)^{-2},$$

and taking into account Theorem 1.6(ii) and Definition 1.7, the result follows.

The proof of (iv) is quite similar, by using Theorem 1.8(i) instead of Theorem 1.6(ii).

Concerning (v), assume first that $V(\cdot,s)x\in C([s,T],E)$. Then in particular by Theorem 1.6(iii), we have as $t\searrow s$,

$$(t-s)A(s)e^{(t-s)A(s)}x = -(t-s)V(t,s)x + O((t-s)^\alpha) = O((t-s)^\alpha),$$

so that, by (4.5), $x\in D_{A(s)}(\alpha,\infty)\subseteq\overline{D_{A(s)}}$. Set now $y:=V(s,s)x$; we shall prove that $x\in D_{A(s)}$ and $A(s)x=-y$. In fact, we have by (1.10)

$$A(s)^{-1}V(t,s)x + x = [1-e^{(t-s)A(s)}]x + [A(s)^{-1}-A(t)^{-1}]V(t,s)x$$
$$- A(t)^{-1}\sum_{n=1}^\infty\int_s^t[R_n(t,r)-R_n(t,s)]A(s)e^{(r-s)A(s)}x\,dr$$
$$- A(t)^{-1}\sum_{n=1}^\infty R_n(t,s)[e^{(t-s)A(s)}-1]x.$$

so that, using (1.3), Theorem 1.6(ii), Lemma 2.6(v)-(iii), (1.4) and Lemma 2.5(iii), we easily get for each $\epsilon\in]0,\delta[$

$$A(s)^{-1}V(t,s)x + x = [1-e^{(t-s)A(s)}]x$$
$$-\int_s^t\frac{d}{dr}A(r)^{-1}[e^{(t-s)A(r)}-e^{(t-r)A(r)}]A(s)e^{(r-s)A(s)}x\,dr$$
$$-\frac{d}{ds}A(s)^{-1}[1-e^{(t-s)A(s)}][e^{(t-s)A(s)}-1]x + O((t-s)^{\delta-\epsilon})$$
$$= [1-e^{(t-s)A(s)}]x + \frac{d}{ds}A(s)^{-1}\cdot(t-s)A(s)e^{(t-s)A(s)}x$$
$$-\frac{d}{ds}A(s)^{-1}[e^{(t-s)A(s)}-1]x + O((t-s)^{\delta-\epsilon}) \quad\text{as } t\searrow s.$$

As $x\in\overline{D_{A(s)}}$, recalling (4.5) we obtain

$$A(s)^{-1}V(t,s)x \to -x \quad\text{as } t\searrow s;$$

thus, since $V(t,s)x\to y$ as $t\searrow s$, we conclude that $x\in D_{A(s)}$ and $A(s)x=-y$. But $y\in\overline{D_{A(s)}}$: this follows by [3, Theorem 2.6], since by (i) $V(\cdot,s)x$ is the classical and strong solution of the problem

$$u'(t) = A(t)u(t),\quad t\in]s,T],\quad u(s)=y.$$

Suppose conversely that $x\in D_{A(s)}$ and $A(s)x\in\overline{D_{A(s)}}$, then by Theorem 1.6(i), we have

$$V(t,s)x = -U(t,s)A(s)x,\quad \forall t\in[s,T], \tag{5.1}$$

and $U(\cdot,s)A(s)x$ is continuous in $[s,T]$: this proves (v).

Finally, let us prove (vi). If $V(\cdot,s)x\in C^\beta([s,T],E)$, with $\beta\in]0,\alpha]$, then, by (v) and Theorem 1.6(i), we deduce that $x\in D_{A(s)}$, $A(s)x\in\overline{D_{A(s)}}$ and (5.1) holds; hence, by (1.6), $V(\cdot,s)x$ satisfies the integral equation

$$V(t,s)x - R_s(V(\cdot,s)x)(t) = -e^{(t-s)A(s)}A(s)x,\quad t\in[s,T]. \tag{5.2}$$

Now it is easy to see that

$$g\in C([s,T],E)\Rightarrow R_sg\in C^\alpha([s,T],E), \tag{5.3}$$

which implies that $e^{(\cdot-s)A(s)}A(s)x\in C^\beta([s,T],E)$; i.e., that $A(s)x\in D_{A(s)}(\beta,\infty)$. Conversely, if $x\in D_{A(s)}$ and $A(s)x\in D_{A(s)}(\beta,\infty)$, then, by (v), $V(\cdot,s)x\in C([s,T],E)$ and (5.1) holds; next, as $e^{(\cdot-s)A(s)}A(s)x\in C^\beta([s,T],E)$, by (5.2) and (5.3) we immediately get $V(\cdot,s)x\in C^\beta([s,T],E)$. This concludes the proof of (vi) and of Theorem 1.9.

