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# INITIAL BOUNDARY VALUE PROBLEMS AND OPTIMAL CONTROL FOR NONAUTONOMOUS PARABOLIC SYSTEMS*

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**Abstract.** A large class of linear nonautonomous parabolic systems in bounded domains is considered, with control acting on the boundary through Dirichlet or Neumann conditions, from the point of view of semigroup theory. The results from [*Rend. Sem. Mat. Univ. Padova*, 78 (1987), pp. 47–107], [*On fundamental solutions for abstract parabolic equations*, Lecture Notes in Math., Vol. 1223, Springer-Verlag, Berlin, Heidelberg, 1986, pp. 1–11] on abstract homogeneous parabolic Cauchy problems allow operators with varying domains and Hölder continuous coefficients to be handled. A representation formula for solutions corresponding to square integrable control functions is derived and used to solve a linear-quadratic regulator problem over finite time horizon, by a direct study of the associated integral Riccati equation.

**Key words.** optimal control, parabolic systems, boundary control, Riccati equation

**AMS(MOS) subject classifications.** 49A22, 49B22, 34H05, 35K50

**1. Introduction.** During the last two decades relevant progress has been made in the theory of boundary control of partial differential equations. In the case of equations of parabolic type, both variational and semigroup methods have been successfully applied (see, for instance, [L2], [DS], [S1], [S2], [Fa], [B1], [La], [LT1], [LT2], [F1], [F2]). Most of these (and other) papers deal with autonomous parabolic equations. Only [L2] and [DS] present results on the boundary control in the nonautonomous case, by variational techniques.

This paper concerns nonautonomous systems of parabolic type, from the point of view of semigroup theory. Our first purpose is to develop a suitable approach to nonhomogeneous initial boundary value problems based on the theory of evolution operators, in view of its application to boundary control problems.

Section 2 is devoted to this basic question. As in the autonomous case we are able to deal with control functions which are only square integrable in time and space. In particular our main goal is to derive a representation formula for solutions, similar to the classical one (see [B2], [La], [LT1]), which will prove to be very useful in the treatment of control problems.

Section 2 is organized as follows. Section 2.1 contains a detailed analysis of two concrete systems of equations of parabolic type with nonhomogeneous Dirichlet or Neumann boundary conditions, which motivate the abstract model to be introduced afterwards. In §§ 2.2 and 2.3 we study an abstract homogeneous nonautonomous parabolic Cauchy problem by the methods of [AT1], [AT2], which allow us to handle operators with variable domains and whose coefficients are just Hölder continuous in time. In § 2.4, by using the properties of the Dirichlet and Neumann maps, we obtain an abstract formulation of the concrete nonhomogeneous problems analyzed in § 2.1. Finally, in § 2.5 we derive the representation formula for solutions of the abstract version of nonhomogeneous initial boundary value problems; this formula is meaningful for nonregular boundary data and will be considered as the state equation for the control problems of § 3.

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The second part of the paper, namely § 3, deals with the linear-quadratic regulator (L-Q-R) problem, over finite time horizon, for an abstract evolution equation which includes the concrete models discussed in § 2. Here we follow the approach of [F2], based on a direct solution of the Riccati equation arising in the L-Q-R problem. However our assumptions on the final state cost operator $P_T$ (see (3.13) below) are weaker than those imposed in [F2], and suggested by the more general results of [DI1]. We are able to solve directly the basic integral Riccati equation under general assumptions, much weaker and more natural for the applications than those imposed in [DS], where the Riccati equation was deduced from the optimality system (see Remark 2.3(iii) below). It turns out that our approach in the more general setting of nonautonomous problems still allows us to employ standard techniques of control theory. This fact lets us hope that many other results on boundary control problems, such as, e.g., infinite horizon optimal control [F2], [LT3], [DI2], [D], and the control of stochastic systems [F2], [I], can also be extended to the nonautonomous framework.

We conclude this section by listing some notation.

If $X$ is a Banach space and $a<b$ we set:

$L^p(a, b; X) :=$ space of strongly measurable functions $f: ]a, b[ \to X$ such that $\int_a^b \|f(t)\|_X^p\, dt < \infty$ $(1 \leqq p < \infty$; obvious modifications for $p = \infty)$; $C^k([a, b], X) :=$ space of functions $f: [a, b] \to X$ which are $k$ times continuously differentiable $(k \in \mathbb{N})$;

$C^{k+\vartheta}([a, b], X) :=$ space of functions $f \in C^k([a, b], X)$ such that $f^{(k)}$ is $\vartheta$-Hölder continuous $(k \in \mathbb{N},\ \theta \in ]0, 1[)$.

If $X$, $Y$ are Banach spaces, we set:

$\mathscr{L}(X, Y) :=$ space of bounded linear operators $T: X \to Y$;

$\mathscr{L}(X) := \mathscr{L}(X, X)$;

$C_s([a, b], \mathscr{L}(X, Y)) :=$ space of operator-valued functions $T(\cdot): [a, b] \to \mathscr{L}(X, Y)$ which are strongly continuous, i.e., $T(\cdot)x \in C^0([a, b], Y)$ for each $x \in X$.

If $H$ is a Hilbert space, we set:

$\Sigma(H) :=$ space of self-adjoint operators $T \in \mathscr{L}(H)$;

$\Sigma^+(H) :=$ space of self-adjoint operators $T \in \mathscr{L}(H)$ which are positive, i.e., $(Tx \mid x)_H \geqq 0$ for each $x \in H$.

If $H$ is a Hilbert space and $T$ is a linear operator in $H$, we set:

$D_T :=$ domain of $T$;

$\sigma(T) :=$ spectrum of $T$;

$\rho(T) :=$ resolvent set of $T$;

$T^* :=$ adjoint operator of $T$ (whenever it exists).

Finally, if $m \in \mathbb{N}^*$ and $\Omega$ is a bounded open set of $\mathbb{R}^n$, we shall use the following spaces of $\mathbb{C}^m$-valued functions:

$$[C^k(\bar{\Omega})]^m, [C^{k+\vartheta}(\bar{\Omega})]^m, [L^p(\Omega)]^m (k \in \mathbb{N},\ \vartheta \in ]0, 1[,\ p \in [1, \infty]),$$

whose definitions are clear, and the usual Sobolev spaces

$[W^{\vartheta,p}(\Omega)]^m$, $[W^{\vartheta,p}(\partial\Omega)]^m (p \in [1, \infty[,\ \vartheta \in \mathbb{R})$,

$[W_0^{\vartheta,p}(\Omega)]^m (p \in [1, \infty[,\ \vartheta \in ]1/p, \infty[)$.

## 2. Nonautonomous parabolic systems.

**2.1. Two classical examples.** We consider in this section two particular types of parabolic initial boundary value problems, namely, two parabolic systems with Dirichlet and Neumann conditions, respectively. We think of them as prototypes of the class of problems which are covered by the general theory of this section.

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, with boundary $\partial\Omega$ of class $C^2$. Fix $T>0$ and let $\{A_{sj}(t, x)\}_{s,j=1,\cdots,n}$ a set of $N \times N$ complex-valued matrices defined in $[0, T] \times \bar{\Omega}$, fulfilling the following hypotheses:

(2.1) (regularity)

$$A_{sj}\in C^{\alpha+1/2}([0,T],[C^0(\bar\Omega)]^{N^2})\cap C^{\alpha}([0,T],[C^1(\bar\Omega)]^{N^2}),$$

(2.2) (strong ellipticity)

$$\operatorname{Re}\sum_{sj=1}^{n}(A_{sj}(t,x)\cdot\eta_j\,|\,\eta_s)_{\mathbb{C}^N}$$
$$\geqq \nu\sum_{s=1}^{n}|\eta_s|^2\quad \forall \eta_1,\cdots,\eta_n\in\mathbb{C}^N,\quad \forall(t,x)\in[0,T]\times\bar\Omega\quad(\nu>0).$$

Under the above assumptions we consider the following problems:

$$(2.3)\qquad \begin{aligned}&D_t y(t,x)-\sum_{sj=1}^{n}D_s[A_{sj}(t,x)\cdot D_j y(t,x)]+y(t,x)=0\quad \text{in } [0,T]\times\bar\Omega,\\ &y(t,x)=u(t,x)\quad \text{in } [0,T]\times\partial\Omega,\\ &y(0,x)=y_0(x)\quad \text{in } \bar\Omega;\end{aligned}$$

$$(2.4)\qquad \begin{aligned}&D_t y(t,x)-\sum_{sj=1}^{n}D_s[A_{sj}(t,x)\cdot D_j y(t,x)]+y(t,x)=0\quad \text{in } [0,T]\times\bar\Omega,\\ &\sum_{sj=1}^{n}A_{sj}(t,x)\cdot D_j y(t,x)\nu_s(x)=u(t,x)\quad \text{in } [0,T]\times\partial\Omega,\\ &y(0,x)=y_0(x)\quad \text{in } \bar\Omega,\end{aligned}$$

where $y_0$, $u$ are prescribed data on the parabolic boundary of $[0,T]\times\bar\Omega$. Here $\nu(x)$ is the unit outward normal vector at $x\in\partial\Omega$. It is well known that if $u$, $y_0$ are sufficiently smooth and fulfill suitable compatibility conditions at $\partial\Omega$ at $t=0$, then problems (2.3), (2.4) possess a unique solution; in addition we want to prove a representation formula for such solutions which will allow us to generalize the concept of solution to the case of less regular data $u$, $y_0$.

Concerning existence and uniqueness, we will invoke the results of Theorem 4.7 of [AT3]; to this purpose we just need that problems (2.3), (2.4) obey the requirements given there, namely, we need that the operator $\{\sum_{sj=1}^{n}D_s(A_{sj}(t,x)\cdot D_j)\}$, with boundary conditions of Dirichlet or of conormal derivative type, satisfies the ellipticity assumptions of [ADN] and [GG]. This is in fact true, as pointed out in Remark 2.3(i) below. Hence we can state the following propositions.

PROPOSITION 2.1. *Under assumptions* (2.1), (2.2), *let* $y_0\in[W^{2,2}(\Omega)]^N$, *and let* $u$ *be the trace on* $[0,T]\times\partial\Omega$ *of a function* $U\in C^{\alpha}([0,T],[W^{1,2}(\Omega)]^N)\cap C^{\alpha+1/2}([0,T],[L^2(\Omega)]^N)$; *assume moreover that*

$$(2.5)\qquad \sum_{sj=1}^{n}A_{sj}(0,x)\cdot D_j y_0(x)\nu_s(x)=u(0,x)\quad \textit{a.e. on } \partial\Omega.$$

*Then problem* (2.4) *has a unique solution* $y$ *such that*

$$(2.6)\qquad y\in C^1([0,T],[L^2(\Omega)]^N)\cap C([0,T],[W^{2,2}(\Omega)]^N).$$

PROPOSITION 2.2. *Under assumptions* (2.1), (2.2) *let* $y_0\in[W^{2,2}(\Omega)]^N$, *and let* $u$ *be the trace on* $[0,T]\times\partial\Omega$ *of a function* $U\in C^{\alpha}([0,T],[W^{2,2}(\Omega)]^N)\cap C^{\alpha+1}([0,T],[L^2(\Omega)]^N)$; *assume moreover that*

$$(2.7)\qquad y_0(x)=u(0,x)\quad \textit{a.e. on } \partial\Omega.$$

*Then problem* (2.3) *has a unique solution* $y$ *such that* (2.6) *holds.*

*Proof.* The proofs of Propositions 2.1, 2.2 follow by Theorem 4.7 of [AT3] with minor modifications (since the operators considered there are not in divergence form). □

*Remark* 2.3. (i) If we confine ourselves to problem (2.3), we may replace hypothesis (2.2) by the weaker one

$$\text{(2.8)}\quad \operatorname{Re} \sum_{sj=1}^{n} (A_{sj}(t,x)\xi_s\xi_j \cdot \eta \mid \eta)_{\mathbb{C}^N} \geqq \nu|\xi|^2|\eta|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \forall \eta \in \mathbb{C}^N, \ \forall (t,x) \in [0,T] \times \bar{\Omega};$$

then the operator $\{\sum_{sj=1}^{n} D_s(A_{sj}(t,x) \cdot D_j)\}$, with Dirichlet boundary conditions, still satisfies the ellipticity assumptions of [ADN] and [GG], as pointed out in [Am, pp. 659–660]. On the other hand, we are not able to prove the same assertion in the case of problem (2.4); that is, in order that the above operator, endowed with boundary conditions of conormal derivative type, satisfies the ellipticity assumptions of [ADN] and [GG], we need the stronger hypothesis (2.2) (this can be seen by adapting the argument of [ADN, p. 44]).

(ii) Adding lower order terms in problem (2.3), or (2.4), does not alter the situation: indeed, the change of unknown $v := e^{\omega t}y$ (for a suitable $\omega \in \mathbb{R}$) leads to a new problem where the new differential operators still enjoy the properties stated in Proposition 2.4 below; in particular, the abstract hypothesis (2.29) is preserved.

(iii) In (2.1) it is assumed that the coefficients of the differential operators satisfy suitable Hölder conditions with respect to time. Such a requirement is necessary in order to fulfill the abstract assumption (2.29)(ii) below, which in turn allows us to construct the evolution operator for the abstract problem (2.28), with its regularity properties (3.4). If the coefficients are just bounded and measurable in $t$, then we can get some results for the concrete problems (2.3), (2.4) (see [LM1]), i.e., for the state equation; however the subsequent step, namely the study of the Riccati equation, seems very difficult and needs stronger hypotheses (see [DS]).

Existence and uniqueness of the solution of problems (2.3) and (2.4) is now guaranteed, at least for smooth data $y_0$, $u$. Our next goal is to establish a representation formula for the solution, which should possess the following features:

(i) It reduces to known representation formulas whenever they hold: see, e.g., [Te] for the autonomous versions of (2.3)–(2.4), [AT1] and [AT2] in the case of homogeneous boundary conditions, [B2] and [La] within the context of control theory;

(ii) It yields "weak" solutions, in some sense, when the data are less smooth;

(iii) It is handy from the point of view of control theory.

In order to construct such a formula, we need to reformulate problems (2.3), (2.4) in an abstract form, and to establish some properties of the evolution operators of the new problem. This will be the object of the next section.

**2.2. The abstract formulation of initial boundary value problems.** Consider again the situation of § 2.1, under assumptions (2.1), (2.2). If we define, for each $t \in [0, T]$, the differential operators

$$\text{(2.9)}\qquad \mathscr{A}(t,x,D)v := \sum_{sj=1}^{n} D_s[A_{sj}(t,x) \cdot D_j v] - v, \qquad x \in \bar{\Omega},$$

$$\text{(2.10)}\qquad \mathscr{B}_0 v := v_{|\partial\Omega},$$

$$\text{(2.11)}\qquad \mathscr{B}_1(t,x,D) := \sum_{sj=1}^{n} A_{sj}(t,x)\nu_s(x) \cdot D_j v, \qquad x \in \partial\Omega,$$

then we can introduce the following linear operators:

$$D_{A_0(t)} := \{v \in [W^{2,2}(\Omega)]^N : \mathscr{B}_0 v = 0\} = [W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)]^N, \qquad A_0(t)v := \mathscr{A}(t, \cdot, D)v, \tag{2.12}$$

$$D_{A_1(t)} := \{v \in [W^{2,2}(\Omega)]^N : \mathscr{B}_1(t, \cdot, D)v = 0\}, \qquad A_1(t)v := \mathscr{A}(t, \cdot, D)v, \tag{2.13}$$

where $t \in [0, T]$.

The main properties of the operators $A_0(t)$, $A_1(t)$ are listed in the following proposition.

