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Fully nonlinear parabolic systems

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Abstract.

THE PAPER CONCERNS THE EXISTENCE OF CONTINUOSLY DIFFERENTIABLE SOLUTIONS OF FULLY NONLINEAR SECOND ORDER PARABOLIC SYSTEMS IN $\Omega\times[0,T]$, Ω BEING A SMOOTH BOUNDED SET OF $\mathbb{R}^n$. FURTHER HÖLDER REGULARITY PROPERTIES OF THE MAXIMAL SOLUTION AND ITS CONTINUOUS DEPENDENCE ON THE INITIAL DATUM ARE ALSO SHOWN. THE PROOF CONSIST IN LINEARIZATION AND USING THE CONTRACTION PRINCIPLE.

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Fully nonlinear parabolic systems

by P. ACQUISTAPACE * – B. TERRENI **.

Preliminaries.

This paper is concerned with existence of continuously differentiable solutions of fully nonlinear parabolic systems. For the sake of simplicity we just consider here second order systems of this kind:

$$(0.1)\qquad \begin{cases} D_t u^h = f^h(t,x,u,Du,D^2u)\ , & (t,x)\in[0,T]\times\overline{\Omega}\ ,\\ g^h(t,x,u,Du)=0\ , & (t,x)\in[0,T]\times\partial\Omega\\ u^h(0,x)=\varphi^h(x)\ , & x\in\overline{\Omega}\qquad (h=1,\dots,N)\ , \end{cases}$$

where $T>0$, Ω is a bounded, smooth domain of $\mathbf{R}^n$ and f,g are smooth $\mathbf{C}^N$-valued functions.

The first result on local existence for fully nonlinear parabolic problems is due to [HUDJAEV, 1963] (see also [EIDEL'MAN, 1964, Section III.4]), by quasi-linearization. The technique of linearization was firstly used by [SOPOLOV, 1970] and [KRUZHKOV, CASTRO, LOPEZ, 1975, 1980, 1982] in the case of boundary conditions of Dirichlet or quasi-linear oblique derivative type. Linearization is also used, in an abstract framework based on semigroup theory, by [DA PRATO – GRISVARD, 1979] and [LUNARDI – SINESTRARI, 1985]; these papers however concern a smaller class of nonlinear parabolic problems.

In this paper we also use linearization, but our technique seems simpler than the previous ones and allows us to consider more general (nonlinear) boundary conditions. It has been previously used [ACQUISTAPACE – TERRENI, 1987, 1988] for quasilinear parabolic systems.

Here is the plan of this paper. After a survey of the main properties of the linear autonomous version of (0.1) (Section 1), we will show in Section 2 that problem (0.1) possesses a unique local solution, whereas in Section 3 we will prove the regularity properties of the maximal solution of (0.1) with respect to the initial datum.

We assume the following hypotheses:

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HYPOTHESIS 0.1 (*Regularity.*) The boundary of Ω and the functions $f(t,x,u,p,q)$, $g(t,x,u,p)$, $\varphi(x)$ satisfy:

$$\text{(0.2)}\qquad \partial\Omega\in C^{2+2\alpha}\,,\ \varphi\in C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)\,,\quad f\in C^2(\Lambda,\mathbf{C}^N)\,,\ g\in C^3(\Lambda',\mathbf{C}^N)\,,\qquad (\alpha\in]0,1/2[)\,,$$

where

$$\text{(0.3)}\qquad \Lambda:=[0,\infty[\times\overline{\Omega}\times\mathbf{C}^N\times\mathbf{C}^{nN}\times\mathbf{C}^{n^2N}\,,\qquad \Lambda':=[0,\infty[\times\overline{\Omega}\times\mathbf{C}^N\times\mathbf{C}^{nN}.$$

HYPOTHESIS 0.2 (*Parabolicity.*) The complex valued functions (s,j=1 ,...,n;h,k=1,...,N)

$$\text{(0.4)}\qquad A^{hk}_{sj}(x):=\frac{\partial f^h}{\partial q^k_{sj}}\left(0,x,\varphi(x),D\varphi(x),D^2\varphi(x)\right)\,,\qquad B^{hk}_j(x):=\frac{\partial g^h}{\partial p^k_j}\left(0,x,\varphi(x),D\varphi(x)\right)$$

satisfy the following ellipticity assumptions: there exists $\theta_0\in]\pi/2,\pi[$ such that:

(0.5) (θ_0*-root condition*) for each $\theta\in]-\theta_0,\theta_0[$ the operator $L_\theta(x,D_t,D_x)$ defined by

$$L_\theta(x,D_t,D_x)v:=\left\{A^{hk}_{sj}(x)D_sD_jv^k(t,x)+e^{i\theta}\delta^{hk}D_t^2v(t,x)\right\}_{h=1,\dots,N}$$

is properly elliptic in $\mathbf{R}\times\overline{\Omega}$ (see [GEYMONAT – GRISVARD 1967, Hypotheses $\widetilde{\text{II}}$, page 163 and $(\widetilde{AN},\theta)$, page 167]);

(0.6) (θ_0*-complementing condition*) for each $\theta\in]-\theta_0,\theta_0[$ the boundary operator $\Gamma(x,D_t,D_x)$ defined by

$$\Gamma(x,D_t,D_x)v:=\left\{B^{hk}_j(x)D_jv^k(t,x)\right\}_{h=1,\dots,N}$$

fulfills the complementing condition with respect to $L_\theta(x,D_t,D_x)$ in $\partial(\mathbf{R}\times\Omega)$ (see [GEYMONAT – GRISVARD, 1967, hypotheses $\widetilde{\text{III}}$, page 164 and $(\widetilde{AN},\theta)$, page 167]);

HYPOTHESIS 0.3 (*Compatibility.*) The initial datum φ satisfies

$$\text{(0.7)}\qquad g^h(0,x,\varphi(x),D\varphi(x))=0\quad\text{on}\quad\partial\Omega\quad,\quad h=1,\dots,N.$$

REMARKS 0.4 (a) The order of the parabolic system (0.1) might be $2m$ rather than 2; this would only require longer calculations. Moreover, the regularity exponent α in (0.2) might be any number in $]0,1[\setminus\{1/2\}$, but when $\alpha>1/2$ a further compatibility condition along $\partial\Omega$ appears. A more general study, covering these cases, will be published elsewhere.