6. The variational case. We consider here problem (0.1) under a different set of assumptions on the operators $A(t)$; these assumptions were introduced in [5] and are independent of the hypotheses of Section 1, as shown in [5, §7]. The attribute "variational" is due to the fact that a large class of concrete parabolic systems in variational form, in fact, fulfills such assumptions. We consider a family of operators $A(t)$ satisfying, instead of (1.1)-(1.4), the following conditions

$$\begin{cases} \text{for each } t \in [0,T],\ A(t) : D_{A(t)} \subseteq E \to E \text{ is} \\ \text{a closed linear operator with dense domain;} \end{cases} \tag{6.1}$$

$$\begin{cases} \text{there exist } \vartheta_0 \in]\pi/2, \pi[\text{ and } M > 0 \text{ such that} \\ \rho(A(t)) \supseteq S(\vartheta_0), \text{ and for each } t \in [0,T] \text{ and } \lambda \in S(\vartheta_0) \\ \|[\lambda - A(t)]^{-1}\|_{\mathcal{L}(E)} + \|[\lambda - A(t)^*]^{-1}\|_{\mathcal{L}(E^*)} \le \frac{M}{1+|\lambda|}; \end{cases} \tag{6.2}$$

$$\begin{cases} \text{there exist } N > 0,\ k \in \mathbb{N}^+ \text{ and } \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k \text{ with} \\ 0 \le \beta_i < \alpha_i \le 2 \text{ such that for each } \lambda \in S(\vartheta_0) \text{ and } t, s \in [0,T] \\ \|A(t)[\lambda - A(t)]^{-1}[A(t)^{-1} - A(s)^{-1}]\|_{\mathcal{L}(E)} + \\ \quad \|A(t)^*[\lambda - A(t)^*]^{-1}[[A(t)^*]^{-1} - [A(s)^*]^{-1}]\|_{\mathcal{L}(E^*)} \\ \le N \sum_{i=1}^k (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1}. \end{cases} \tag{6.3}$$

Here E^* denotes the antidual space of E; i.e., the Banach space of bounded antilinear functionals on E. We remark that it is not restrictive to assume in (6.3)

$$\delta := \min\{\alpha_i - \beta_i : i = 1 \dots, k\} \in]0, 1[. \tag{6.4}$$

By [1, Theorem 2.3] we get:

Proposition 6.1. *Under assumptions* (6.1), (6.2) *and* (6.3), *parts* (i), (ii) *and* (iii) *of Proposition* 1.3 *hold true.*

The evolution operator $U(t,s)$ associated to $\{A(t)\}$ can be represented in the following way [1, formula (2.6)]:

$$U(t,s) = e^{(t-s)A(s)} + \int_s^t Z(r,s)\,dr, \quad 0 \le s \le t \le T, \tag{6.5}$$

where

$$\begin{aligned} Z(r,s) := {} & \{(1 - Q_s)^{-1}[A(\cdot)e^{(\cdot-s)A(\cdot)} - A(s)e^{(\cdot-s)A(s)}]\}(r) \\ & + \sum_{n=1}^\infty \int_s^t [Q_n(r,q) - Q_n(r,s)]A(s)e^{(q-s)A(s)}dq \\ & + \sum_{n=1}^\infty Q_n(r,s)[e^{(r-s)A(s)} - 1], \quad 0 \le s < r \le T; \end{aligned} \tag{6.6}$$

here Q_s is the Volterra integral operator

$$[Q_s g](t) := \int_s^t Q(t,r)g(r)dr, \quad t \in [s,T], \ g \in L^1(s,T;E), \tag{6.7}$$

with kernel

$$Q(t,r) := A(t)^2 e^{(t-r)A(t)}[A(t)^{-1} - A(r)^{-1}], \quad 0 \le r < t \le T, \tag{6.8}$$

and $Q_n(t,r)$ is inductively defined by

$$Q_1(t,r) := Q(t,r), \qquad Q_n(t,r) := \int_r^t Q_{n-1}(t,q)Q(q,r)\,dr, \quad \forall n > 1. \tag{6.9}$$

Remark 6.2. (i) In order to get the results of Proposition 6.1, it is sufficient to assume (6.1)-(6.3) just for the family $\{A(t)\}$, and the density of the domains $D_{A(t)}$ is not necessary; but for proving the existence of the derivative $dU(t,s)/ds$, we need that (6.1)-(6.3) are true symmetrically both for $\{A(t)\}$ and $\{A(t)^*\}$ (and, of course, we need density of domains in order to define $A(t)^*$). On the other hand, assumptions (1.1)-(1.4), which also guarantee all results concerning $dU(t,s)/ds$, are in themselves symmetric with respect to $\{A(t)\}$ and $\{A(t)^*\}$.

(ii) In [5, 6, 1] assumption (6.2) for $\{A(t)\}$ was required just for $0 \le s \le t \le T$; thus, in particular, Lemmas 1.9, 1.10 and 1.11 of [5, §1-2] can now be correspondingly improved, with no restrictions on the points r, s, σ, $t \in [0,T]$. However, we do not have significant examples where (6.2) holds for $0 \le s \le t \le T$ and not for all $s, t \in [0,T]$.