PROPOSITION 2.4. *Under assumptions* (2.1), (2.2) *we have for* $r = 0, 1$:

(i) *For each* $t \in [0, T]$, $A_r(t)$ *is the infinitesimal generator of an analytic semigroup in* $[L^2(\Omega)]^N$;

(ii) *for each* $t \in [0, T]$, $D_{A_r(t)}$ *is dense in* $[L^2(\Omega)]^N$;

(iii) *the family* $\{A_r(t)\}_{t\in[0,T]}$ *satisfies Hypothesis* II *of* [AT1], *i.e., there exists* $\vartheta_0 \in ]\pi/2, \pi]$ *such that*

$$\|A_r(t)[\lambda - A_r(t)]^{-1}[A_r(t)^{-1} - A_r(\tau)^{-1}]\|_{\mathscr{L}([L^2(\Omega)]^N)} \leq c\frac{|t-\tau|^{\alpha+1/2}}{|\lambda|^{1/2}} \quad \forall t, \tau \in [0, T], \tag{2.14}$$

*provided* $\lambda$ *belongs to the sector* $S_{\vartheta_0} := \{z \in \mathbb{C} : |\arg z| < \vartheta_0\}$.

*Proof.* (i) It is well known (see [Am], [GG]) that the resolvent set of the operators $A_r(t)$ contains the sector

$$S_{\vartheta_0} + \omega = \{z \in \mathbb{C} : |\arg (z - \omega)| < \vartheta_0\}$$

for suitable $\vartheta_0 \in ]\pi/2, \pi]$ and $\omega \in \mathbb{R}$; we want to show here that we can choose $\omega = 0$ and that

$$\|[\lambda - A_r(t)]^{-1}\|_{\mathscr{L}([L^2(\Omega)]^N)} \leq \frac{c}{1+|\lambda|} \quad \forall \lambda \in \overline{S_{\vartheta_0}}. \tag{2.15}$$

Suppose first $r = 0$. Fix $t \in [0, T]$ and let $\lambda \in \mathbb{C}$ be such that either $\operatorname{Re} \lambda > 0$ or $(M/\nu)|\operatorname{Re} \lambda| \leq \frac{1}{2}|\operatorname{Im} \lambda|$. For $v \in [W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)]^N$ set

$$f(x) := \lambda v(x) - \mathscr{A}(t, x, D)v, \qquad x \in \Omega.$$

Multiplying by $v$ (with respect to the inner product of $[L^2(\Omega)]^N$) and integrating by parts, we get

$$(1+\lambda)\int_\Omega |v|^2\,dx + \int_\Omega \sum_{sj=1}^n (A_{sj}(t,x)\cdot D_j v(x) \,|\, D_s v(x))_{\mathbb{C}^N}\,dx = \int_\Omega (f(x)\,|\,g(x))_{\mathbb{C}^N}\,dx. \tag{2.16}$$

By taking the real part, we obtain by (2.2)

$$(1+\operatorname{Re}\lambda)\int_\Omega |v|^2\,dx + \nu\int_\Omega |Dv|^2\,dx \leq \|f\|_{[L^2(\Omega)]^N}\cdot\|v\|_{[L^2(\Omega)]^N}; \tag{2.17}$$

on the other hand, by taking the imaginary part in (2.16),

$$|\mathrm{Im}\,\lambda| \int_\Omega |v|^2\,dx \leqq \|f\|_{[L^2(\Omega)]^N} \cdot \|v\|_{[L^2(\Omega)]^N} + M \int_\Omega |Dv|^2\,dx,$$

where

$$M := \sum_{sj=1}^{n} \sup_{[0,T]\times\bar{\Omega}} |A_{sj}(t,x)|. \tag{2.18}$$

Hence by (2.17)

$$|\mathrm{Im}\,\lambda| \int_\Omega |v|^2\,dx \leqq \left(1+\frac{M}{\nu}\right) \|f\|_{[L^2(\Omega)]^N} \cdot \|v\|_{[L^2(\Omega)]^N} + \frac{M}{\nu} |\mathrm{Re}\,\lambda| \int_\Omega |Dv|^2\,dx.$$

Consequently, if $(M/\nu)|\mathrm{Re}\,\lambda| \leqq \frac{1}{2}|\mathrm{Im}\,\lambda|$ we deduce that

$$|\mathrm{Im}\,\lambda| \int_\Omega |v|^2\,dx \leqq 2\left(1+\frac{M}{\nu}\right) \|f\|_{[L^2(\Omega)]^N} \cdot \|v\|_{[L^2(\Omega)]^N}, \tag{2.19}$$

whereas if $(M/\nu)\,\mathrm{Re}\,\lambda > \frac{1}{2}|\mathrm{Im}\,\lambda|$ (2.17) yields

$$(1+\mathrm{Re}\,\lambda) \int_\Omega |v|^2\,dx \leqq \|f\|_{[L^2(\Omega)]^N} \cdot \|v\|_{[L^2(\Omega)]^N}. \tag{2.20}$$

Combining (2.19) and (2.20) we get the estimate

$$\|v\|_{[L^2(\Omega)]^N} \leqq \frac{c}{1+|\lambda|} \|\lambda v - \mathscr{A}(t,\cdot,D)v\|_{[L^2(\Omega)]^N} \quad \forall \lambda \in \overline{S_{\vartheta_0}},$$

where

$$\vartheta_0 = \pi - \mathrm{arctg}\,\frac{2M}{\nu}, \qquad c = 2\left(1+\frac{M}{\nu}\right)\sqrt{1+(\nu/2M)^2}+1;$$

since we already know that $\rho(A_0(t))$ is not empty, the desired estimate (2.15) for $r=0$ follows from the above inequality by standard arguments.

The case $r=1$ is completely analogous and we find the same constants $\vartheta_0$ and $c$.

The proof of part (ii) is obvious in both cases $r=0, 1$.

(iii) Consider the case $r=0$. Fix $f \in [L^2(\Omega)]^N$, and set $v := [A_0(\tau)]^{-1} f$, $w := [\lambda - A_0(t)]^{-1}[\lambda - A_0(\tau)]v$; then we must estimate

$$v - w = A_0(t)[\lambda - A_0(t)]^{-1}[A_0(t)^{-1} - A_0(\tau)^{-1}]f.$$

The function $v-w$ solves the problem

$$\lambda(v-w) - \mathscr{A}(t,\cdot,D)(v-w) = \sum_{sj=1}^{n} D_s([A_{sj}(t,\cdot) - A_{sj}(\tau,\cdot)] \cdot D_j v) \quad \text{in } \Omega,$$

$$v - w \in [W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)]^N.$$

Multiplying by $v-w$ (in $[L^2(\Omega)]^N$), an integration by parts yields

$$(1+\lambda) \int_\Omega |v-w|^2\,dx + \int_\Omega \sum_{sj=1}^{n} (A_{sj}(t,x) \cdot D_j(v-w) \,|\, D_s(v-w))_{\mathbb{C}^N}\,dx$$

$$= -\int_\Omega \sum_{sj=1}^{n} ([A_{sj}(t,x) - A_{sj}(\tau,x)] \cdot D_j v \,|\, D_s(v-w))_{\mathbb{C}^N}\,dx,$$

which implies

$$(2.21)\qquad (1+\operatorname{Re}\lambda)\int_\Omega |v-w|^2\,dx+\frac{\nu}{2}\int_\Omega |D(v-w)|^2\,dx \leqq \frac{1}{2\nu}\int_\Omega \sum_{sj=1}^n |A_{sj}(t,\cdot)-A_{sj}(\tau,\cdot)|^2|Dv|^2\,dx,$$

$$(2.22)\qquad |\operatorname{Im}\lambda|\int_\Omega |v-w|^2\,dx \leqq \frac{1}{2}\int_\Omega \sum_{sj=1}^n |A_{sj}(t,\cdot)-A_{sj}(\tau,\cdot)|^2|Dv|^2\,dx + \left(M+\frac{1}{2}\right)\int_\Omega |D(v-w)|^2\,dx.$$

If we set

$$(2.23)\qquad N:=\sum_{sj=1}^n \sup_{0\leqq\tau<t\leqq T}\ \sup_{x\in\Omega}\frac{|A_{sj}(t,x)-A_{sj}(\tau,x)|}{|t-\tau|^{\alpha+1/2}},$$

then by (2.22) and (2.21) we easily get

$$|\operatorname{Im}\lambda|\int_\Omega |v-w|^2\,dx \leqq N^2\left[\frac{1}{2}+\left(M+\frac{1}{2}\right)\nu^{-2}\right]|t-\tau|^{2\alpha+1}\int_\Omega |Dv|^2\,dx + \frac{2}{\nu}\left(M+\frac{1}{2}\right)|\operatorname{Re}\lambda|\int_\Omega |v-w|^2\,dx.$$

Hence if $(2/\nu)(M+\frac{1}{2})|\operatorname{Re}\lambda|\leqq\frac{1}{2}|\operatorname{Im}\lambda|$

$$(2.24)\qquad |\operatorname{Im}\lambda|\int_\Omega |v-w|^2\,dx \leqq N^2\left[1+\frac{2M+1}{\nu^2}\right]|t-\tau|^{2\alpha+1}\int_\Omega |Dv|^2\,dx,$$

whereas if $(2/\nu)(M+\frac{1}{2})\operatorname{Re}\lambda>\frac{1}{2}|\operatorname{Im}\lambda|$

$$(2.25)\qquad \operatorname{Re}\lambda\int_\Omega |v-w|^2\,dx \leqq \frac{N^2}{2\nu}|t-\tau|^{2\alpha+1}\int_\Omega |Dv|^2\,dx.$$

Recalling that, by (2.17),

$$\int_\Omega |Dv|^2\,dx \leqq \frac{1}{\nu}\int_\Omega |f|^2\,dx,$$

we conclude that

$$|\lambda|\int_\Omega |v-w|^2\,dx \leqq \frac{N^2}{\nu}\left(1+\frac{2M+1}{\nu^2}\right)\sqrt{1+[\nu/(4M+2)]^2}\,|t-\tau|^{2\alpha+1}\int_\Omega |f|^2\,dx,$$

and (2.14) follows for $r=0$, with

$$\vartheta_0=\pi-\operatorname{arctg}\frac{4M+2}{\nu},\qquad c=N\nu^{-1/2}\left(1+\frac{2M+1}{\nu^2}\right)^{1/2}\left[1+\left(\frac{\nu}{4M+2}\right)^2\right]^{1/4}.$$

Concerning the case $r=1$, we proceed similarly and we find that $v-w$ now solves the problem

$$\lambda(v-w)-\mathscr{A}(t,\cdot,D)(v-w)=\sum_{sj=1}^n D_s([A_{sj}(t,\cdot)-A_{sj}(\tau,\cdot)]\cdot D_j v)\quad \text{in }\Omega,$$

$$\sum_{sj=1}^n A_{sj}(t,\cdot)\nu_s\cdot D_j(v-w)=\sum_{sj=1}^n [A_{sj}(\tau,\cdot)-A_{sj}(t,\cdot)]\nu_s\cdot D_j v\quad \text{on }\partial\Omega;$$

arguing as above, and taking into account the boundary conditions, we obtain the result with the same constants $\vartheta_0$ and $c$. The proof of Proposition 2.4 is complete. □

*Remark* 2.5. The estimates (2.15) and (2.14) do not need that $A_{sj}$ belongs to $C^\alpha([0, T], [C^1(\bar{\Omega})]^N)$.

Consider now the operators $A_r(t)^*$, i.e., the adjoint operators of $A_r(t)$ ($t \in [0, T]$, $r = 0, 1$). It is easy to verify that they are defined by

$$D_{A_0(t)^*} := [W^{1,2}(\Omega) \cap W_0^{1,2}(\Omega)]^N,$$

(2.26)
$$A_0(t)^* y := \overline{\mathscr{A}(t, \cdot, D)} y = \sum_{sj=1}^n D_j[\overline{{}^tA_{sj}(t, \cdot)} \cdot D_s y] - y,$$

$$D_{A_1(t)^*} := \left\{ y \in [W^{2,2}(\Omega)]^N : \overline{\mathscr{B}_1(t, \cdot, D)} y = \sum_{sj=1}^n \overline{{}^tA_{sj}(t, \cdot)} \nu_s \cdot D_j y = 0 \right\},$$

(2.27)
$$A_1(t)^* y := \overline{\mathscr{A}(t, \cdot, D)} y,$$

where $\overline{{}^tA_{sj}}$ is the matrix whose elements are the conjugates of the elements of the transposed ${}^tA_{sj}$ of $A_{sj}$. Consequently, it is clear that the following result holds.

PROPOSITION 2.6. *All statements of Proposition 2.4 are true if $A_r(t)$ is replaced by $A_r(t)^*$.*

The results of Propositions 2.4 and 2.6 allow us to apply to the operators $\{A_r(t)\}_{t\in[0,T]}$, $\{A_r(t)^*\}_{t\in[0,T]}$ the abstract theory of [AT1], [AT2], and [Ac] concerning linear nonautonomous parabolic Cauchy problems of the following kind:

(2.28)
$$u'(t) - A(t)u(t) = f(t), \qquad t \in [0, T],$$
$$u(0) = x,$$

where $f \in C([0, T], E)$, $x \in E$ ($E$ being a general Banach space) and $\{A(t)\}_{t\in[0,T]}$ fulfills (2.15) and (2.14). In the next section we will recall some facts concerning a problem such as (2.28).

**2.3. The study of the abstract problem.** We now consider problem (2.28), but we restrict our considerations to the case of a Hilbert space $H$, which is enough for our successive applications. We assume that:

(2.29) $\{A(t)\}_{t\in[0,T]}$ is a family of closed linear operators in $H$, such that:

(i) $$\|[\lambda - A(t)]^{-1}\|_{\mathscr{L}(H)} \leqq \frac{M}{1+|\lambda|} \quad \forall \lambda \in \overline{S_{\vartheta_0}}, \quad \forall t \in [0, T],$$

(ii) $$\|A(t)[\lambda - A(t)]^{-1}[A(t)^{-1} - A(s)^{-1}]\|_{\mathscr{L}(H)} \leqq B \frac{|t-s|^{\alpha+1/2}}{|\lambda|^{1/2}}$$

$$\forall \lambda \in S_{\vartheta_0}, \quad \forall t, s \in [0, T],$$

where $\vartheta_0 \in ]\pi/2, \pi]$ and $\alpha$, $M$, $B > 0$.

In particular, $A(t)$ generates an analytic semigroup $e^{\xi A(t)}$ which can be represented as a Dunford integral:

(2.30)
$$e^{\xi A(t)} = (2\pi i)^{-1} \int_\Gamma e^{\xi\lambda} [\lambda - A(t)]^{-1} \, d\lambda,$$

$\Gamma$ being a smooth path contained in $S_{\vartheta_0}$ and joining $+\infty\, e^{-i\vartheta}$ to $+\infty\, e^{i\vartheta}$, $\vartheta \in ]\pi/2, \vartheta_0[$. Moreover, the fractional powers $[-A(t)]^\gamma$ are well defined and we have the representations

$$(2.31)\qquad [-A(t)]^{-\gamma} = (2\pi i)^{-1} \int_{\Gamma'} (-\lambda)^{-\gamma}[\lambda - A(t)]^{-1}\, d\lambda, \qquad \gamma > 0,$$

$$(2.32)\qquad [-A(t)]^{-\gamma}\, e^{\xi A(t)} = (2\pi i)^{-1} \int_{\Gamma'} (-\lambda)^{-\gamma}\, e^{\xi\lambda}[\lambda - A(t)]^{-1}\, d\lambda$$

where $\Gamma' \subset S_{\theta_0}$ joins $+\infty\, e^{-i\theta}$ to $+\infty\, e^{i\theta}$ leaving 0 on its right-hand side. We also recall the well-known continuous inclusions

$$(2.33)\qquad D_{A(t)}(\gamma+\varepsilon, \infty) \subset D_{[-A(t)]^\gamma} \subset D_{A(t)}(\gamma, \infty) \qquad \forall \gamma \in ]0, 1[, \quad \forall \varepsilon \in ]0, 1-\gamma[, \quad \forall t \in [0, T];$$

here $D_{A(t)}(\gamma, p)$, $1 \leqq p \leqq \infty$, is the real interpolation space $(D_{A(t)}, H)_{1-\gamma,\infty}$ introduced in [LP], which can be characterized in the following way:

$$(2.34)\qquad D_{A(t)}(\gamma, p) = \{x \in H : \xi \to \xi^{-\gamma}\|[e^{\xi A(t)} - 1]x\|_H \in L^p(0, \infty; d\xi/\xi)\}.$$

We need the following lemma.