Clearly, by (6.1)-(6.3) it follows that we can also construct the evolution operator in E^* relative to $\{A(t)^*\}$. More precisely, for each fixed $t_0 \in]0,T]$ and for each $0 \le s \le \tau \le t_0$ we denote by $W(t_0;\tau,s)$ the evolution operator associated to the family $\{A(t_0 - \tau)^*, \tau \in [0,t_0]\}$. There is a strict connection between $W(t_0;t,s)$ and the adjoint $U(t,s)^*$ of $U(t,s)$: namely, arguing as in [2, Proposition 2.9] we obtain

$$U(t,s)^* = W(t;\tau,0)\big|_{\tau=t-s}, \quad 0 \le s < t \le T. \tag{6.10}$$

Thus $U(t,s)^*$ enjoys all properties deduced in [1] for the evolution operator relative to any family $\{A(t)\}$ fulfilling (6.1), (6.2) and (6.3). In particular, by [1, Theorem 2.3(v)],

$$\exists\, \frac{d}{ds}U(t,s)^* = -A(s)^* U(t,s)^* \quad \text{in } \mathcal{L}(E^*), \ \forall (t,s) \in \Delta \tag{6.11}$$

and

$$\|\frac{d}{ds}U(t,s)^*\|_{\mathcal{L}(E^*)} \le c(M,N,\delta,\vartheta_0,T)(t-s)^{-1}, \quad \forall (t,s) \in \Delta. \tag{6.12}$$

Now if we denote by J the canonical injection $E \to E^{**}$, by (6.11) we easily deduce that $[A(s)^*U(t,s)^*]^*Jx$ is in the range of J for each $x \in E$, and moreover we get

$$\exists\, \frac{d}{ds}U(t,s) = -J^{-1}[A(s)^*U(t,s)^*]^*J \quad \text{in } \mathcal{L}(E), \ \forall (t,s) \in \Delta. \tag{6.13}$$

Remark 6.3. (i) If E is reflexive, then we can identify E and E^{**}, obtaining

$$\frac{d}{ds}U(t,s) = -[A(s)^*U(t,s)^*]^* \quad \text{in } \mathcal{L}(E), \ \forall (t,s) \in \Delta.$$

Of course, this is the case when E is Hilbert space: compare with [2].

Let us extend now the results of the preceding sections to the present situation. We start with the analogue of Theorem 1.6.

Theorem 6.4. *Under assumptions (6.1)-(6.3), the operator*

$$V(t,s) := -J^{-1}[A(s)^*U(t,s)^*]^*J$$

satisfies, for each $(t,s) \in \Delta$, properties (i)-(ii)-(iii) of Theorem 1.6 (with δ defined in (6.4)).

Proof. (i)-(ii) By (6.13), (6.11) and (6.12), we clearly get

$$\|J^{-1}[A(s)^*U(t,s)^*]^*J\|_{\mathcal{L}(E)} = \|\frac{d}{ds}U(t,s)\|_{\mathcal{L}(E)} \le c(M,N,\delta,\vartheta_0,T)(t-s)^{-1}, \quad (6.14)$$

$\forall (t,s) \in \Delta$; in particular, using (6.13), it is easy to obtain

$$\frac{d}{ds}U(t,s)x = -U(t,s)A(s)x, \quad \forall x \in D_{A(s)}, \ \forall t \in]s,T], \quad (6.15)$$

and the results follow.

(iii) Let us write the representation formula (6.5) applied to the evolution operator $W(t;\tau,0)$ (with $0 \le \tau \le t$, $t \in]0,T]$):

$$W(t;\tau,0) = e^{\tau A(t)^*} + \int_0^\tau \hat{Z}(r,0)dr;$$

here $\hat{Z}(r,0)$ is defined as in (6.6), replacing everywhere $A(\cdot)$ by $A(t-\cdot)^*$. Then

$$\frac{d}{d\tau}W(t;\tau,0) = A(t)^*e^{\tau A(t)^*} + \hat{Z}(\tau,0),$$

and hence by (6.10) we derive

$$\frac{d}{ds}U(t,s)^* + A(s)^*e^{(t-s)A(s)^*} = -[A(t)^*e^{(t-s)A(t)^*} - A(s)^*e^{(t-s)A(s)^*}] + \hat{Z}(t-s,0);$$

thus by [5, Lemma 1.10(i)] and [1, Lemma 2.2(i)], we get

$$\begin{aligned}\|\frac{d}{ds}U(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(E)} &= \|\frac{d}{ds}U(t,s)^* + A(s)^*e^{(t-s)A(s)^*}\|_{\mathcal{L}(E^*)} \\ &\le c(M,N,\delta,\vartheta_0,T)(t-s)^{\delta-1},\end{aligned} \quad (6.16)$$

which proves (iii). The proof is complete.

We want now to prove the analogue of Theorem 1.8; i.e.,

Theorem 6.5. *Under assumptions (6.1)-(6.3), the operator $V(t,s) := dU(t,s)/ds$ satisfies properties (i)-(iv) of Theorem 1.8; in addition the analogue of (v), (vi) hold in the following form*

(v) *$s \to V(t,s)x \in C([0,t],E)$ if and only if $x \in D_{A(t)}$, and in this case*

$$\left[\frac{d}{ds}U(t,s)x\right]_{s=t} = -A(t)x;$$

(vi) *$s \to V(t,s)x \in C^\beta([0,t],E)$ ($\beta \in]0,\delta[$) if and only if $x \in D_{A(t)}$ and $A(t)x \in D_{A(t)}(\beta,\infty)$.*

Proof. (i) First, assume $x \in D_{A(s)}(\vartheta,\infty)$, $\vartheta \in]0,1]$; then by (6.15) we have

$$\frac{d}{ds}U(t,s)x = \frac{d}{ds}U(t,s)[1 - e^{(t-s)A(s)}]x - U(t,s)A(s)e^{(t-s)A(s)}x,$$

and hence by (6.14) and (4.1)

$$\|\frac{d}{ds}U(t,s)x\|_E \le c(\vartheta,M,N,\delta,\vartheta_0,T)(t-s)^{\vartheta-1}\|x\|_{D_{A(s)}(\vartheta,\infty)}, \quad (6.17)$$

which proves (i).