LEMMA 2.7. *Under assumption* (2.29) *we have for each* $t, s \in [0, T]$:

(i) $$\|[-A(t)]^{-\vartheta} - [-A(s)]^{-\vartheta}\|_{\mathscr{L}(H)} \leqq \begin{cases} c(\vartheta, \alpha)|t-s|^{\alpha+1/2} & \text{if } \vartheta > \frac{1}{2}, \\ c(\vartheta, \alpha, \sigma)|t-s|^{\sigma(\alpha+1/2)} \quad \forall \sigma \in ]0, 2\vartheta[ & \text{if } \vartheta \in ]0, \frac{1}{2}]; \end{cases}$$

(ii) $$\|[-A(t)]^{\vartheta}\, e^{\xi A(t)}\|_{\mathscr{L}(H)} \leqq c(\vartheta)\xi^{-\vartheta} \quad \forall \xi > 0;$$

(iii) $$\|[-A(t)]^{\vartheta}\, e^{\xi A(t)} - [-A(s)]^{\vartheta}\, e^{\xi A(s)}\|_{\mathscr{L}(H)} \leqq c(\vartheta, \alpha)|t-s|^{\alpha+1/2}\xi^{-\vartheta-1/2} \quad \forall \xi > 0.$$

*Proof.* (i) An easy check shows that

$$[\lambda - A(t)]^{-1} - [\lambda - A(s)]^{-1} = -A(t)[\lambda - A(t)]^{-1}[A(t)^{-1} - A(s)^{-1}]A(s)[\lambda - A(s)]^{-1};$$

hence by (2.31) and (2.29) we get

$$\|[-A(t)]^{-\vartheta} - [-A(s)]^{-\vartheta}\|_{\mathscr{L}(H)} \leqq c \int_{\Gamma'} |\lambda|^{-\vartheta} \left[\frac{M}{1+|\lambda|}\right]^{1-\sigma} \left[\frac{B|t-s|^{\alpha+1/2}}{|\lambda|^{1/2}}\right]^{\sigma} |d\lambda| \qquad \forall \sigma \in [0, 1],$$

which easily leads to the result.

Parts (ii) and (iii) follow similarly by (2.32) and (2.29). □

We are ready to state the main result concerning problem (2.28).

PROPOSITION 2.8. *Under assumption* (2.29), *the evolution operator* $U(t, s) \in \mathscr{L}(H, D_{A(t)})$, *associated to problem* (2.28), *exists and possesses the following properties*:

(i) $(t, s) \to U(t, s) \in C(\Delta, \mathscr{L}(H)) \cap C_s(\bar{\Delta}, \mathscr{L}(H))$, *where* $\Delta := \{(t, s) \in [0, T] : s < t\}$, *and*

$$U(t, t) = 1, \quad U(t, \tau)U(\tau, s) = U(t, s) \quad \forall \tau \in [s, t];$$

(ii) $(t, s) \to A(t)U(t, s) \in C(\Delta, \mathscr{L}(H))$ *and*

$$A(t)U(t, s) = \frac{\partial}{\partial t} U(t, s), \quad \|A(t)U(t, s)\|_{\mathscr{L}(H)} \leqq M_1(t-s)^{-1} \quad \forall 0 \leqq s \leqq t \leqq T;$$

(iii) *If* $s \in [0, t[$ *and* $x \in D_{A(s)}$, *then* $\partial/\partial s U(t, s)x = -U(t, s)A(s)x$ *in the following sense*:

$$h^{-1}[U(t, s+h) - U(t, s)]x \to -U(t, s)A(s)x \quad \text{in } H \quad \text{as } h \to 0_+,$$

$$h^{-1}[U(t, s+h) - U(t, s)]A(s+h)^{-1}A(s)x \to -U(t, s)A(s)x \quad \text{in } H \quad \text{as } h \to 0_-;$$

(iv) *If* $\gamma, \beta \in [0, 1]$, *then* $(t, s) \to [-A(t)]^\gamma U(t, s)[-A(s)]^{-\beta} \in C(\Delta, \mathscr{L}(H))$ *and*

$$\|[-A(t)]^\gamma U(t, s)[-A(s)]^{-\beta}\|_{\mathscr{L}(H)} \leqq M_{\gamma\beta}[(t-s)^{\beta-\gamma} + 1] \quad \forall 0 \leqq s \leqq t \leqq T;$$

(v) *If* $0 \leqq \gamma \leqq \beta \leqq 1$, *then* $(t, s) \to [-A(t)]^\gamma U(t, s)[-A(s)]^{-\beta} \in C(\bar{\Delta}, \mathscr{L}(H))$.

*Proof.* Parts (i)–(iii) are proved in Theorem 3.2 of [Ac] (recall that the domains $D_{A(t)}$ are dense in $H$ here), with the exception of the assertion $(t, s) \to U(t, s) \in C_s(\bar{\Delta}, \mathscr{L}(H))$. In order to show this property, we first recall that by the density of domains and by Lemma 1.9(i) of [AT1] we have

$$\lim_{n\to\infty} \|x - n[n - A(\tau)]^{-1}x\|_H = 0 \quad \forall x \in H, \quad \forall \tau \in [0, T], \tag{2.35}$$

$$\|A(s)[n - A(s)]^{-1} - A(\tau)[n - A(\tau)]^{-1}\|_{\mathscr{L}(H)} \leqq Bn^{1/2}|\tau - s|^{\alpha+1/2} \quad \forall n \in \mathbb{N}^+, \quad \forall \tau, s \in [0, T]. \tag{2.36}$$

Now let $(\tau, \tau) \in \partial\Delta$, $x \in H$; then we have (see [Ac, formula (2.6)]):

$$\begin{aligned} U(t, s)x - x &= [e^{(t-s)A(s)}x - x] + \int_s^t Z(r, s)x\, dr \\ &= [e^{(t-s)A(s)} - 1]\{[x - n[n - A(\tau)]^{-1}x] \\ &\quad + [n[n - A(\tau)]^{-1} - n[n - A(s)]^{-1}]x\} \\ &\quad + \int_0^{t-s} e^{\sigma A(s)}[nA(s)[n - A(s)]^{-1} - nA(\tau)[n - A(\tau)]^{-1}]x\, d\sigma \\ &\quad + \int_0^{t-s} e^{\sigma A(s)} nA(\tau)[n - A(\tau)]^{-1}x\, d\sigma + \int_s^t Z(r, s)x\, dr; \end{aligned}$$

hence by (2.36) and Lemma 2.2(i) of [Ac] we easily obtain

$$\begin{aligned} \|U(t, s)x - x\|_H \leqq c(M, \beta, \alpha)\{&(1 + n(t-s))\|x - n[n - A(\tau)]^{-1}x\|_H \\ &+ \|x\|_H[(1 + n(t-s))n^{1/2}|\tau - s|^{\alpha+1/2} + (t-s)^\alpha]\}. \end{aligned}$$

By (2.35) there exists $\nu_\varepsilon \in \mathbb{N}^*$ such that

$$\|x - \nu_\varepsilon[\nu_\varepsilon - A(\tau)]^{-1}x\|_H < \tfrac{1}{2}\varepsilon^2[c(M, \beta, \alpha)]^{-1};$$

choosing $n = \nu_\varepsilon$ and $\delta_\varepsilon > 0$ such that

$$c(M, \beta, \alpha)[(1 + \nu_\varepsilon\delta_\varepsilon)\tfrac{1}{2}\varepsilon^2[c(M, \beta, \alpha)]^{-1} + \|x\|_H(1 + \nu_\varepsilon\delta_\varepsilon)\nu_\varepsilon^{1/2}\delta_\varepsilon^{\alpha+1/2} + \|x\|_H\delta_\varepsilon^\alpha] < \varepsilon,$$

we immediately get

$$\|U(t, s)x - x\|_H \leqq \varepsilon \quad \text{if } |t - \tau| + |\tau - s| < \delta_\varepsilon.$$

Note that, in particular, the above proof shows that

$$(t, s) \to e^{(t-s)A(s)} \in C_s(\bar{\Delta}, \mathscr{L}(H)). \tag{2.37}$$

Let us prove (iv). We write

$$[-A(t)]^\gamma U(t, s)[-A(s)]^{-\beta} = -[-A(t)]^{\gamma-1}[A(t)U(t, s)][-A(s)]^{-\beta};$$

since each operator in the right-hand side is in $C(\Delta, \mathscr{L}(H))$, we get that the left-hand side also belongs to $C(\Delta, \mathscr{L}(H))$. In order to prove the estimate, we remark that if $x \in H$, then $t \to U(t, s)[-A(s)]^{-\beta}x$ is the classical solution [AT1, Def. 1.6] of the problem

$$\begin{aligned} &u'(t) - A(t)u(t) = 0, \qquad t \in ]s, T], \\ &u(0) = [-A(s)]^{-\beta}x, \end{aligned} \tag{2.38}$$

and consequently [AT1, Thm. 6.3(i)] $t \to [A(t)U(t, s)][-A(s)]^{-\beta}x$ solves the integral equation

$$v(t) - [Q_s v](t) = A(t)\, e^{(t-s)A(t)}[-A(s)]^{-\beta}x, \qquad t \in [s, T], \tag{2.39}$$

where the integral operator $Q_s$ is defined [Ac, (2.1)–(2.2)] by

$$[Q_s v](t) := \int_s^t A(t)^2\, e^{(t-\tau)A(t)}[A(t)^{-1} - A(\tau)^{-1}]v(\tau)\, d\tau, \qquad t \in [s, T]. \tag{2.40}$$

Hence we can write

$$\begin{aligned} &[-A(t)]^{\gamma} U(t, s)[-A(s)]^{-\beta}x \\ &\quad = -[-A(t)]^{\gamma-1}[Q_s([A(t)U(t, s)][-A(s)]^{-\beta}x)](t) \\ &\qquad - [-A(t)]^{\gamma}\, e^{(t-s)A(t)}[-A(s)]^{-\beta}x \\ &\quad = -\int_s^t [-A(t)]^{\gamma+1}\, e^{(t-\tau)A(t)}[A(t)^{-1} - A(\tau)^{-1}]A(\tau)U(\tau, s)[-A(s)]^{-\beta}x\, d\tau \\ &\qquad - [[-A(t)]^{\gamma}\, e^{(t-s)A(t)} - [-A(s)]^{\gamma}\, e^{(t-s)A(s)}][-A(s)]^{-\beta}x \\ &\qquad - [-A(s)]^{\gamma-\beta}\, e^{(t-s)A(s)}x, \end{aligned} \tag{2.41}$$

and by Lemma 2.7 we readily obtain the result.

Finally, we prove (v). By (2.41) it is enough to show that if $(t, s) \to (\tau, \tau)$ in $\Delta$ and $x \in H$, then

$$\|([-A(s)]^{\gamma-\beta}\, e^{(t-s)A(s)} - [-A(\tau)]^{\gamma-\beta})x\|_H \to 0.$$

If $\beta = \gamma$ this follows by (2.37); otherwise we can write

$$\begin{aligned} &([-A(s)]^{\gamma-\beta}\, e^{(t-s)A(s)} - [-A(t)]^{\gamma-\beta})x \\ &\quad = \int_0^{t-s} [-A(s)]^{\gamma-\beta+1}\, e^{\xi A(s)}x\, d\xi + [[-A(s)]^{\gamma-\beta} - [-A(\tau)]^{\gamma-\beta}]x, \end{aligned}$$

which by Lemma 2.7 implies the result. □

Assume now that the adjoint operator $A(t)^*$ of $A(t)$ also satisfies (2.29), i.e.,

$$\begin{aligned} &\text{(i)} \quad \|[\lambda - A(t)^*]^{-1}\|_{\mathscr{L}(H)} \leqq \frac{M}{1 + |\lambda|} \quad \forall \lambda \in \overline{S_{\vartheta_0}}, \quad \forall t \in [0, T], \\ &\text{(ii)} \quad \|A(t)^*[\lambda - A(t)^*]^{-1}[[A(t)^*]^{-1} - [A(s)^*]^{-1}]\|_{\mathscr{L}(H)} \\ &\qquad \leqq B\frac{|t-s|^{\alpha+1/2}}{|\lambda|^{1/2}} \quad \forall \lambda \in S_{\vartheta_0}, \quad \forall t, s \in [0, T]. \end{aligned} \tag{2.42}$$

Then Proposition 2.8 also holds for $A(t)^*$.

The next result concerns the adjoint operator $U(t, s)^*$ of the evolution operator $U(t, s)$ relative to $A(t)$.

PROPOSITION 2.9. *Under assumptions* (2.29), (2.42) *let* $U(t, s)$ *be the evolution operator of problem* (2.28). *Then*:

(i) $U(t, s)^* \in \mathscr{L}(H, D_{A(s)^*})$, *for all* $s \in [0, t[$;

(ii) *For each* $\varphi \in H$, $s \to U(t, s)^* \varphi$ *solves the problem*

$$\frac{d}{ds} U(t, s)^* \varphi = -A(s)^* U(t, s)^* \varphi, \qquad s \in [0, t[, \tag{2.43}$$
$$U(t, t)^* \varphi = \varphi.$$

*Proof.* First of all, we show that the solution of (2.43) exists. Fix $t_0 \in ]0, T]$ and set

$$V(t_0; t, s) := \text{the evolution operator relative to } B(t) := A(t_0 - t)^*, \ t \in [0, t_0]. \tag{2.44}$$

This means that

$$\frac{d}{dt} V(t_0; t, s) \varphi = A(t_0 - t)^* V(t_0; t, s) \varphi, \qquad t \in ]s, t_0], \tag{2.45}$$
$$V(t_0; s, s) \varphi = \varphi.$$

Set $W(t, s) := V(t; t - s, 0)$, $s \in [0, t]$. Then, applying Proposition 2.8 to problem (2.45), we get $W(t, s) \in \mathscr{L}(H, D_{A(s)^*})$ and

$$\frac{d}{ds} W(t, s) \varphi = -\left[\frac{d}{d\tau} V(t; \tau, 0) \varphi\right]_{\tau = t - s} = -[A(t - \tau)^* V(t; \tau, 0) \varphi]_{\tau = t - s}$$
$$= -A(s)^* W(t, s) \varphi, \qquad s \in [0, t[,$$

$$W(t, t) \varphi = V(t; 0, 0) \varphi = \varphi,$$

i.e., $W(t, s)$ solves (2.43). The proof will be complete by showing that

$$V(t; t - s, 0) = W(t, s) = U(t, s)^*. \tag{2.46}$$

Indeed for $r \in ]s, t[$ we have

$$\frac{d}{dr} (W(t, r) \varphi \mid U(r, s) x)_H$$
$$= -(A(r)^* W(t, r) \varphi \mid U(r, s) \varphi)_H + (W(t, r) \varphi \mid A(r) U(r, s) \varphi)_H = 0,$$

so that $(W(t, r) \varphi \mid U(r, s) x)_H = \text{const.}$ for all $r \in [s, t]$. As $r \to t^-$ and $r \to s^+$ we get

$$(\varphi \mid U(r, s) x)_H = (W(t, r) \varphi \mid x)_H \quad \forall \varphi, x \in H,$$

i.e., $W(t, s) = U(t, s)^*$. □

COROLLARY 2.10. *Under assumptions* (2.29), (2.42) *we have for* $\gamma, \beta \in [0, 1]$:

$$\|[-A(s)^*]^\gamma U(t, s)^* [-A(t)^*]^{-\beta}\|_{\mathscr{L}(H)} \leqq M_{\gamma\beta}[(t - s)^{\beta - \gamma} + 1] \quad \forall 0 \leqq s < t \leqq T.$$

*Proof.* We have by (2.44) and (2.46)

$$[-A(s)^*]^\gamma U(t, s)^* [-A(t)^*]^{-\beta} = [[-B(\tau)]^\gamma V(t; \tau, 0)[-B(0)]^{-\beta}]_{\tau = t - s};$$

hence the result follows by applying Proposition 2.8 to problem (2.45). □

COROLLARY 2.11. *Under assumptions* (2.29), (2.42) *let* $\beta, \gamma \in [0, 1]$. *Then for* $0 \leqq s < t \leqq T$ *the closed linear operator*

$$[-A(t)]^{-\beta} U(t, s) [-A(s)]^\gamma$$

*possesses an extension* $\overline{[-A(t)]^{-\beta}U(t,s)[-A(s)]^{\gamma}} \in \mathscr{L}(H)$, *which satisfies*

$$\|\overline{[-A(t)]^{-\beta}U(t,s)[-A(s)]^{\gamma}}\|_{\mathscr{L}(H)} \leqq M_{\gamma\beta}[(t-s)^{\beta-\gamma}+1] \quad \forall 0 \leqq s < t \leqq T.$$

*Proof.* As

$$[[-A(s)]^{\gamma}]^* = [[-A(s)]^*]^{\gamma} \quad \forall \gamma \in [0,1],$$

if $x \in D_{[-A(s)]^{\gamma}}$ and $\varphi \in H$ we have

$$([-A(t)]^{-\beta}U(t,s)[-A(s)]^{\gamma}x \,|\, \varphi)_H = (x \,|\, [-A(s)^*]^{\gamma}U(t,s)^*[-A(t)^*]^{-\beta}\varphi)_H,$$

and by Corollary 2.10

$$|([-A(t)]^{-\beta}U(t,s)[-A(s)]^{\gamma}x \,|\, \varphi)_H| \leqq M_{\gamma\beta}[(t-s)^{\beta-\gamma}+1]\|x\|_H \|\varphi\|_H;$$

choosing $y := [-A(t)]^{-\beta}U(t,s)[-A(s)]^{\gamma}x$ and $\varphi := y/\|y\|_H$, by the density of $D_{[-A(s)]^{\gamma}}$ in $H$ we get the result. □

The study of the abstract problem (2.28) (which concerns homogeneous boundary conditions) is complete. In the next section we will introduce nonhomogeneous boundary data in the abstract framework.

**2.4. The Dirichlet and Neumann maps.** Let us go back to problems (2.3), (2.4): we will examine the regularity properties of the Dirichlet and Neumann maps $G_0(t)$, $G_1(t)$ which are defined by (see (2.9)–(2.11)):

$$u := G_0(t)g \Leftrightarrow \begin{cases} \mathscr{A}(t,\cdot,D)u = 0 & \text{in } \Omega, \\ \mathscr{B}_0 u = g & \text{on } \partial\Omega, \end{cases} \tag{2.47}$$

$$u := G_1(t)g \Leftrightarrow \begin{cases} \mathscr{A}(t,\cdot,D)u = 0 & \text{in } \Omega, \\ \mathscr{B}_1(t,\cdot,D)u = g & \text{on } \partial\Omega. \end{cases} \tag{2.48}$$

PROPOSITION 2.12. *Let* $A_0(t)$, $A_1(t)$ *be defined by* (2.12), (2.13), *respectively. If* $r=0,1$ *the operator* $G_r(t)$ *is well defined from* $[L^2(\partial\Omega)]^N$ *into* $D_{[-A_r(t)]^{\vartheta}}$, *for each* $\vartheta \in ]0, \alpha_r[$, *where* $\alpha_0 := \frac{1}{4}$ *and* $\alpha_1 := \frac{3}{4}$. *Moreover,*

$$t \to [-A_r(t)]^{\vartheta}G_r(t) \in L^{\infty}(0,T; \mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N)) \quad \forall \vartheta \in ]0, \alpha_r[.$$

*Proof.* This result was pointed out in [La] assuming $\partial\Omega \in C^{\infty}$; here we give an independent proof.

Let us start with the case $r=0$. Fix $t \in [0,T]$, let $g \in [W^{1/2,2}(\partial\Omega)]^N$, and consider the variational problem corresponding to (2.9), (2.10):

$$\begin{aligned} &\mathscr{A}(t,\cdot,D)u_0 = 0 && \text{in } \Omega, \\ &u_0 = g && \text{on } \partial\Omega, \end{aligned} \tag{2.49}$$

which means

$$\begin{aligned} &\sum_{sj=1}^{n} \int_{\Omega} [(A_{sj}(t,x)\cdot D_j u_0 \,|\, D_s\varphi)_{\mathbf{C}^N} + (u_0 \,|\, \varphi)_{\mathbf{C}^N}]\,dx = 0 \quad \forall \varphi \in [C_0^{\infty}(\Omega)]^N, \\ &u_0 - G \in [W_0^{1,2}(\Omega)]^N, \end{aligned} \tag{2.50}$$

where $G$ is an element of $[W^{1,2}(\Omega)]^N$ whose trace on $\partial\Omega$ is $g$, and such that

$$\|g\|_{[W^{1/2,2}(\partial\Omega)]^N} \leqq c_0\|G\|_{[W^{1,2}(\Omega)]^N} \leqq c_1\|g\|_{[W^{1/2,2}(\partial\Omega)]^N}. \tag{2.51}$$

By Poincaré inequality and Lax–Milgram theorem, problem (2.50) is uniquely solvable: we denote its solution $u_0$ by $S_0(t)g$, and we easily get the estimate

$$\|S_0(t)g\|_{[W^{1,2}(\Omega)]^N} \leqq c(M, \nu, c_0, c_1)\|g\|_{[W^{1/2,2}(\partial\Omega)]^N} \quad \forall g \in [W^{1/2,2}(\partial\Omega)]^N, \tag{2.52}$$

where $M := \sum_{sj=1}^n \|A_{sj}\|_{C([0,T],[C^1(\bar\Omega)]^{N^2})}$.

Note that if $g \in [W^{3/2,2}(\partial\Omega)]^N$, then by (2.1) and the classical results of [ADN] we have $S_0(t) \in [W^{2,2}(\Omega)]^N$ and

$$\|S_0(t)g\|_{[W^{2,2}(\Omega)]^N} \leqq c\|g\|_{[W^{3/2,2}(\partial\Omega)]^N} \quad \forall g \in [W^{3/2,2}(\partial\Omega)]^N. \tag{2.53}$$

We want now to estimate $S_0(t)$ in a lower norm. For $g \in [W^{1/2,2}(\partial\Omega)]^N$ set $u_0 := S_0(t)g$ and let $\psi$ be the variational solution of

$$\begin{aligned} &\sum_{sj=1}^n \int_\Omega [(\overline{{}^tA_{sj}(t,x)} \cdot D_s\psi \mid D_j\varphi)_{\mathbb{C}^N} + (\psi \mid \varphi)_{\mathbb{C}^N}]\, dx = \int_\Omega (u_0 \mid \varphi)_{\mathbb{C}^N}\, dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall \varphi \in [C_0^\infty(\Omega)]^N, \\ &\psi = 0 \quad \text{on } \partial\Omega; \end{aligned} \tag{2.54}$$

as $u_0 \in [W^{1,2}(\Omega)]^N \subset [L^2(\Omega)]^N$, we have $\psi \in [W^{2,2}(\Omega)]^N \cap [W_0^{1,2}(\Omega)]^N$, and

$$-\sum_{sj=1}^n D_j(\overline{{}^tA_{sj}(t,\cdot)} \cdot D_s\psi) + \psi = u_0 \quad \text{a.e. in } \Omega; \tag{2.55}$$

in addition

$$\|\psi\|_{[W^{3/2,2}(\partial\Omega)]^N} \leqq c\|\psi\|_{[W^{2,2}(\Omega)]^N} \leqq c\|u_0\|_{[L^2(\Omega)]^N}. \tag{2.56}$$

By density we may choose $\varphi = \psi$ in (2.50); an integration by parts yields

$$\int_\Omega \left(u_0 \,\Big|\, \psi - \sum_{sj=1}^n D_j[\overline{{}^tA_{sj}(t,x)} \cdot D_s\psi]\right)_{\mathbb{C}^N} dx = -\int_{\partial\Omega} \left(u_0 \,\Big|\, \sum_{sj=1}^n \overline{{}^tA_{sj}(t,x)} \cdot D_s\psi\nu_j\right)_{\mathbb{C}^N} d\sigma,$$

and by (2.55) (since $u_0 = g$ on $\partial\Omega$)

$$\int_\Omega |u_0|^2\, dx = -\int_{\partial\Omega} \left(g \,\Big|\, \sum_{sj=1}^n \overline{{}^tA_{sj}(t,x)} \cdot D_s\psi\nu_j\right)_{\mathbb{C}^N} d\sigma. \tag{2.57}$$

Now, as $\partial\Omega \in C^2$, the function $d(x)$, i.e., the distance of $x \in \Omega$ from $\partial\Omega$, is of class $C^2$ in a neighbourhood of $\partial\Omega$ and $Dd(x) = -\nu(x)$ on $\partial\Omega$ (see [GT, Appendix]); moreover we can clearly modify $d(x)$ inside $\Omega$ in order to get $d \in C^2(\bar\Omega)$. Hence by (2.57) and (2.56) it follows that

$$\begin{aligned} \|u_0\|^2_{[L^2(\Omega)]^N} &= |\langle g, \overline{{}^tA_{sj}(t,x)} \cdot D_s\psi\nu_j\rangle_{[W^{-1/2,2}(\partial\Omega)]^N, [W^{1/2,2}(\partial\Omega)]^N}| \\ &\leqq \|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \|\overline{{}^tA_{sj}} \cdot D_s\psi\nu_j\|_{[W^{1/2,2}(\partial\Omega)]^N} \\ &\leqq c\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \|\overline{{}^tA_{sj}} \cdot D_s\psi D_jd\|_{[W^{1,2}(\Omega)]^N} \\ &\leqq c(M,\Omega)\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \|\psi\|_{[W^{2,2}(\Omega)]^N} \\ &\leqq c\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \|u_0\|_{[L^2(\Omega)]^N}, \end{aligned}$$

that is,

$$\|S_0(t)g\|_{[L^2(\Omega)]^N} \leqq c\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N} \quad \forall g \in [W^{1/2,2}(\partial\Omega)]^N. \tag{2.58}$$

We now interpolate between (2.58) and (2.52), using Theorems 7.7 and 9.4 of [LM]: the proof of such theorems requires $\partial\Omega \in C^\infty$, but it can be readily adapted to our case. The result of interpolation is the estimate

$$\|S_0(t)g\|_{[W^{1/2,2}(\partial\Omega)]^N} \leqq c\|g\|_{[L^2(\partial\Omega)]^N} \quad \forall g \in [W^{1/2,2}(\partial\Omega)]^N, \tag{2.59}$$

which shows that the linear operator $S_0(t)$ may be boundedly extended to an operator $G_0(t)\in\mathscr{L}([L^2(\partial\Omega)]^N, [W^{1/2,2}(\Omega)]^N)$ defined by (compare with (2.47))

$$(2.60)\qquad \begin{aligned} &G_0(t):[L^2(\partial\Omega)]^N\to[W^{1/2,2}(\Omega)]^N,\\ &G_0(t)g:=S_0(t)g\quad \forall g\in[W^{1/2,2}(\partial\Omega)]^N. \end{aligned}$$

We now turn to the case $r=1$. Fix $t\in[0,T]$, let $g\in[W^{1/2,2}(\partial\Omega)]^N$, and consider the problem corresponding to (2.9), (2.11):

$$(2.61)\qquad \begin{aligned} &\mathscr{A}(t,x,D)u_1=0\quad \text{in }\Omega,\\ &\mathscr{B}_1(t,x,D)u_1=g\quad \text{on }\partial\Omega, \end{aligned}$$

which, by [ADN], has a unique solution $u_1:=S_1(t)g\in[W^{2,2}(\Omega)]^N$, such that

$$(2.62)\qquad \|S_1(t)g\|_{[W^{2,2}(\Omega)]^N}\le c\|g\|_{[W^{1/2,2}(\partial\Omega)]^N}\quad \forall g\in[W^{1/2,2}(\partial\Omega)]^N.$$

Multiply by $u_1$ in $[L^2(\Omega)]^N$ in (2.61) and integrate by parts: the result is

$$\begin{aligned} \nu\int_\Omega |Du_1|^2\,dx+\int_\Omega |u_1|^2\,dx &\le \int_\Omega [(A_{sj}(t,x)\cdot D_ju_1\,|\,D_su_1)_{\mathbb{C}^N}+(u_1\,|\,u_1)_{\mathbb{C}^N}]\,dx\\ &=\int_{\partial\Omega}(g\,|\,u_1)_{\mathbb{C}^N}\,d\sigma=|\langle g,u_1\rangle_{[W^{-1/2,2}(\partial\Omega)]^N,[W^{1/2,2}(\partial\Omega)]^N}|\\ &\le c\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N}\|u_1\|_{[W^{1,2}(\Omega)]^N}, \end{aligned}$$

which implies

$$(2.63)\qquad \|S_1(t)g\|_{[W^{1,2}(\Omega)]^N}\le c\|g\|_{[W^{-1/2,2}(\partial\Omega)]^N}\quad \forall g\in[W^{1/2,2}(\partial\Omega)]^N.$$

Interpolation between (2.63) and (2.62) (see the remark after (2.58)) yields

$$(2.64)\qquad \|S_1(t)g\|_{[W^{3/2,2}(\Omega)]^N}\le c\|g\|_{[L^2(\partial\Omega)]^N}\quad \forall g\in[W^{1/2,2}(\partial\Omega)]^N,$$

i.e., $S_1$ may be boundedly extended to an operator $G_1(t)\in\mathscr{L}([L^2(\partial\Omega)]^N,[W^{3/2,2}(\Omega)]^N)$ defined by (compare with (2.48)):

$$(2.65)\qquad \begin{aligned} &G_1(t):[L^2(\partial\Omega)]^N\to[W^{3/2,2}(\Omega)]^N,\\ &G_1(t)g:=S_1(t)g\quad \forall g\in[W^{1/2,2}(\partial\Omega)]^N. \end{aligned}$$

Now we recall that by Theorem 3.1 of [L1] we have for $r=0,1$ (see (2.34)):

$$(2.66)\qquad D_{[-A_r(t)]^\vartheta}=D_{A_r(t)}(\vartheta,2)\quad \forall\vartheta\in\,]0,1[,$$

$$(2.67)\qquad D_{[-A_r(t)^*]^\vartheta}=D_{A_r(t)^*}(\vartheta,2)\quad \forall\vartheta\in\,]0,1[.$$

On the other hand, the real interpolation spaces $D_{A_r(t)}(\vartheta,2)$ and $D_{A_r(t)^*}(\vartheta,2)$ can be characterized in the following way:

$$(2.68)\qquad \begin{aligned} D_{A_0(t)}(\vartheta,2)&=D_{A_0(t)^*}(\vartheta,2)\\ &=\begin{cases} [W^{2\vartheta,2}(\Omega)]^N & \text{if }\vartheta\in\,]0,\tfrac14[,\\ \left\{u\in[W^{1/2,2}(\Omega)]^N:\int_\Omega d(x)^{-1}|u(x)|^2\,dx<\infty\right\} & \text{if }\vartheta=\tfrac14,\\ [W_0^{2\vartheta,2}(\Omega)]^N & \text{if }\vartheta\in\,]\tfrac14,1[\setminus\{\tfrac12\},\\ [B_0^{1,2}(\Omega)]^N & \text{if }\vartheta=\tfrac12; \end{cases} \end{aligned}$$

(2.69)

$$D_{A_1(t)}(\vartheta, 2) = \begin{cases} [W^{2\vartheta,2}(\Omega)]^N & \text{if } \vartheta \in ]0, \tfrac{3}{4}[\setminus\{\tfrac{1}{2}\}, \\ [B^{1,2}(\Omega)]^N & \text{if } \vartheta = \tfrac{1}{2}, \\ \left\{u \in [W^{3/2,2}(\Omega)]^N : \int_\Omega d(x)^{-1} \left| \sum_{sj=1}^n A_{sj}(t,x) \cdot D_j u(x) D_s d(x) \right|^2 dx < \infty \right\} & \text{if } \vartheta = \tfrac{3}{4}, \\ \{u \in [W^{2\vartheta,2}(\Omega)]^N : \mathscr{B}_1(t,\cdot,D)u = 0 \text{ on } \partial\Omega\} & \text{if } \vartheta \in ]\tfrac{3}{4}, 1[; \end{cases}$$

(2.70)

$$D_{A_1(t)^*}(\vartheta, 2) = \begin{cases} [W^{2\vartheta,2}(\Omega)]^N & \text{if } \vartheta \in ]0, \tfrac{3}{4}[\setminus\{\tfrac{1}{2}\}, \\ [B^{1,2}(\Omega)]^N & \text{if } \vartheta = \tfrac{1}{2}, \\ \left\{u \in [W^{3/2,2}(\Omega)]^N : \int_\Omega d(x)^{-1} \left| \sum_{sj=1}^n \overline{{}^t A_{sj}(t,x)} \cdot D_s u(x) D_j d(x) \right|^2 dx < \infty \right\} & \text{if } \vartheta = \tfrac{3}{4}, \\ \{u \in [W^{2\vartheta,2}(\Omega)]^N : \overline{\mathscr{B}_1(t,\cdot,D)}u = 0 \text{ on } \partial\Omega\} & \text{if } \vartheta \in ]\tfrac{3}{4}, 1[. \end{cases}$$

Here $[B^{1,2}(\Omega)]^N$ is the Besov-Nikol'skij space. A proof of the results (2.68)–(2.70) is in Theorem 7.5 of [Gr] (see also [Tr, Thm. 4.3.3]) in the case $N=1$ and $\partial\Omega \in C^\infty$, but the same argument works in our situation.