(ii) Similarly, from the identity

$$\begin{aligned}\left[\frac{d}{ds}U(t,s) + A(s)e^{(t-s)A(s)}\right]x &= \left[\frac{d}{ds}U(t,s) + A(s)e^{(t-s)A(s)}\right][1 - e^{(t-s)A(s)}]x \\ &\quad + \left[U(t,s) + e^{(t-s)A(s)}\right]A(s)e^{(t-s)A(s)}x,\end{aligned}$$

we derive

$$\|\frac{d}{ds}U(t,s) + A(s)e^{(t-s)A(s)}\|_{\mathcal{L}(E)} \le c(\vartheta,M,N,\delta,\vartheta_0,T)(t-s)^{\vartheta+\delta-1}\|x\|_{D_{A(s)}(\vartheta,\infty)}, \quad (6.18)$$

which proves (ii).

(iii) Fix $0 \le s \le p \le q \le (t+s)/2 < t \le T$ and set $\tau := t-p$, $\sigma := t-q$, so that $0 < (t-s)/2 \le \sigma \le \tau \le (t-s) \le t \le T$. Then we have, by [5, Theorem 6.5] and [1, Theorem 2.3(i)],

$$\begin{aligned}&\|\frac{d}{dq}U(t,q) - \frac{d}{dp}U(t,p)\|_{\mathcal{L}(E)} = \|\frac{d}{dq}U(t,q)^* - \frac{d}{dq}U(t,p)^*\|_{\mathcal{L}(E^*)} \\ &= \|\frac{d}{d\tau}W(t;\tau,0) - \frac{d}{d\sigma}W(t;\sigma,0)\|_{\mathcal{L}(E^*)} \\ &\le \|\frac{d}{d\tau}W(t;\tau,\frac{t-s}{4}) - \frac{d}{d\sigma}W(t;\sigma,\frac{t-s}{4})\|_{\mathcal{L}(E^*)}\|W(t;\frac{t-s}{4},0)\|_{\mathcal{L}(E^*)} \\ &\le c(M,N,\delta,\vartheta_0,T)(\tau-\sigma)^\delta(t-s)^{-1-\delta} \\ &= c(M,N,\delta,\vartheta_0,T)(q-p)^\delta(t-s)^{-1-\delta},\end{aligned} \quad (6.19)$$

and (iii) follows.

(iv) For $0 \le s \le p \le q \le (t+s)/2 < t \le T$, we write

$$\begin{aligned}&[V(t,q) - V(t,p)]x = \\ &[V(t,q) - V(t,p)][1 - e^{(t-s)A(p)}]x + V(t,q)[e^{(t-s)A(p)} - e^{(t-s)A(q)}]x \\ &+ [V(t,q)e^{(t-s)A(q)} - V(t,p)e^{(t-s)A(p)}]x =: \sum_{i=1}^3 I_i;\end{aligned}$$

now, using (iii), (4.1) and [5, Lemma 1.10(i)] (with Remark 6.2(ii)), we get

$$\|I_1\|_E \le c(M,N,\delta,\vartheta_0,T)(q-p)^\delta(t-s)^{\vartheta-\delta-1};$$

whereas by (ii), (4.1) and [5, Lemma 1.10(i)]

$$\|I_2\|_E \le c(M,N,\delta,\vartheta_0,T)(q-p)^\delta (t-s)^{\vartheta-1};$$

finally using (6.13) [1, Theorem 2.3(iv)], [5, Lemma 1.10(ii)-(i)] and Proposition 6.1(iii), we deduce

$$\begin{aligned}\|I_3\|_E &\le \sup_{\|\phi\|_{E^*}=1} \left|\langle \phi, [U(t,q)A(q)e^{(t-s)A(q)} - U(t,p)A(p)e^{(t-s)A(p)}]x\rangle_{E^*,E}\right| \\ &\le \|U(t,q)-U(t,p)\|_{\mathcal{L}(E)}\|A(q)e^{(t-s)A(q)}x\|_E \\ &\quad + \|U(t,p)\|_{\mathcal{L}(E)}\|[A(q)e^{(t-s)A(q)} - A(p)e^{(t-s)A(p)}]x\|_E \\ &\le C(M,N,\delta,\vartheta_0,T)(q-p)^\delta (t-s)^{\vartheta-\delta-1}.\end{aligned}$$

Summing up, recalling (6.17) we obtain that $s \to V(t,s)x$ belongs to the space $Z_{1-\vartheta,\delta}([0,t[,\mathcal{L}(D_{A(t)}(\vartheta,\infty),E)$, and (iv) is proved.