The above results (namely, (2.66)–(2.69) together with (2.60), (2.65)) show that

$$\text{(2.71)} \qquad \begin{aligned} &G_0(t) \in \mathscr{L}([L^2(\partial\Omega)]^N, D_{[-A_0(t)]^\vartheta}) \quad \forall \vartheta \in ]0, \tfrac{1}{4}[, \\ &G_1(t) \in \mathscr{L}([L^2(\partial\Omega)]^N, D_{[-A_1(t)]^\vartheta}) \quad \forall \vartheta \in ]0, \tfrac{3}{4}[; \end{aligned}$$

the norms of $G_0(t)$, $G_1(t)$ are bounded independently of $t \in [0, T]$ in view of (2.59), (2.64).

On the other hand, if we set

$$F_n(t) := [-A_r(t)]^\vartheta \exp\left(\frac{1}{n} A_r(t)\right) G_r(t),$$

we have $F_n \in C([0, T], \mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N))$ by Lemma 2.7(ii); in addition, choosing $\rho \in ]0, \alpha_r - \vartheta[$ (with $\alpha_0 = \frac{1}{4}$, $\alpha_1 = \frac{3}{4}$) we see that

$$\begin{aligned} &\|F_n(t) - [-A_r(t)]^\vartheta G_r(t)\|_{\mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N)} \\ &\quad \leq \left\| \int_0^{1/n} [-A_r(t)]^{1-\rho} \exp(\xi A_r(t))\, d\xi \right\|_{\mathscr{L}([L^2(\Omega)])^N} \\ &\qquad \cdot \|[-A_r(t)]^{\vartheta+\rho} G_r(t)\|_{\mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N)} \leq \frac{c}{n^\rho}, \end{aligned}$$

so that $F_n(t) \to [-A_r(t)]^\vartheta G_r(t)$ in $\mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N)$ as $n \to \infty$, uniformly with respect to $t$; thus $[-A_r(\cdot)]^\vartheta G_r(\cdot)$ is a continuous function. This shows that

$$\text{(2.72)} \qquad [-A_0(\cdot)]^\vartheta G_0(\cdot) \in C([0, T], \mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N)) \quad \forall \vartheta \in ]0, \tfrac{1}{4}[,$$

$$\text{(2.73)} \qquad [-A_1(\cdot)]^\vartheta G_1(\cdot) \in C([0, T], \mathscr{L}([L^2(\partial\Omega)]^N, [L^2(\Omega)]^N)) \quad \forall \vartheta \in ]0, \tfrac{3}{4}[,$$

and, in particular, the proof is complete. □

We are ready to write a representation formula for (regular) solutions of problems (2.3), (2.4), which depends just on low-order norms, and hence can be extended to the case of less smooth data. This construction will be performed in the next section.

**2.5. The representation formula.** Consider again problems (2.3), (2.4) with smooth data: our representation formula for their solution is provided by the following proposition.

PROPOSITION 2.13. *Assume* (2.1), (2.2), *let* $y_0 \in [W^{2-r,2}(\Omega)]^N$ *and* $u \in C^\alpha([0,T],[W^{2-r,2}(\Omega)]^N) \cap C^{\alpha+1-r/2}([0,T],[L^2(\Omega)]^N)$ $(r=0$ *or* $r=1)$, *and suppose moreover that the compatibility conditions* (2.7) *or* (2.5) *hold. Then the solution of problem* (2.3), *or* (2.4), *is given by*

$$y(t,\cdot)=U_r(t,0)y_0+\int_0^t [[-A_r(s)^*]^{1-\vartheta}U_r(t,s)^*]^*[-A_r(s)]^\vartheta G_r(s)u(s,\cdot)\,ds, \tag{2.74}$$

$$t\in[0,T] \quad (\vartheta\in]0,\alpha_r[).$$

*Proof.* By Proposition 2.2 or 2.1 we know that problems (2.3) or (2.4) have a unique solution

$$y\in C^0([0,T],[W^{2,2}(\Omega)]^N)\cap C^1([0,T],[L^2(\Omega)]^N).$$

Consider the function $y-G_r(t)u$: by (2.47), (2.48), (2.53), and (2.62) we get (see (2.9))

$$\begin{aligned} &y(t,\cdot)-G_r(t)u(t,\cdot)\in D_{A_r(t)},\\ &A_r(t)[y(t,\cdot)-G_r(t)u(t,\cdot)]=\mathscr{A}(t,\cdot,D)y(t,\cdot). \end{aligned} \tag{2.75}$$

Next, denoting by $U_r(t,s)$ the evolution operator associated to $\{A_r(t)\}_{t\in[0,T]}$, we have by Corollary 2.10

$$\|[-A_r(s)^*]^\gamma U_r(t,s)^*\|_{\mathscr{L}([L^2(\Omega)]^N)}\leqq M_\gamma(t-s)^{-\gamma} \quad \forall\gamma\in]0,1[, \quad \forall 0\leqq s<t\leqq T,$$

and consequently

$$\|[[-A_r(s)^*]^\gamma U_r(t,s)^*]^*\|_{\mathscr{L}([L^2(\Omega)]^N)}\leqq M_\gamma(t-s)^{-\gamma} \quad \forall\gamma\in]0,1[, \quad \forall 0\leqq s<t\leqq T. \tag{2.76}$$

Now fix $t\in[0,T]$, let $z\in D_{A_r(t)^*}$, and define

$$h(s):=(y(s,\cdot)\,|\,U_r(t,s)^*z)_{[L^2(\Omega)]^N}, \qquad s\in[0,t[. \tag{2.77}$$

By Proposition 2.9 and (2.75) we may compute

$$\begin{aligned} h'(s)&=(D_sy(s,\cdot)\,|\,U_r(t,s)^*z)-(y(s,\cdot)-G_r(s)u(s,\cdot)\,|\,A_r(s)^*U_r(t,s)^*z)\\ &\qquad -(G_r(s)u(s,\cdot)\,|\,A_r(s)^*U_r(t,s)^*z)\\ &=(A(s,\cdot,D)y(s,\cdot)\,|\,U_r(t,s)^*z)\\ &\qquad -(A_r(s)[y(s,\cdot)-G_r(s)u(s,\cdot)]\,|\,U_r(t,s)^*z)-(G_r(s)u(s,\cdot)\,|\,A_r(s)^*U_r(t,s)^*z)\\ &=-(G_r(s)u(s,\cdot)\,|\,A_r(s)^*U_r(t,s)^*z). \end{aligned}$$

On the other hand, by (2.72), (2.73) we may write for $\vartheta\in]0,\alpha_r[$ (with $\alpha_0=\frac14$, $\alpha_1=\frac34$)

$$\begin{aligned} h'(s)&=([-A_r(s)]^\vartheta G_r(s)u(s,\cdot)\,|\,[-A_r(s)^*]^{1-\vartheta}U_r(t,s)^*z)\\ &=([[-A_r(s)^*]^{1-\vartheta}U_r(t,s)^*]^*[-A_r(s)]^\vartheta G_r(s)u(s,\cdot)\,|\,z), \qquad s\in]0,t[, \end{aligned}$$

and $h'\in L^2(0,t)$. Hence by integrating in $]0,t[$ we get

$$\begin{aligned} &(y(t,\cdot)\,|\,z)_{[L^2(\Omega)]^N}-(U_r(t,0)y_0(\cdot)\,|\,z)_{[L^2(\Omega)]^N}\\ &\quad=\left(\int_0^t [[-A_r(s)^*]^{1-\vartheta}U_r(t,s)^*]^*[-A_r(s)]^\vartheta G_r(s)u(s,\cdot)\,ds\,\Big|\,z\right)_{[L^2(\Omega)]^N}, \end{aligned}$$

and finally by density we deduce (2.74). □

*Remark* 2.14. (i) The representation formula (2.74) makes sense for any $y_0 \in [L^2(\Omega)]^N$ and $u \in [L^2(]0, T[\times \partial\Omega)]^N$, since by Proposition 2.2(i), (2.76), and (2.72), (2.73) we have

$$(2.78) \qquad \|y(t,\cdot)\|_{[L^2(\Omega)]^N} \leqq c\left\{\|y_0\|_{[L^2(\Omega)]^N} + \int_0^t (t-s)^{\vartheta-1}\|u(s,\cdot)\|_{[L^2(\partial\Omega)]^N}\,ds\right\},$$

which implies

$$(2.79) \qquad \|y\|^2_{[L^2(]0,T[\times\Omega)]^N} \leqq c\left\{T\|y_0\|^2_{[L^2(\Omega)]^N} + \frac{T^{2\vartheta}}{\vartheta^2}\|u\|^2_{[L^2(]0,T[\times\partial\Omega)]^N}\right\}.$$

(ii) We may rewrite formula (2.74) in a shorter, although improper, form, namely

$$(2.80) \qquad y(t) = U_r(t,0)y_0 - \int_0^t U_r(t,s)A_r(s)G_r(s)u(s)\,ds, \qquad t \in [0, T],$$

where the integrand is to be understood as in (2.74). In the foregoing section we will study an abstract version of (2.80) (see (3.1) below) within the context of control theory.

## 3. The L-Q-R problem over finite-time horizon.

**3.1. State problem and cost functional.** This section concerns the classical linear-quadratic regulator (L-Q-R) problem, over finite horizon $[0, T]$, for a class of abstract evolution equations corresponding to nonautonomous parabolic systems with boundary control. As we have shown in § 2, an equation of the form

$$(3.1) \qquad y(t) = U(t,0)y_0 - \int_0^t U(t,s)A(s)G(s)u(s)\,ds, \qquad t \in [0, T],$$

is appropriate to cover a wide class of concrete problems. In § 2 we derived in two concrete examples equation (2.80), which is an equation of the form (3.1), under hypotheses (2.1) and (2.2) (or, from the abstract point of view, (2.29) and (2.42)). Such assumptions will not be directly needed in most part of the next results on control problems; thus, in order to identify those properties which are really relevant from the control point of view, and to point out both analogies and novelties of the nonautonomous case with respect to the autonomous one (treated, e.g., in [B1], [La], [LT1], [LT2], [F1], [F2]), we will hereafter impose explicitly only assumptions (3.2)-(3.5) listed below.

Let $H$, $U$ two separable (for simplicity) complex Hilbert spaces. In (3.1) we shall take $y_0 \in H$ and $u \in L^2(0, T; U)$. Here is our list of hypotheses:

(3.2) $\{A(t)\}_{t\in[0,T]}$ is a family of closed linear operators in $H$ with (dense) domains $D_{A(t)}$, such that $A(t)$ generates an analytic semigroup in $H$ and $0 \in \rho(A(t))$.

(3.3) $\{U(t,s)\}_{0\leqq s\leqq t\leqq T}$ is the (strongly continuous) evolution operator in $H$ associated to $\{A(t)\}_{t\in[0,T]}$; in particular,

$$\|U(t,s)\|_{\mathscr{L}(H)} \leqq M_0, \text{ for all } (t,s)\in\bar{\Delta}, \text{ where } \Delta := \{(t,s)\in[0,T]^2 : s<t\}.$$

(3.4) The operator-valued function $(t,s) \to U(t,s)^*$ belongs to $C_s([0,T], \mathscr{L}(H))$; moreover, for each $\eta \in [0,1]$ and $(t,s) \in \Delta$, $U(t,s)^* \in \mathscr{L}(H, D_{[-A(s)^*]^\eta})$, the map $(t,s) \to [-A(s)^*]^\eta U(t,s)^*$ is strongly measurable and satisfies

$$\|[-A(s)^*]^\eta U(t,s)^*[-A(t)^*]^{-\mu}\|_{\mathscr{L}(H)} \leqq M_{\eta\mu}[(t-s)^{\mu-\eta}+1]$$
$$\forall (t,s)\in\Delta, \quad \forall \eta,\mu\in[0,1].$$

(3.5) $\{G(t)\}_{t\in[0,T]}$ is a family of operators in $\mathscr{L}(U, H)$ such that there exists $\alpha \in ]0, 1]$ with the following properties: $G(t) \in \mathscr{L}(U, D_{[-A(t)^*]^\alpha})$ for each $t \in [0, T]$ and the map $t \to [-A(t)]^\alpha G(t)$ belongs to $L^\infty(0, T; \mathscr{L}(U, H))$.

*Remark* 3.1. (i) The above assumptions can be relaxed in various directions, with minor consequences on the subsequent results. So, for instance, (3.4) is needed only for $\eta = 1-\alpha$: in this case we would obtain slightly weaker regularity results for the Riccati equation. However, the applications discussed in § 2 do not motivate a further level of generality.

(ii) Condition (3.4) with $\mu > 0$ is not necessary to give sense to equation (3.1): just a much weaker version of it is needed in order to define the L-Q-R problem (3.10) below. However it will be used in (more or less) this generality as a technical tool in the study of the Riccati equation. Except for (3.4) with $\mu > 0$, all the other assumptions are the natural (and minimal, in a sense) ones in order to give a meaning to equation (3.1) and problem (3.10).

(iii) In the examples of § 2, we have under assumptions (2.1), (2.2):

$H = [L^2(\Omega)]^N$, $U = [L^2(\partial\Omega)]^N$;
$\{A(t)\}$, defined by (2.12) or (2.13), fulfills (3.2) by Proposition 2.4;
The existence of $\{U(t, s)\}$ with the properties (3.3) is guaranteed by Proposition 2.8(i);
Conditions (3.4) for $\{U(t, s)^*\}$ are proved in Corollary 2.10;
$\{G(t)\}$, defined by (2.47) or (2.48), satisfies (3.5) in view of Proposition 2.12.

As at the end of § 2, we agree that the formal notation $U(t, s)A(s)G(s)$ stands for $[[-A(s)^*]^{1-\alpha}U(t, s)^*]^*[-A(s)]^\alpha G(s)$, which is well defined as an element of $\mathscr{L}(U, H)$ for each $(t, s) \in \Delta$, by (3.4)–(3.5). More precisely we have Lemma 3.2.

LEMMA 3.2. *The operator-valued function*

$$U(t, s)A(s)G(s) := [[-A(s)^*]^{1-\alpha}U(t, s)^*]^*[-A(s)]^\alpha G(s), \qquad 0 \leqq s < t \leqq T, \tag{3.6}$$

*is strongly measurable with respect to* $s \in [0, t[$ *for each fixed* $t \in ]0, T]$, *and strongly continuous with respect to* $t \in ]s, T]$ *for each fixed* $s \in [0, T[$. *Moreover,*

$$\| U(t, s)A(s)G(s)\|_{\mathscr{L}(U,H)} \leqq c(t-s)^{\alpha-1} \quad \forall (t, s) \in \Delta. \tag{3.7}$$

*Proof.* The first assertion follows directly by (3.4), (3.5). Concerning the second one, let $s \in [0, T[$ and $t_0 \in ]s, T]$ be fixed: it is easy to verify that if $t \in ](s+t_0)/2, T]$ we have

$$U(t, s)A(s)G(s) = U(t, (s+t_0)/2)[U((s+t_0)/2, s)A(s)G(s)];$$

but $t \to U(t, (s+t_0)/2)$ is strongly continuous, whereas the bounded operator $U((s+t_0)/2, s)A(s)G(s)$ does not depend on $t$, so that $U(t, s)A(s)G(s)$ is strongly continuous at $t = t_0$. Finally, the estimate (3.7) follows by (3.4) and (3.5). □

The next lemma gives a precise interpretation of the function (3.1).

LEMMA 3.3. (i) *If* $u \in L^2(0, T; U)$, *then* (3.1) *defines a function* $y \in L^2(0, T; H)$ *and*

$$\|y\|_{L^2(0,T;H)} \leqq c\{\|y_0\|_H + \|u\|_{L^2(0,T;U)}\}. \tag{3.8}$$

(ii) *If* $u \in L^p(0, T; U)$ *for some* $p > 1/\alpha$, *then* $y \in C([0, T], H)$ *and*

$$\|y\|_{C([0,T],H)} \leqq c\{\|y_0\|_H + \|u\|_{L^p(0,T;U)}\}. \tag{3.9}$$

*Proof.* Part (i) is an easy consequence of (3.3), (3.7) and Young's inequality.

(ii) If $p>1/\alpha$, by (3.7) we have for $0\leqq r<t\leqq T$

$$\left\|\int_r^t U(t,s)A(s)G(s)u(s)\,ds\right\|_H \leqq \left[\int_r^t c(t-s)^{-(1-\alpha)p/(p-1)}\,ds\right]^{(p-1)/p}\|u\|_{L^p(r,T;U)}$$

$$\leqq c\frac{p-1}{\alpha p-1}(t-r)^{\alpha-1/p}\|u\|_{L^p(0,T;U)},$$

which, together with (3.30), implies in particular (3.9). Moreover, if $t_0\in\,]0,T]$ and $\varepsilon>0$, we have for small $\delta>0$

$$\left\|\int_{t_0-\delta}^t U(t,s)A(s)G(s)u(s)\,ds\right\|_H \leqq \varepsilon \quad \forall t\in[t_0-\delta,t_0+\delta].$$

Therefore by (3.3) we get for $|t-t_0|\leqq\delta$:

$$\begin{aligned}&\|y(t)-y(t_0)\|_H\\ &\leqq\|U(t,0)y_0-U(t_0,0)y_0\|_H\\ &\quad+\left\|[U(t,t_0-\delta)-U(t_0,t_0-\delta)]\int_0^{t_0-\delta}U(t_0-\delta,s)A(s)G(s)u(s)\,ds\right\|_H\\ &\quad+\left\|\int_{t_0-\delta}^t U(t,s)A(s)G(s)u(s)\,ds\right\|_H+\left\|\int_{t_0-\delta}^{t_0}U(t_0,s)A(s)G(s)u(s)\,ds\right\|_H\\ &\leqq\|U(t,0)y_0-U(t_0,0)y_0\|_H+(2M_0+2)\varepsilon,\end{aligned}$$

and the result follows by the strong continuity of $t\to U(t,0)$. The case $t_0=0$ is even simpler. □

We can now define the optimal control problem which is the object of our study in this section. We shall consider the following L-Q-R problem:

(3.10) Minimize

$$J(u):=\int_0^T[(M(t)y(t)|y(t))_H+(N(t)u(t)|u(t))_U]\,dt+(P_Ty(T)|y(T))_H$$

over all controls $u\in L^2(0,T;U)$ subject to the state equation (3.1).

Here we assume:

(3.11) $M(t)\in\Sigma^+(H)$, for all $t\in[0,T]$ and $M\in L^\infty(0,T;\mathscr{L}(H))$;

(3.12) $N(t)\in\Sigma^+(U)$ with $N(t)\geqq\nu>0$, for all $t\in[0,T]$ and $N\in C_s([0,T],\mathscr{L}(U))$;

(3.13) $P_T\in\Sigma^+(H)$, and there exists $\beta\in\,](\frac{1}{2}-\alpha),\frac{1}{2}]\cap[0,\frac{1}{2}]$ such that $P_T\in\mathscr{L}(H,D_{[-A(T)^*]^{2\beta}})$.

*Remark* 3.4. Due to Lemma 3.3 of [F1], assumption (3.13) implies that the operator $[-A(T)^*]^{\beta-\varepsilon}P_T[-A(T)]^{\beta-\varepsilon}$ ($\varepsilon\in\,]0,\beta]$) can be extended to an operator $L_\varepsilon\in\mathscr{L}(H)$.

Note that $y$, given by (3.1), is not continuous in general, but only in $L^2(0,T;H)$: hence the term $(P_Ty(T)|y(T))_H$ is not well defined a priori for all controls $u\in L^2(0,T;U)$, but only for controls in a dense subspace of $L^2(0,T;U)$, by Lemma 3.3(ii). However, the regularity property (3.13), along with (3.4), yields Lemma 3.5.

LEMMA 3.5. *The mapping* $u\to(P_Ty(T)|y(T))_H$, *defined (for instance) from* $C([0,T],U)$ *into* $\mathbb{R}$, *is locally uniformly continuous with respect to the topology of* $L^2(0,T;U)$, *and hence it can be extended to* $L^2(0,T;U)$.

*Proof.* Let $L_\varepsilon$ be the bounded extension to $H$ of the operator $[-A(T)^*]^{\beta-\varepsilon}P_T[-A(T)]^{\beta-\varepsilon}$ (see Remark 3.4). If $u\in C([0,T],U)$, choosing $\varepsilon\in]0,\beta-(\frac{1}{2}-\alpha)[$ we have by (3.1), (3.3), (3.4), and (3.5)

$$
\begin{aligned}
&(P_Ty(T)|y(T))_H\\
&\quad=\|L_\varepsilon^{1/2}[-A(T)]^{-\beta+\varepsilon}y(T)\|_H^2\\
&\quad\leqq\|L_\varepsilon^{1/2}\|^2_{\mathscr{L}(H)}\Big\{\|[-A(T)]^{-\beta+\varepsilon}U(T,0)y_0\|_H\\
&\qquad\qquad+\int_0^T\|[[-A(s)^*]^{1-\alpha}U(T,s)^*[-A(T)^*]^{-\beta+\varepsilon}]^*[-A(s)]^\alpha G(s)u(s)\|_H\,ds\Big\}^2\\
&\quad\leqq c\Big\{\|y_0\|_H^2+\Big[\int_0^T[1+(T-s)^{\beta+\alpha-\varepsilon-1}]\|u(s)\|_U\,ds\Big]^2\Big\}\\
&\quad\leqq c\{\|y_0\|_H^2+(T+T^{2\beta+2\alpha-2\varepsilon-1})\|u\|^2_{L^2(0,T;U)}\}.
\end{aligned}
$$

Hence if $u_1,u_2\in C([0,T],U)$ and $y_1,y_2$ are the corresponding functions (3.1) with initial state $y_0$ we have

$$
\begin{aligned}
&|(P_Ty_1(T)|y_1(T))_H-(P_Ty_2(T)|y_2(T))_H|\\
&\quad\leqq[\|L_\varepsilon^{1/2}[-A(T)]^{-\beta+\varepsilon}y_1(T)\|_H+\|L_\varepsilon^{1/2}[-A(T)]^{-\beta+\varepsilon}y_2(T)\|_H]\\
&\qquad\cdot\|L_\varepsilon^{1/2}[-A(T)]^{-\beta+\varepsilon}[y_1(T)-y_2(T)]\|_H\\
&\quad\leqq c\{\|y_0\|_H+\|u_1\|_{L^2(0,T;U)}+\|u_2\|_{L^2(0,T;U)}\}\|u_1-u_2\|^2_{L^2(0,T;U)}. \qquad\square
\end{aligned}
$$

*Remark* 3.6. The initial state $y_0$ can be taken in a space larger than $H$ without changing the main results of this and subsequent sections. More precisely, we need to fulfill two essential requirements, namely (1°) $y\in L^2(0,T;H)$, and (2°) $[-A(T)]^{-\beta}y(T)$ is well defined; in order to get them, it is sufficient that $[-A(0)]^{-\delta}y_0\in H$ for some $\delta\in]0,\frac{1}{2}[$, i.e., that $y_0$ belongs to the dual of $D_{[-A(0)^*]^\delta}$ with respect to $H$ (indeed $[-A(0)]^{-\delta}$ can be extended to an isomorphism between the dual of $D_{[-A(0)^*]^\delta}$ and $H$). In this case the condition $y\in L^2(0,T;H)$ is satisfied because of (3.4) (with $\mu=0$), since we have

$$
\begin{aligned}
(U(t,0)y_0|x)_H&=([-A(0)]^{-\delta}y_0|[-A(0)^*]^\delta U(t,0)^*x)_H\\
&\leqq\|[-A(0)]^{-\delta}y_0\|_H M_{\delta 0}t^{-\delta}\|x\|_H\quad\forall x\in H,
\end{aligned}
$$

i.e., $\|U(t,0)y_0\|_H\leqq ct^{-\delta}$, $\delta\in]0,\frac{1}{2}[$; on the other hand, $[-A(T)]^{-\beta}y(T)$ is well defined (even if $\delta\in[\frac{1}{2},1[$) by (3.4), since

$$[-A(T)]^{-\beta}U(T,0)y_0=[[-A(0)^*]^\delta U(T,0)^*[-A(T)^*]^{-\beta}]^*[-A(0)]^{-\delta}y_0.$$

**3.2. The Riccati equation.** The main step in the solution of problem (3.10) is the direct study of the associated Riccati equation, which takes the form

$$
\begin{aligned}
P(t)&=U(T,t)^*P_TU(T,t)+\int_t^T U(s,t)^*\\
&\qquad\cdot[M(s)-P(s)A(s)G(s)N(s)^{-1}G(s)^*A(s)^*P(s)]U(s,t)\,ds.
\end{aligned}\tag{3.14}
$$

The nonlinear term in (3.14) is not well defined in the present form. For this reason we consider the following version of (3.14):

$$P(t) = U(T, t)^* P_T U(T, t) + \int_t^T U(s, t)^* \cdot [M(s) - [[-A(s)^*]^{1-\alpha} P(s)]^* K(s) [-A(s)^*]^{1-\alpha} P(s)] U(s, t)\, ds, \tag{3.15}$$

where

$$K(s) := [-A(s)]^\alpha G(s) N(s)^{-1} [[-A(s)]^\alpha G(s)]^*. \tag{3.16}$$

By (3.5) and (3.12) we have

$$K(\cdot) \in L^\infty(0, T; \Sigma^+(H)). \tag{3.17}$$

Note that the integration in (3.15) is performed in the strong sense.

PROPOSITION 3.7 (local solution). *There exist an interval* $[T_0, T]$ *and a unique function* $P \in C_s([T_0, T], \Sigma(H))$ *such that*:

(i) $[-A(\cdot)^*]^{1-\alpha} P(\cdot)$ *is well defined and strongly measurable from* $[T_0, T]$ *into* $\mathscr{L}(H)$,

(ii) $\|[-A(t)^*]^{1-\alpha} P(t)\|_{\mathscr{L}(H)} \leqq c(T-t)^{-(1-\alpha-2\beta)\vee 0}$, *for all* $t \in [T_0, T[$,

(iii) $P(\cdot)$ *solves* (3.15) *in* $[T_0, T]$.

*Proof.* For any $T_0 \in [0, T[$ denote by $B_\gamma(T_0, T)$ the Banach space of all strongly measurable functions $Q : [T_0, T[ \to \mathscr{L}(H)$ such that

$$\|Q\|_{B_\gamma(T_0,T)} := \sup_{T_0 \leqq t \leqq T} (T-t)^\gamma \|Q(t)\|_{\mathscr{L}(H)} < \infty,$$

where $\gamma := (1-\alpha-2\beta) \vee 0$. For $Q \in B_\gamma(T_0, T)$, define

$$\Gamma_{T_0}(Q)(t) := [-A(t)^*]^{1-\alpha} U(T, t)^* P_T U(T, t) + \int_t^T [-A(t)^*]^{1-\alpha} U(s, t)^* [M(s) - Q(s)^* K(s) Q(s)] U(s, t)\, ds, \quad t \in [T_0, T].$$

Let us show that $\Gamma_{T_0}$ maps $B_\gamma(T_0, T)$ into itself. By (3.4), (3.13), (3.3), (3.11), and (3.17) we get

$$\begin{aligned}
&\|\Gamma_{T_0}(Q)(t)\|_{\mathscr{L}(H)} \\
&\leqq M_{1-\alpha,2\beta}[1 + (T-t)^{2\beta+\alpha-1}] \|[-A(T)^*]^{2\beta} P_T\|_{\mathscr{L}(H)} \|U(T, t)\|_{\mathscr{L}(H)} \\
&\quad + M_{1-\alpha,0} \int_t^T (s-t)^{\alpha-1} [\|M(s)\|_{\mathscr{L}(H)} \\
&\quad + \|K(s)\|_{\mathscr{L}(H)} (T-s)^{-2\gamma} \|Q\|^2_{B_\gamma(T_0,T)}] \|U(s, t)\|_{\mathscr{L}(H)}\, ds \\
&\leqq c[(T-t)^{-\gamma} + (T-t)^\alpha + (T-t)^{\alpha-2\gamma} \|Q\|^2_{B_\gamma(T_0,T)}] \quad \forall t \in [T_0, T[;
\end{aligned}$$

this shows that $\Gamma_{T_0}(Q) \in B_\gamma(T_0, T)$ and

$$\|\Gamma_{T_0}(Q)\|_{B_\gamma(T_0,T)} \leqq c_1 + c_2 (T - T_0)^{\alpha-\gamma} \|Q\|^2_{B_\gamma(T_0,T)}. \tag{3.18}$$

Next, we show that $\Gamma_{T_0}$ is a contradiction in the ball

$$B_\gamma(T_0, T; \rho) := \{Q \in B_\gamma(T_0, T) : \|Q\|_{B_\gamma(T_0,T)} \leqq \rho\}$$

for a suitable $\rho > 0$. Indeed if $Q_1, Q_2 \in B_\gamma(T_0, T; \rho)$ we have as before:

$$\|\Gamma_{T_0}(Q_1)(t) - \Gamma_{T_0}(Q_2)(t)\|_{\mathscr{L}(H)}$$
$$\leqq c \int_t^T (s-t)^{\alpha-1}[\|Q_1(s)\|_{\mathscr{L}(H)} + \|Q_2(s)\|_{\mathscr{L}(H)}]\|Q_1(s) - Q_2(s)\|_{\mathscr{L}(H)}\, ds$$
$$\leqq c(T-t)^{\alpha-2\gamma}\rho\|Q_1 - Q_2\|_{B_\gamma(T_0,T)}, \quad \forall t \in [T_0, T[,$$

which implies

$$\|\Gamma_{T_0}(Q_1) - \Gamma_{T_0}(Q_2)\|_{B_\gamma(T_0,T)} \leqq c\rho(T-t)^{\alpha-\gamma}\|Q_1 - Q_2\|_{B_\gamma(T_0,T)}. \tag{3.19}$$

By (3.18) and (3.19) we see that it is possible to choose a (large) $\rho > 0$ and a $T_0 \in [0, T[$ (close to $T$), such that $\Gamma_{T_0}$ maps $B_\gamma(T_0, T; \rho)$ into itself and is a contraction in $B_\gamma(T_0, T; \rho)$. Thus we get a unique solution of the equation

$$Q = \Gamma_{T_0}(Q) \quad \text{in } [T_0, T[.$$

Hence $P := [-A(\cdot)^*]^{\alpha-1}Q(\cdot)$ is the unique operator-valued function that satisfies (i)–(iii). The property $P \in C_s([T_0, T], \mathscr{L}(H))$ follows by (3.15), whereas the property $P(t) \in \Sigma(H)$ is a consequence of the fact that $P(\cdot)^*$ is also in $C_s([T_0, T], \mathscr{L}(H))$ and satisfies (i)–(iii), so that $P(t)^* \equiv P(t)$ in $[T_0, T]$. □

*Remark* 3.8. Since $[-A(T_0)]^{1-\alpha}P(T_0) \in \mathscr{L}(H)$, for each $\beta \in ]\frac{1}{2} - \alpha, (1-\alpha)/2[ \cap [0, (1-\alpha)/2[$ the operator $[-A(T_0)^*]^\beta P(T_0)[-A(T_0)]^\beta$ is continuous with respect to the topology of $H$ (see [F1, Lemma 3.3]).