(v) Fix $x \in D_{A(t)}$; then by (6.15)

$$\begin{aligned}V(t,s)x + A(t)x &= V(t,s)A(t)^{-1}[1-e^{(t-s)A(t)}]A(t)x \\ &+ V(t,s)[e^{(t-s)A(t)} - e^{(t-s)A(s)}]x - U(t,s)[A(s)e^{(t-s)A(s)} - A(t)e^{(t-s)A(t)}]x \\ &- U(t,s)[e^{(t-s)A(t)}-1]A(t)x - [U(t,s)-1]A(t)x = \sum_{i=1}^5 I_i;\end{aligned}$$

on the other hand, by (iv), (i), [5, Lemma 1.10(i)], Proposition 6.1(iii), we easily obtain

$$\begin{aligned}\|I_1\|_E + \|I_4\|_E &\le c(M,N,\delta,\vartheta_0,T)\|[1-e^{(t-s)A(t)}]A(t)x\|_E, \\ \|I_2\|_E + \|I_3\|_E &\le c(M,N,\delta,\vartheta_0,T)(t-s)^\delta, \\ \|I_5\|_E &= \|[U(t,s)-1]A(t)x\|_E,\end{aligned}$$

so that by (4.5) and [1, Theorem 2.3(ii)], we find (since $\overline{D_{A(t)}} = E$)

$$V(t,s)x + A(t)x = o(1) \quad \text{as } s \nearrow t. \tag{6.20}$$

As, by (iv), $s \to V(t,s)x \in C([0,t[,E)$, we get the first part of (v).

To prove the second one, let $x \in E$ be such that $s \to V(t,s)x$ is a continuous function, and set $y := V(t,t)x$. We will show that $A(t)^{-1}V(t,s)x \to -x$ as $s \nearrow t$; since $A(t)$ is closed, this will imply that $x \in D_{A(t)}$ and that $A(t)x = -y$. Indeed, by (6.13) we have, setting $\tau := t-s$,

$$\begin{aligned}\|A(t)^{-1}V(t,s)x + x\|_E &= \sup_{\|\phi\|_{E^*}=1}\left|\langle -A(s)^*U(t,s)^*[A(t)^*]^{-1}\phi + \phi, x\rangle_{E^*,E}\right| \\ &\le \sup_{\|\phi\|_{E^*}=1}\left|\langle (1-A(t-\tau)^*W(t;\tau,0)[A(0)^*]^{-1})\phi, x\rangle_{E^*,E}\right| \\ &\le \sup_{\|\phi\|_{E^*}=1}\left|\langle [1-W(t;\tau,0)]\phi, x\rangle_{E^*,E}\right| \\ &+ \sup_{\|\phi\|_{E^*}=1}\left|\langle (W(t;\tau,0) - A(t-\tau)^*W(t;\tau,0)[A(0)^*]^{-1})\phi, x\rangle_{E^*,E}\right| \\ &\le \|[1-U(t,s)]x\|_E + c(M,N,\delta,\vartheta_0,T)(t-s)^\delta,\end{aligned}$$

so that by [1, Theorem 2.3(viii)-(ii)] (since $\overline{D_{A(t)}} = E$)

$$\|A(t)^{-1}V(t,s)x + x\|_E = o(1) \quad \text{as } s \nearrow t,$$

which proves the second part of (v). Note that, in particular, the above calculation shows that

$$\|A(t)^{-1}V(t,s) + U(t,s)\|_{\mathcal{L}(E)} \le c(M,N,\delta,\vartheta_0,T)(t-s)^\delta. \tag{6.21}$$

(vi) The first part is easy: if $x \in D_{A(t)}$ and $A(t)x \in D_{A(t)}(\beta,\infty)$ (with $\beta \in]0,\delta]$), then as in the proof of (6.20), we get by (4.1),

$$\begin{aligned}\|V(t,s)x + A(t)x\|_E &\le c(M,N,\delta,\vartheta_0,T)(t-s)^\delta + \|[e^{(t-s)A(t)}-1]A(t)x\|_E \\ &\le c(M,N,\delta,\vartheta_0,T)(t-s)^\beta.\end{aligned}$$

Conversely let $x \in E$ be such that $V(t,\cdot)x \in C^\beta([0,t],E)$; then we know by (v) that $x \in D_{A(t)}$ and $A(t)x = -V(t,t)x$. We have to show that $A(t)x \in D_{A(t)}(\beta,\infty)$. Now we can write by Theorem 6.4(iii), [5, Lemma 1.10(i)], (6.5) and [1, Lemma 2.2(i)],

$$\begin{aligned}&O((t-s)^\beta) = V(t,s)x + A(t)x = \\ &[V(t,s) + A(s)e^{(t-s)A(s)}][[1-e^{(t-s)A(t)}]x + [e^{(t-s)A(t)} - e^{(t-s)A(s)}]x] \\ &- [U(t,s) - e^{(t-s)A(s)}][A(s)e^{(t-s)A(s)} - A(t)e^{(t-s)A(t)}]x \\ &- [U(t,s) - e^{(t-s)A(s)}]e^{(t-s)A(t)}A(t)x - [A(s)e^{(t-s)A(s)} - A(t)e^{(t-s)A(t)}]x \\ &- [e^{(t-s)A(t)}-1]A(t)x = O((t-s)^\delta) - [e^{(t-s)A(t)}-1]A(t)x \quad \text{as } s \nearrow t;\end{aligned}$$

i.e.,

$$[e^{(t-s)A(t)}-1]A(t)x = O((t-s)^\beta) \quad \text{as } s \nearrow t,$$

which by (4.1) yields the second part of (vi). The proof of Theorem 6.5 is complete.