The result of Proposition 3.7 justifies the following definition:

DEFINITION 3.9. Let $J$ be an interval in $[0, T]$ such that $T \in J$. We say that $P$ is a solution of (3.15) in $J$ if:

(i) $P \in C_s(J, \Sigma(H))$, $[-A(\cdot)^*]^{1-\alpha}P(\cdot)$ is well defined and strongly measurable from $J$ into $\mathscr{L}(H)$,

(ii) For each $\tau \in J\backslash\{T\}$ there exists a constant $c(\tau)$ such that

$$\|[-A(t)^*]^{1-\alpha}P(t)\|_{\mathscr{L}(H)} \leqq c(\tau)(T-t)^{-\gamma} \quad \forall t \in [\tau, T[,$$

where $\gamma = (1 - \alpha - 2\beta) \vee 0$,

(iii) $P(\cdot)$ satisfies (3.15) in $J$.

We must prove the existence and uniqueness of a global solution, i.e., a solution in $[0, T]$, of (3.15). The proof will be based on an a priori bound; to this purpose we introduce an evolution operator which will be related to the optimal trajectories of problem (3.10).

LEMMA 3.10. *Let $P$ be a solution of* (3.15) *in $J$. Consider the integral equation*

$$\Phi(t,s)x = U(t,s)x + \int_s^t [[-A(r)^*]^{1-\alpha}U(t,r)^*]^*K(r) \cdot [[-A(r)^*]^{1-\alpha}P(r)]^*\Phi(r,s)x\, dr, \qquad t \in J, \tag{3.20}$$

*where $x \in H$. Then there exists a unique operator-valued function $\Phi: \overline{\Delta_J} \to \mathscr{L}(H)$, with $\Delta_J := \{(t,s) \in J^2 : t > s\}$, such that $\Phi(t,s)x$ is a solution in $C([s,T], H)$ of* (3.20) *for each $x \in H$ and $s \in J$. Moreover, $\Phi$ is a strongly continuous evolution operator.*

*Proof.* Fix $s \in J$. If $\Phi(\cdot, s)x \in C([s, T], H)$, then

$$[-A(\cdot)^*]^{1-\alpha}P(\cdot)\Phi(\cdot, s)x \in L^p(s, T; H)$$

for some $p > 1/\alpha$ (by Definition 3.9(ii)); thus (3.16), (3.5), (3.12), and Lemma 3.3(i) imply that the right-hand side of (3.20) is in $C([s, T], H)$. Therefore it is standard to apply the contraction principle to (3.20), in order to get existence of a unique solution of (3.20) in $C([s, T], H)$, denoted by $\Phi(\cdot, s)x$. The proof that $\Phi$ is a strongly continuous evolution operator is classical. □

Using the evolution operator $\Phi$ it is possible to rewrite the Riccati equation (3.15) in two alternative integral forms as follows.

LEMMA 3.11. *If $P$ is a solution of* (3.15) *in $J$ and $\bar{T} \in J$, then for each $t \in J \cap [0, \bar{T}]$*

$$P(t) = \Phi(\bar{T}, t)^* P(\bar{T}) \Phi(\bar{T}, t) + \int_t^{\bar{T}} \Phi(s, t)^* \cdot [M(s) + [[-A(s)^*]^{1-\alpha} P(s)]^* K(s) [-A(s)^*]^{1-\alpha} P(s)] \Phi(s, t)\, ds, \tag{3.21}$$

$$P(t) = U(\bar{T}, t)^* P(\bar{T}) \Phi(\bar{T}, t) + \int_t^{\bar{T}} U(\sigma, t)^* M(\sigma) \Phi(\sigma, t)\, d\sigma. \tag{3.22}$$

*Proof.* The proof is classical (see, e.g., [Gi], [LT1]). □

We are now able to prove the following a priori bound, which is the key point in showing global existence.

LEMMA 3.12. *There exists $c > 0$ with the following property: if $P$ is a solution of* (3.15) *in some interval $J$, then*

$$\|[-A(t)^*]^{1-\alpha} P(t)\|_{\mathscr{L}(H)} \leqq c(T-t)^{-\gamma} \quad \forall t \in J \setminus \{T\}, \tag{3.23}$$

*where $\gamma = (1 - \alpha - 2\beta) \vee 0$.*

*Proof.* Of course (3.23) is obvious if $t \in J \cap [T_0, T[$, with $T_0$ given by Proposition 3.7. Thus we may confine ourselves to consider the interval $J \cap [0, T_0]$.

Our first step consists in showing that there exists $c > 0$, independent of $P$ and $J$, such that

$$\|P(t)\|_{\mathscr{L}(H)} \leqq c \quad \forall t \in J. \tag{3.24}$$

Indeed, choose $\bar{T} = T$ in Lemma 3.11: by (3.21) we have

$$P(t) \geqq 0 \quad \forall t \in J; \tag{3.25}$$

moreover, by (3.15) we get

$$(P(t)x|x)_H \leqq \|P_T^{1/2} U(T, t)x\|_H^2 + \int_t^T \|M(s)^{1/2} U(s, t)x\|_H^2\, ds$$
$$\leqq c\|x\|_H^2 \quad \forall x \in H, \quad \forall t \in J,$$

with $c$ independent of $P$ and $J$. Thus (3.24) follows by (3.25). Next, by (3.22) we deduce for $s, t \in J \cap [0, T_0]$, $s \leqq t$:

$$\begin{aligned}[-A(t)^*]^{1-\alpha} P(t) \Phi(t, s) &= [-A(t)^*]^{1-\alpha} U(T_0, t)^* P(T_0) \Phi(T_0, s) \\ &\quad + \int_t^{T_0} [-A(t)^*]^{1-\alpha} U(\sigma, t)^* M(\sigma) \Phi(\sigma, s)\, d\sigma \\ &=: I_1(t, s) + I_2(t, s),\end{aligned} \tag{3.26}$$

where $T_0$ is taken as in Proposition 3.7. By (3.21) and (3.24) we obtain a first estimate:

$$\int_s^{T_0} \|K(t)^{1/2} [-A(t)^*]^{1-\alpha} P(t) \Phi(t, s)x\|_H^2\, dt \leqq c\|x\|_H^2 \quad \forall x \in H, \quad \forall s \in J \cap [0, T_0], \tag{3.27}$$

with $c$ independent of $s \in J \cap [0, T_0]$ and $x \in H$.

The proof now proceeds in the following manner. Starting from (3.27), we will apply a bootstrap process in order to get more and more summability for the function $[-A(\cdot)^*]^{1-\alpha}P(\cdot)\Phi(\cdot, s)x$ in the interval $J\cap[s, T_0]$, where $s\in J\cap[0, T_0]$. Our final goal is the estimate

$$(3.28)\quad \|[-A(t)^*]^{1-\alpha}P(t)\Phi(t, s)x\|_H\leqq c\|x\|_H \quad \forall x\in H, \quad \forall s, t\in J\cap[0, T_0], \quad s\leqq t,$$

with $c$ independent of $x$, $P$, $s$, $t$, and $J$: choosing in (3.28) $s=t$, (3.23) will follow, thus completing the proof of Lemma 3.12.

The bootstrap procedure works as follows. Let $p\in[2, \infty[$ be given, and set

$$p_0:=\begin{cases}\dfrac{p}{1-\alpha p} & \text{if } p\in\left[2, \dfrac{1}{\alpha}\right],\\[2mm] +\infty & \text{if } p\geqq\dfrac{1}{\alpha}.\end{cases}$$

Clearly, if $\alpha\in[\frac{1}{2}, 1]$ we have $1/\alpha\leqq 2$ so that $p_0=+\infty$ whatever be $p$. If otherwise $\alpha\in]0, \frac{1}{2}[$, then

$$(3.29)\qquad p_0-p=\frac{\alpha p^2}{1-\alpha p}\geqq\frac{4\alpha}{1-2\alpha}>0 \quad \forall p\in\left[2, \frac{1}{\alpha}\right].$$

Assuming the truth of the estimate ($c$ independent of $x$, $P$, $s$, $J$)

$$(3.30)\quad \|[-A(\cdot)^*]^{1-\alpha}P(\cdot)\Phi(\cdot, s)x\|_{L^p(s,T_0;H)}\leqq c\|x\|_H \quad \forall x\in H, \quad \forall s\in J\cap[0, T_0],$$

we will prove the same estimate with $p$ replaced by $p_0$. This argument starts with $p=2$, in which case we assume (3.27) instead of (3.30), and stops after a finite number of iterations (by virtue of (3.29)), the final estimate being (3.28). Suppose that (3.30) holds for a certain $p\geqq 2$: by (3.26) it is enough to show that

$$(3.31)\qquad \|I_1(\cdot, s)\|_{L^{p_0}(s,T_0;H)}\leqq c\|x\|_H,$$

$$(3.32)\qquad \|I_2(\cdot, s)\|_{L^{p_0}(s,T_0;H)}\leqq c\|x\|_H.$$

Concerning (3.31), by (3.27), using (3.3), (3.4), and (3.17) we get

$$\int_s^{T_0}\|\Phi(t, s)x\|_H^{p_0}\,dt$$

$$\leqq c\left\{\|x\|_H^{p_0}+\int_s^{T_0}\left[\int_s^t (t-r)^{\alpha-1}\|[-A(r)^*]^{1-\alpha}P(r)\Phi(r, s)x\|_H\,dr\right]^{p_0}dt\right\}$$

and a Young-type estimate [HLP, Thm. 383], along with our assumption (3.30), yields

$$\int_s^{T_0}\|\Phi(t, s)x\|_H^{p_0}\,dt\leqq c\|x\|_H^{p_0},$$

with $c$ independent of $P$ and $J$ (with obvious modifications if $p_0=\infty$). The bound (3.31) now follows by (3.4) and (3.11), applying the simplest version of Young's inequality.

Let us verify (3.32). First we observe that for any $\eta\in[0, 1[$ and $\varepsilon\in]0, 1-\eta[$ the operator $[-A(T_0)^*]^{1-\eta-\varepsilon}P(T_0)[-A(T_0)]^{\eta}$ can be uniquely extended to a bounded linear operator in $H$. Indeed, by (3.15) and (3.13) we have for each $x \in D_{[-A(T_0)]^\eta}$

$$\begin{aligned}
&[-A(T_0)^*]^{1-\eta-\varepsilon} P(T_0)[-A(T_0)]^\eta x \\
&\quad = [[-A(T_0)^*]^{1-\eta-\varepsilon} U(T, T_0)^*[-A(T)^*]^{-2\beta}][[-A(T)^*]^{2\beta} P_T] \\
&\qquad \cdot [[-A(T_0)^*]^\eta U(T, T_0)^*]^* x + \int_{T_0}^{T} [-A(T_0)^*]^{1-\eta-\varepsilon} U(s, T_0)^* \\
&\qquad \cdot [M(s) - [[-A(s)^*]^{1-\alpha} P(s)]^* K(s)[-A(s)^*]^{1-\alpha} P(s)] \\
&\qquad \cdot [[-A(T_0)^*]^\eta U(s, T_0)^*]^* x \, ds,
\end{aligned}$$

and hence by (3.4), (3.13), (3.11), and Proposition 3.7

$$\begin{aligned}
&\|[-A(T_0)^*]^{1-\eta-\varepsilon} P(T_0)[-A(T_0)]^\eta x\|_H \\
&\quad \leqq c\left\{[1+(T-T_0)^{2\beta-1+\eta+\varepsilon}][1+(T-T_0)^{-\eta}] + \int_{T_0}^{T} (s-T_0)^{\varepsilon-1}(T-s)^{-2\gamma}\, ds\right\} \|x\|_H \\
&\quad \leqq c\|x\|_H
\end{aligned}$$

(with $c$ independent of $x$, $P$, $J$). Moreover, if $\eta > 1/p - \alpha$ we have by (3.20), (3.3), (3.4), (3.17), and (3.30)

$$\begin{aligned}
\|[-A(T_0)]^{-\eta}\Phi(T_0, s)x\|_H &\leqq c\|x\|_H + c\left[\int_s^{T_0} (T_0 - r)^{(\alpha-1+\eta)p/(p-1)}\, dr\right]^{(p-1)/p} \\
&\quad \cdot \|[-A(\cdot)]^{1-\alpha} P(\cdot)\Phi(\cdot, s)x\|_{L^p(0,T_0;H)} \leqq c\|x\|_H
\end{aligned}$$

($c$ independent of $x$, $P$, $s$, $J$). Therefore we can rewrite $I_1(t, s)$ as

$$\begin{aligned}
I_1(t, s) &= [[-A(t)^*]^{1-\alpha} U(T_0, t)^*[-A(T_0)^*]^{\eta+\varepsilon-1}] \\
&\quad \cdot [[-A(T_0)^*]^{1-\eta-\varepsilon} P(T_0)[-A(T_0)]^\eta][[-A(T_0)]^{-\eta}\Phi(T_0, s)];
\end{aligned}$$

hence if we take $\eta \in ]1/p - \alpha, 1/p[$ and $\varepsilon \in ]0, 1/p[$ we see using (3.4) that

$$\begin{aligned}
&\int_s^{T_0} \|[-A(t)^*]^{1-\alpha} U(T_0, t)^*[-A(T_0)^*]^{\eta+\varepsilon-1} x\|_H^{p_0}\, dt \\
&\qquad \leqq c\int_s^{T_0} (T_0 - t)^{(\alpha-\eta-\varepsilon)p_0}\, dt \|x\|_H^{p_0} \leqq c\|x\|_H^{p_0},
\end{aligned}$$

since $(\eta + \varepsilon - \alpha)p_0 < 1$. Thus we immediately obtain (3.32). Hence we get (3.30) with $p$ replaced by $p_0$; consequently (3.28) follows, and the proof of Lemma 3.12 is complete. □

We can now prove the main result of this section.

THEOREM 3.13 (global solution). *There exists a unique solution $P$ of equation* (3.15) *in* $[0, T]$. *Moreover, it has the following properties*:

(i) $P(t) \geqq 0$ *for each* $t \in [0, T]$;

(ii) *$P$ satisfies the integral Riccati equations* (3.21) *and* (3.22);

(iii) *$P$ satisfies the bounds* (3.24) *in* $[0, T]$ *and* (3.23) *in* $[0, T[$;

(iv) *For each* $\eta \in [0, 1[$, *the linear operator* $[-A(t)^*]^\eta P(t)$, $t \in [0, T[$, *is well defined, strongly measurable in t, and equibounded on compact subsets of* $[0, T[$.

*Proof.* Let $T_0$ be given by Proposition 3.7. For each $T_1 \in [0, T_0[$, consider the Banach space $L^\infty(T_1, T_0; \mathscr{L}(H))$ and the balls

$$B(T_1, T_0; \rho) := \{Q \in L^\infty(T_1, T_0; \mathscr{L}(H)) : \|Q\|_{L^\infty(T_1,T_0;\mathscr{L}(H))} \leqq \rho\}, \qquad \rho > 0.$$

Define the mapping $\Gamma_{T_1,T_0}$ on $L^\infty(T_1, T_0; \mathscr{L}(H))$ by

$$\Gamma_{T_1,T_0}(Q)(t) := [[-A(t)^*]^{1-\alpha}U(T_0, t)^*[-A(T_0)^*]^{\alpha-1}][[-A(T_0)^*]^{1-\alpha}P(T_0)U(T_0, t)] + \int_t^{T_0} [-A(t)^*]^{1-\alpha}U(s, t)^*[M(s)-Q(s)^*K(s)Q(s)]U(s, t)\,ds, \quad t\in[T_1, T_0[,$$

$$\Gamma_{T_1,T_0}(Q)(T_0) := [-A(T_0)^*]^{1-\alpha}P(T_0).$$

As in the proof of Proposition 3.7, we have

$$\|\Gamma_{T_1,T_0}(Q)\|_{L^\infty(T_1,T_0;\mathscr{L}(H))} \leqq c_1\|[-A(T_0)^*]^{1-\alpha}P(T_0)\|_{\mathscr{L}(H)} + c_2 + c_3(T_0-T_1)^\alpha\|Q\|^2_{L^\infty(T_1,T_0;\mathscr{L}(H))}, \quad \forall Q\in L^\infty(T_1, T_0; \mathscr{L}(H)), \tag{3.33}$$

$$\|\Gamma_{T_1,T_0}(Q_1)-\Gamma_{T_1,T_0}(Q_2)\|_{L^\infty(T_1,T_0;\mathscr{L}(H))} \leqq c_4\rho(T_0-T_1)^\alpha\|Q_1-Q_2\|_{L^\infty(T_1,T_0;\mathscr{L}(H))} \quad \forall Q_1, Q_2\in L^\infty(T_1, T_0; \mathscr{L}(H)); \tag{3.34}$$

where $c_1, \cdots, c_4$ are constants independent of $T_1$, $T_0$. Using the a priori bound (3.23), by (3.33) and (3.34) we see that we can select $\rho>0$ and $T_1\in]0, T_0[$ such that:

(a) $T_0-T_1$ and $\rho$ are independent of $T_0$;

(b) $\Gamma_{T_1,T_0}$ is a contraction which maps $B(T_1, T_0; \rho)$ into itself. Thus there exists a unique solution $Q$ of the equation

$$Q=\Gamma_{T_1,T_0}(Q)$$

in the space $L^\infty(T_1, T_0; \mathscr{L}(H))$, and this procedure can be repeated in the interval $[T_1-(T_0-T_1), T_1]$, and so on, with constant step. As in the proof of Proposition 3.7, we conclude that there exists a unique solution $P$ of (3.15) in $[0, T]$.

Finally, property (i) follows by (3.25), and similarly properties (ii) and (iii) are proved in Lemmas 3.11 and 3.12. As to (iv), it is sufficient to use (3.4), (3.11) and the last assertion of Lemma 3.10 in equation (3.22) with $\bar{T}=T$. □

**3.3. Synthesis.** The results of the preceding section lead to the following theorem.

THEOREM 3.14. *Let $y_0\in H$ be given. Then*:

(i) *There exists a unique optimal control $\hat{u}_0\in L^2(0, T; U)$ for problem* (3.10);

(ii) *Denoting by $P(\cdot)$ the solution of the Riccati equation* (3.15), *we have*

$$J(\hat{u}_0)=(P(0)y_0|y_0)_H; \tag{3.35}$$

(iii) *If $\hat{y}_0\in L^2(0, T; H)$ is the optimal trajectory, i.e., the solution of the state equation* (3.1) *corresponding to $\hat{u}_0(\cdot)$, we have the feedback formula for $\hat{u}_0(\cdot)$*:

$$\hat{U}_0(t)=N(t)^{-1}G(t)^*A(t)^*P(t)\hat{Y}_0(t), \qquad t\in[0, T[; \tag{3.36}$$

(iv) *The optimal trajectory $\hat{y}_0(\cdot)$ is expressed by*

$$\hat{y}_0(t)=\Phi(t, 0)y_0, \tag{3.37}$$

*where $\Phi(t, s)$ is defined by the integral equation* (3.20) *with $J=[0, T]$*;

(v) *The optimal pair $(\hat{u}_0, \hat{y}_0)$ is characterized by the following optimality system*:

$$\begin{aligned}&\hat{Y}_0(t)=U(t, 0)Y_0-\int_0^t U(t, s)A(s)G(s)\hat{U}_0(s)\,ds,\\ &\hat{u}_0(t)=N(t)^{-1}G(t)^*A(t)^*p(t), \qquad t\in[0, T[,\\ &p(t)=U(T, t)^*P_T\hat{y}_0(T)+\int_t^T U(s, t)^*M(s)\hat{y}_0(s)\,ds.\end{aligned} \tag{3.38}$$

In (3.36) and (3.38) we have set

$$G(t)^*A(t)^* := -[[-A(t)]^\alpha G(t)]^*[-A(t)^*]^{1-\alpha}; \tag{3.39}$$

as both operators $P(t)$, $U(r, t)^*$ (with $r > t$) have their range contained in $D_{[-A(t)^*]^\eta}$ for each $\eta \in [0, 1[$, both (3.36) and (3.38) are meaningful.

*Proof.* Recalling (3.39), set

$$\hat{u}_0(t) := N(t)^{-1}G(t)^*A(t)^*P(t)\Phi(t, 0)Y_0; \tag{3.40}$$

note that $\hat{u}_0 \in L^2(0, T; U)$ because of (3.23) (since $2(1-\alpha-2\beta)<1$) and Lemma 3.10 (with $J=[0, T]$). Now let $\hat{y}_0(\cdot)$ be the function (3.1) corresponding to $\hat{u}_0(\cdot)$. Then $\hat{y}_0 \in L^2(0, T; H)$ by Lemma 3.3(i). Moreover, comparing (3.1) with (3.20), and taking into account (3.16), we see that (3.37) holds. Consequently (3.40) implies (3.36). In addition, evaluating $P(0)Y_0$ by means of (3.21) with $\bar{T}=T$, we easily check that (3.35) also holds. Next, setting $p(t) := P(t)\hat{Y}_0(t)$, (3.40) and (3.37) immediately yield

$$\hat{U}_0(t) = N(t)^{-1}G(t)^*A(t)^*p(t);$$

on the other hand, by (3.37) and (3.22) with $\bar{T}=T$ we obtain the last equation in (3.38), so that the pair $(\hat{u}_0, \hat{y}_0)$ satisfies (3.38).

In order to conclude the proof of the theorem, it is sufficient to show that:

(a) If $(\hat{u}_0, \hat{y}_0)$ is a solution of the system (3.38) in $L^2(0, T; U) \times L^2(0, T; H)$, then $\hat{U}_0$ is an optimal control;

(b) The optimal control is unique.

From the equality

$$(z_1|z_1)-(z_2|z_2)=(z_1-z_2|z_1-z_2)+2\,\mathrm{Re}\,(z_2|z_1-z_2),$$

which holds true for any inner product, we derive for each $u \in L^2(0, T; U)$, denoting by $y(\cdot)$ the corresponding function (3.1)

$$J(u)-J(\hat{u}_0)=I_1(u, \hat{u}_0)+I_2(u, \hat{u}_0), \tag{3.41}$$

where $J(\cdot)$ is the cost functional appearing in (3.10) and

$$\begin{aligned} I_1(u, \hat{u}_0) := \int_0^T &\{(M(t)[y(t)-\hat{Y}_0(t)]|y(t)-\hat{Y}_0(t))_H \\ &+(N(t)[u(t)-\hat{u}_0(t)]|u(t)-\hat{u}_0(t))_U\}\,dt \\ &+(P_T[y(T)-\hat{Y}_0(T)]|y(T)-\hat{Y}_0(T))_H, \end{aligned}$$

$$\begin{aligned} I_2(u, \hat{u}_0) := 2\,\mathrm{Re} \int_0^T &\{(M(t)\hat{Y}_0(t)|y(t)-\hat{Y}_0(t))_H+(N(t)\hat{u}_0(t)|u(t)-\hat{u}_0(t))_U\}\,dt \\ &+2\,\mathrm{Re}\,(P_T\hat{Y}_0(T)|y(T)-\hat{Y}_0(T))_H. \end{aligned}$$

Now, using (3.1) and integrating by parts,

$$\begin{aligned} I_2(u, \hat{u}_0) = 2\,\mathrm{Re} \int_0^T \Big\{ -\int_s^T &(M(t)\hat{Y}_0(t)|U(t, s)A(s)G(s)[u(s)-\hat{u}_0(s)])_H\,dt \\ &+(N(s)\hat{u}_0(s)|u(s)-\hat{u}_0(s))_U \\ &-(P_T\hat{Y}_0(T)|U(T, s)A(s)G(s)[u(s)-\hat{u}_0(s)])_H \Big\}\,ds, \end{aligned}$$

and by the last two identities in (3.38) we easily get

$$I_2(u, \hat{u}_0) = 2\,\mathrm{Re} \int_0^T (-G(s)^* A(s)^* p(s) + N(s)\hat{u}_0(s) \,|\, u(s) - \hat{u}_0(s))_U \, ds = 0.$$

On the other hand, clearly, $I_1(u, \hat{u}_0) \geqq 0$, so that (3.41) yields

$$J(u) \geqq J(\hat{u}_0) \quad \forall u \in L^2(0, T; U),$$

i.e., $\hat{u}_0$ is an optimal control. This proves (a).

Finally, if $\bar{u}$ is another optimal control, the equality $J(\hat{u}_0) = J(\bar{u})$ implies

$$I_1(\bar{u}, \hat{u}_0) = 0,$$

and by the uniform coerciveness of $N(t)$ (see (3.12)) we obtain $\bar{u} = \hat{u}_0$. This proves (b). The proof of Theorem 3.14 is complete. □

## REFERENCES


[Ac] P. Acquistapace, *Evolution operators and strong solutions of abstract linear parabolic equations*, Differential and Integral Equations, 1 (1988), pp. 433–457.

[AT1] P. Acquistapace and B. Terreni, *A unified approach to abstract linear non-autonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova, 78 (1987), pp. 47–107.

[AT2] ———, *On fundamental solutions for abstract parabolic equations*, in Differential Equations in Banach Spaces, Proc. Bologna 1985, A. Favini and E. Obrecht, eds., Lecture Notes in Mathematics, Vol. 1223, Springer-Verlag, Berlin, Heidelberg, 1986, pp. 1–11.

[AT3] ———, *On quasilinear parabolic systems*, Math. Ann., 282 (1988), pp. 315–335.

[ADN] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary of solutions of elliptic partial differential equations satisfying general boundary conditions*, II, Comm. Pure Appl. Math., 17 (1964), pp. 35–92.

[Am] H. Amann, *Existence and regularity for semilinear parabolic evolution equations*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4), 11 (1984), pp. 593–676.

[B1] V. A. Balakrishnan, *Boundary control of parabolic equations:* L-Q-R *theory*, in Theory of Nonlinear Operators, Proc. 5th Internat. Summer School, Berlin 1977, R. Kluge, ed., Abh. Akad. Wiss. DDR, Abh. Math. Naturwiss. Tech., 6N (1978), pp. 11–23.

[B2] ———, *Applied Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

[D] G. Da Prato, *Synthesis of optimal control for an infinite-dimensional periodic problem*, SIAM J. Control Optim., 25 (1987), pp. 706–714.

[DI1] G. Da Prato and A. Ichikawa, *Riccati equations with unbounded coefficients*, Ann. Mat. Pura Appl., 140 (1985), pp. 209–221.

[DI2] ———, *Bounded solutions on the real line to non-autonomous Riccati equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 79 (1985), pp. 107–112.

[DLT] G. Da Prato, I. Lasiecka, and R. Triggiani, *A direct study of the Riccati equation arising in hyperbolic boundary control problems*, J. Differential Equations, 64 (1986), pp. 26–47.

[DS] M. C. Delfour and M. Sorine, *The linear quadratic optimal control problem for parabolic systems with boundary control through a Dirichlet condition*, in Control of Distributed Parameter Systems, Proc. 3rd IFAC Symposium, Toulouse 1982, J. P. Babary and L. Le Letty eds., Pergamon Press, Oxford, 1983, pp. 87–90.

[Fa] H. O. Fattorini, *Boundary control systems*, SIAM J. Control, 6 (1968), pp. 349–385.

[F1] F. Flandoli, *Riccati equation arising in a boundary control problem with distributed parameters*, SIAM J. Control Optim., 22 (1984), pp. 76–86.

[F2] ———, *Algebraic Riccati equation arising in boundary control problems*, SIAM J. Control Optim., 25 (1987), pp. 612–636.

[GG] G. Geymonat and P. Grisvard, *Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici*, Rend. Sem. Mat. Univ. Padova, 38 (1967), pp. 121–173.

[Gi] J. S. Gibson, *The Riccati integral equations for optimal control problems on Hilbert spaces*, SIAM J. Control Optim., 17 (1979), pp. 537–565.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[Gr] P. GRISVARD, *Équations différentielles abstraites*, Ann. Sci. École Norm. Sup. (4), 2 (1969), pp. 311–395.

[HLP] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[I] A. ICHIKAWA, *Filtering and control of stochastic differential equations with unbounded coefficients*, Stochastic Anal. Appl., 4 (1986), pp. 187–212.

[La] I. LASIECKA, *Unified theory for abstract parabolic boundary problems—a semigroup approach*, Appl. Math. Optim., 6 (1980), pp. 287–383.

[LT1] I. LASIECKA AND R. TRIGGIANI, *Dirichlet boundary control problem for parabolic equations with quadratic cost: analyticity and Riccati's feedback synthesis*, SIAM J. Control Optim., 21 (1983), pp. 41–67.

[LT2] ———, *Stabilization and structural assignment of Dirichlet boundary feedback parabolic equations*, SIAM J. Control Optim., 21 (1983), pp. 766–803.

[L1] J.-L. LIONS, *Espaces d'interpolation et domaines des puissances fractionnaires d'opérateurs*, J. Math. Soc. Japan, 14 (1962), pp. 233–242.

[L2] ———, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, Heidelberg, 1971.

[LM1] J.-L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes*, I, Dunod, Paris, 1968.

[LP] J.-L. LIONS AND J. PEETRE, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math., 19 (1964), pp. 5–68.

[S1] M. SORINE, *Un resultat d'existence et unicité pour l'équation de Riccati stationnaire*, Rapport CRMA no. 984, University of Montreal, Montreal, Quebec, Canada, 1980.

[S2] ———, *Sur le semigroupe non linéaire associé a l'équation de Riccati*, Rapport CRMA no. 1055, University Montreal, Montreal, Quebec, Canada, 1981.

[Te] B. TERRENI, *Non-homogeneous initial-boundary value problems for linear parabolic systems*, Studia Math., 92 (1989), pp. 141–175.

[Tr] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, New York, Oxford, 1978.