Finally, we prove the analogue of Theorem 1.9.

Theorem 6.6. *Under assumptions* (6.1)-(6.3), *the operator* $V(t,s) := dU(t,s)/ds$ *satisfies properties* (i)-(vi) *of Theorem* 1.9 (*with* $\beta \in]0,\delta]$ *in* (vi)).

Proof. (i)-(ii)-(iii)-(iv) They follow as in the proof of Theorem 1.9 (see Section 5), using Theorems 6.4, 6.5 instead of Theorems 1.6, 1.8.

(v) The "only if" part is just like that of Theorem 1.9(v). (Recall that $\overline{D_{A(s)}} = E$ now.) Conversely, assume $V(\cdot,s)x \in C([s,T],E)$; then by (6.3) and (6.21)

$$\begin{aligned}\|A(s)^{-1}V(t,s)x + x\|_E &\le \|[A(s)^{-1} - A(t)^{-1}]V(t,s)x\|_E + \|A(t)^{-1}V(t,s)x + x\|_E \\ &\le c(M,N)(t-s)^\delta\|V(\cdot,s)x\|_{C([s,T],E)} + \|[1-U(t,s)]x\|_E = o(1) \quad \text{as } t \searrow s,\end{aligned}$$

so that by the closedness of $A(s)$ we deduce that $x \in D_{A(s)}$ and $A(s)x = -V(s,s)x$. This proves (v).

(vi) If $x \in D_{A(s)}$ and $A(s)x \in D_{A(s)}(\beta,\infty)$ ($\beta \in]0,\delta]$), then by Theorem 6.4(i) and [1, Theorem 4.1(iii)]

$$V(\cdot,s)x = -U(\cdot,s)A(s)x \in C^\beta([s,T],E);$$

on the other hand, if $V(\cdot,s)x \in C^\beta([s,T],E)$ then by (v) $x \in D_{A(s)}$, so that $V(\cdot,s)x = -U(\cdot,s)A(s)x$ and hence, by [1, Theorem 4.1(iii)], we get $A(s)x \in D_{A(s)}(\beta,\infty)$. This proves (vi) and concludes the proof of Theorem 6.6.

Appendix. We prove here the following result, stated (without proof) as Proposition 3.1(i)-(iii) in [7]. We remark that part of that statement was proved in Section 1 (see (1.12), (1.13)).

Propostion A.1. *Under assumptions (1.1)-(1.4) let $U(t,s)$, $V(t,s)$ be defined by (1.6), (1.10), respectively. Then we have for each $(t,s) \in \Delta$ and $\vartheta,\beta \in [0,1]$,*

(i) $\|[-A(t)]^\vartheta U(t,s)[-A(s)]^{-\beta}\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,\beta,\vartheta)[1+|t-s|^{\beta-\vartheta}]$;

(ii) *the unbounded operator $[-A(t)]^{-\beta}U(t,s)[-A(s)]^\vartheta$ has an extension to $\mathcal{L}(E)$ which is bounded by*

$$c(M,L,N,\alpha,\eta,\vartheta_0,\beta,\vartheta)[1+|t-s|^{\beta-\vartheta}],$$

provided we assume in addition that the domains $D_{A(t)}$ are dense in E for each $t \in [0,T]$.

Proof. (i) We need several steps. First, we remark that $U(t,s)$ has another representation formula introduced in [3, (0.3)], namely

$$U(t,s) = e^{(t-s)A(t)} - \int_s^t e^{(t-r)A(r)}[(1+P_s)^{-1}P(\cdot,s)](r)dr, \quad 0 \le s \le t \le T, \tag{A.1}$$

where

$$P(t,s) := \int_\gamma e^{(t-s)\lambda}\frac{d}{dt}[\lambda - A(t)]^{-1}d\lambda, \quad 0 \le s < t \le T, \tag{A.2}$$

$$(P_s g)(t) := \int_s^t P(t,\sigma)g(\sigma)d\sigma, \quad g \in L^1(s,T;E), \quad 0 \le s \le t \le T. \tag{A.3}$$

Second, it is proved in [3, (1.3) and Lemma 3.2] that if $0 \le s < r \le t \le T$,

$$\|P(t,s)\|_{\mathcal{L}(E)} \le c(L,\alpha,\vartheta_0)(t-s)^{\alpha-1} \le c(L,\alpha,\vartheta_0)(t-s)^{\delta-1},$$

$$\|P(t,s) - P(r,s)\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,\epsilon)(t-r)^{\delta-\epsilon}(r-s)^{\epsilon-1}, \quad \forall \epsilon \in]0,\delta[;$$

i.e., according to Definition 1.7,

$$P(\cdot,s) \in Z_{1-\delta,\delta-\epsilon}(]s,T],\mathcal{L}(E)), \quad \forall \epsilon \in]0,\delta[; \tag{A.4}$$

thus, in particular,

$$\|P(\cdot,s)\|_{B_{1-\delta}(]s,T],\mathcal{L}(E))} \le c(L,\alpha,\vartheta_0). \tag{A.5}$$

On the other hand, we will show in a moment that for each $0 \le s < r \le t \le T$,

$$\|P(t,s)A(s)^{-1}\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0), \tag{A.6}$$

$$\|[P(t,s) - P(r,s)]A(s)^{-1}\|_{\mathcal{L}(E)} \le c(M,L,N,\alpha,\eta,\vartheta_0,\epsilon)(t-r)^{\delta-\epsilon}(r-s)^{\epsilon-\delta}, \tag{A.7}$$

$\forall \epsilon \in]0,\delta[$; i.e.,

$$P(\cdot,s)A(s)^{-1} \in Z_{0,\delta-\epsilon}(]s,T],\mathcal{L}(E)), \quad \forall \epsilon \in]0,\delta[; \tag{A.8}$$

thus, in particular,

$$\|P(\cdot,s)A(s)^{-1}\|_{B_0(]s,T],\mathcal{L}(E))} \le c(M,L,N,\alpha,\eta,\vartheta_0). \tag{A.9}$$

Indeed, we can write, recalling (1.8),

$$P(t,s)A(s)^{-1} = \int_\gamma e^{(t-s)\gamma}\left[\frac{d}{dt}[\lambda - A(t)]^{-1} - \frac{d}{ds}[\lambda - A(s)]^{-1}\right]A(s)^{-1}d\lambda + R(t,s)A(s)^{-1},$$

and (A.6) follows by Lemma 2.2(iii), (1.8) and (2.3); concerning (A.7), we write

$$[P(t,s) - P(r,s)]A(s)^{-1} = \int_\gamma e^{(t-s)\lambda}\left[\frac{d}{dt}[\lambda - A(t)]^{-1} - \frac{d}{dr}[\lambda - A(r)]^{-1}\right]A(s)^{-1}d\lambda$$
$$+ \int_{r-s}^{t-s}\int_\gamma \lambda e^{\lambda\xi}\frac{d}{dr}[\lambda - A(r)]^{-1}A(s)^{-1}d\lambda\, d\xi.$$

and by Lemma 2.2(iii), (2.2) and Lemma 2.1(iii), we get (A.7).

Now interpolating between (A.4) and (A.8) and between (A.5) and (A.9), we easily find

$$P(\cdot,s)[-A(s)]^{-\beta} \in Z_{(1-\delta)(1-\beta),\delta-\epsilon}(]s,T],\mathcal{L}(E)), \quad \forall \epsilon \in]0,\delta[, \tag{A.10}$$

$$\|P(\cdot,s)[-A(s)]^{-\beta}\|_{B_{(1-\delta)(1-\beta)}(]s,T],\mathcal{L}(E))} \le c(M,L,N,\alpha,\eta,\vartheta_0,\beta). \tag{A.11}$$

Next, consider the operator-valued function

$$g(t) := (1+P_s)^{-1}(P(\cdot,s)[-A(s)]^{-\beta})(t), \quad t \in [s,T], \tag{A.12}$$

which, by definition, solves the integral equation

$$g + P_s g = P(\cdot,s)[-A(s)]^{-\beta} \quad \text{in }]s,T];$$

now, due to the mollifying properties of the integral operator P_s, it is a straightforward task to verify that g has the same regularity as $P(\cdot,s)[-A(s)]^{-\beta}$: thus

$$g \in Z_{(1-\delta)(1-\beta),\delta-\epsilon}(]s,T],\mathcal{L}(E)), \quad \forall \epsilon \in]0,\delta[, \tag{A.13}$$

$$\|g\|_{B_{(1-\delta)(1-\beta)}(]s,T],\mathcal{L}(E))} \le c(M,L,N,\alpha,\eta,\vartheta_0,\beta). \tag{A.14}$$

Finally, using (A.1) and (A.12), we have

$$[-A(t)]^{\vartheta}U(t,s)[-A(s)]^{-\beta} =$$

$$[-A(t)]^{\vartheta}e^{(t-s)A(t)}[-A(s)]^{-\beta} - \int_s^t [-A(t)]^{\vartheta}e^{(t-r)A(t)}g(r)dr;$$

but it is easily seen (arguing as in [2, Lemma 2.7(iii)-(ii)]) that

$$\left\|[-A(t)]^{\vartheta}e^{(t-s)A(t)}[-A(s)]^{-\beta}\right\|_{\mathcal{L}(E)} \leq$$

$$\left\|\left([-A(t)]^{\vartheta}e^{(t-s)A(t)} - [-A(s)]^{\vartheta}e^{(t-s)A(s)}\right)[-A(s)]^{-\beta}\right\|_{\mathcal{L}(E)}$$

$$+\left\|[-A(s)]^{\vartheta-\beta}e^{(t-s)A(s)}\right\|_{\mathcal{L}(E)}$$

$$\leq c(M,L,\vartheta_0,\beta,\vartheta)[(t-s)^{\beta-\vartheta}+1],$$

$$\left\|\int_s^t [-A(t)]^{\vartheta}e^{(t-r)A(t)}g(r)\,dr\right\|_{\mathcal{L}(E)} \leq$$

$$c(M,L,N,\alpha,\eta,\vartheta_0)\int_s^t (t-r)^{-\vartheta}(r-s)^{-(1-\delta)(1-\beta)}dr$$

$$\leq c(M,L,N,\alpha,\eta,\vartheta_0,\beta,\vartheta)(t-s)^{\beta-\vartheta+\delta(1-\beta)},$$

and (i) follows.

(ii) First of all we remark that since the domains $D_{A(t)}$ are all dense in E, hypotheses (1.1)-(1.4) hold in E^* for the operators $A(t)^*$ as well; hence, by [3] and the results of §§1-5, we see that for fixed $t \in]0,T]$ there exists the evolution operator $W(t;\tau,s)$ relative to the family $\{B(\tau)\}$, with $B(\tau) := A(t-\tau)^*$, $\tau \in [0,t]$, in the Banach space E^*: in particular

$$\frac{d}{d\tau}W(t;\tau,s) = B(\tau)W(t;\tau,s), \quad \forall \tau \in]s,t], \ \forall s \in [0,t[. \qquad (A.15)$$

Now, as in (6.10), we have

$$U(t,s)^* = W(t;t-s,0), \quad \forall (t,s) \in \Delta; \qquad (A.16)$$

indeed, by (A.15) and Proposition 1.3(ii), if $r \in]s,t[$, we have, for each $x \in E$ and $\phi \in E^*$:

$$\frac{d}{dr}\langle W(t;t-r,0)\phi, U(r,s)x\rangle_{E^*,E} =$$

$$-\langle A(r)^*W(t;t-r,0)\phi, U(r,s)x\rangle_{E^*,E} + \langle W(t;t-r,0)\phi, A(r)U(r,s)x\rangle_{E^*,E} = 0,$$

which implies, as $r \searrow s$ and $r \nearrow t$,

$$\langle W(t;t-s,0)\phi, x\rangle_{E^*,E} = \langle \phi, U(t,s)x\rangle_{E^*,E}, \quad \forall 0 \leq s < t \leq T,$$

and (A.16) is proved.

Let us now just apply (i) to the evolution operator $W(t;\tau,0)$, with $\tau = t-s$; the conclusion is that

$$\left\|[-A(s)^*]^{\vartheta}U(t,s)^*[-A(t)^*]^{-\beta}\right\|_{\mathcal{L}(E^*)} \leq c(M,L,N,\alpha,\eta,\vartheta_0,\beta,\vartheta)[(t-s)^{\beta-\vartheta}+1],$$

and (ii) follows easily. This concludes the proof of Propostion 6.1.

REFERENCES

[1] P. Acquistapace, *Evolution operators and strong solutions of abstract linear parabolic equations*, Diff. Int. Eq., 1 (1988), 433–457.

[2] P. Acquistapace, F. Flandoli and B. Terreni, *Initial-boundary value problems and optimal control for non-autonomous parabolic systems*, SIAM J. Control Optimiz., 29 (1991), 89–118.

[3] P. Acquistapace and B. Terreni, *Some existence and regularity results for abstract non autonomous parabolic equations*, J. Math. Anal. Appl., 99 (1984), 9–64.

[4] P. Acquistapace and B. Terreni, *An approach to Ito linear equations in Hilbert spaces by approximation of white noise by coloured noise*, Stoch. Anal. Appl., 2 (1984), 131–186.

[5] P. Acquistapace and B. Terreni, *A unified approach to abstract linear non-autonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova, 78 (1987), 47–107.

[6] P. Acquistapace and B. Terreni, *On fundamental solutions for abstract parabolic equations* in "Differential Equations in Banach Spaces," Proceedings, Bologna 1985, A. Favini and E Obrecht, editors, Lecture Notesin Math., No. 1223, Springer, Berlin, 1986, 1–11.

[7] P. Acquistapace and B. Terreni, *Boundary control problems for non-autonomous parabolic systems*, in "Stability of Flexible Structures," Proceedings, Montpellier, 1989, J.P. Zolésio editor, Lect. Notes in Control and Information Sciences, n^0 147, Springer, Berlin 1990, 156–166.

[8] M. Fuhrman, *Bounded solutions for time-periodic abstract parabolic equations with non-constant domains*, Differential Integral Equations, 4 (1991), 493–518.

[9] T. Kato, *Abstract evolution equations of parabolic type in Banach and Hilbert spaces*, Nagoya Math. J., 19 (1961), 93–125.

[10] T. Kato and H. Tanabe, *On the abstract evolution equations*, Osaka Math. J., 14 (1962), 107–133.

[11] A. Lunardi, *On the local dynamical system associated to a fully nonlinear abstract parabolic equation*, in "Nonlinear Analysis and Applications," Proceedings, Arlington, 1986, V.A. Lakshmikantham, editor, Lecture Notes in P.A.M., No. 109, M. Dekker, New York, 1987, 319–326.

[12] A. Lunardi, *On the evolution operator for abstract parabolic equations*, Israel J. Math., 60 (1987), 281–314.

[13] A. Lunardi, *Differentiability with respect to (t, s) of the parabolic evolution operator*, Israel J. Math., 68 (1989), 161–184.

[14] E. Sinestrari, *On the abstract Cauchy problem of parabolic type in spaces of continuous functions*, J. Math. Anal. Appl., 107 (1985), 16–66.

[15] P.E. Sobolevskii, *Coerciveness inequalities for abstract parabolic equations*, Soviet Math. Dokl., 20 II (1964), 894–897.

[16] P.E. Sobolevskii, *Equations of parabolic type in a Banach space*, Amer. Math. Soc. Transl., 49 (1966), 1–62.

[17] H. Tanabe, "Equations of Evolution," Pitman, London, 1979.