(b) In (0.4) and (0.6) it is tacitly supposed that the function $g(t,x,u,p)$ really depends on p, i.e. the boundary operator is of first order; however it would be possible to consider boundary operators of different orders, such as:

$$g^h=\begin{cases} g^h(t,x,u,p)\quad, & h=1,\dots,r_0\,,\\ g^h(t,x,u)\quad, & h=r_0+1,\dots,N\end{cases}$$

(see [TERRENI, 1987] for the linear case).
(c) We shall need in Section 3 a stronger version of hypotheses 0.1, 0.2.

We will look for solutions u of (0.1) in the Banach space

$$\text{(0.8)}\qquad \begin{aligned}E_T:=&C^{1+\alpha,2+2\alpha}\left([0,T]\times\overline{\Omega},\mathbf{C}^N\right)\\ \equiv&\, C^{1+\alpha}\left([0,T],C^0(\overline{\Omega},\mathbf{C}^N)\right)\cap L^\infty\left(0,T;C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)\right)\end{aligned}$$

for a suitable $T>0$), endowed with the norm (obviously equivalent to the usual one)

$$\text{(0.9)}\qquad \|u\|_{E_T}:=\sup_{t\in[0,T]}\|u(t,\cdot)\|_C+[D_tu]_{C^\alpha(C)}+\sup_{t\in[0,T]}\left[D_x^2u(t,\cdot)\right]_{C^{2\alpha}}.$$

Here $C, C^\alpha(C), C^{2\alpha}$ stand for $C(\overline{\Omega},\mathbf{C}^N), C^\alpha\left([0,T],C(\overline{\Omega},\mathbf{C}^N)\right), C^{2\alpha}(\overline{\Omega},\mathbf{C}^N)$.

We start with introducing an auxiliary function z which will be useful later on. First of all consider the differential operator

$$\text{(0.10)}\qquad L(x,D_x)v:=\left\{A^{hk}_{sj}(x)D_sD_jv(t,x)\right\}_{h=1,\dots,N}\,,$$

where the functions A^{hk}_{sj} were defined in (0.4).

LEMMA 0.5 *There exists a function* $z:[0,T]\times\overline{\Omega}\to\mathbf{C}^N$ *such that:*

$$\text{(0.11)}\qquad z(0,\cdot)=\varphi,\quad D_tz(0,\cdot)=h:=f(0,\cdot,\varphi,D\varphi,D^2\varphi)\qquad in\ \overline{\Omega}$$

$$\text{(0.12)}\qquad \|z\|_{C^{1+\alpha,2+2\alpha}}+\|D_tz\|_{C^{\alpha,2\alpha}}+\|D_xz\|_{C^{1/2+\alpha,1+2\alpha}}+\|D_x^2z\|_{C^{\alpha,2\alpha}}\le\ \le C_0(\alpha,\Omega,f,\varphi,T).$$

PROOF. As, by (0.2), (0.3) and (0.11), $h\in C^{2\alpha}(\overline{\Omega},\mathbf{C}^N)$, we can find a function $H\in C^{2\alpha}(\mathbf{R}^n,\mathbf{C}^N)$ which extends h and is such that

$$\text{(0.13)}\qquad \|H\|_{C^{2\alpha}(\mathbf{R}^n,\mathbf{C}^N)}\le C_1(\alpha,\Omega)\|h\|_{C^{2\alpha}(\overline{\Omega},\mathbf{C}^N)}.$$

Define now

$$\text{(0.14)}\qquad \begin{aligned}z(t,x):=&\varphi(x)+t^{\frac{2-n}{2}}\int_{\mathbf{R}^n}H(y)\eta\left(t^{-1/2}(x-y)\right)dy\\ =&\,\varphi(x)+t\cdot(\eta_{\sqrt{t}}*H)(x)\end{aligned}$$

where η is a non-negative, even function from $C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \eta(x)dx = 1$; the desired properties then follow in a straightforward way.

1. The linear autonomous problem.

Consider the problem

$$\text{(1.1)} \quad \begin{cases} D_t u^h - A_{sj}^{hk}(x) D_s u^k = f^h(t,x)\,, & (t,x) \in [0,T] \times \overline{\Omega} \\ B_j^{hk}(x) D_j u^k = g^h(t,x)\,, & (t,x) \in [0,T] \times \partial\Omega \\ u^h(0,x) = \varphi^h(x)\,, & x \in \overline{\Omega} \qquad (h = 1,\dots,N). \end{cases}$$

Suppose that:

$$\text{(1.2)} \quad A_{sj}^{hk} \in C^{2\alpha}(\overline{\Omega})\,,\ B_j^{hk} \in C^{1+2\alpha}(\overline{\Omega}) \qquad \text{and (0.5), (0.6) hold,}$$

$$\text{(1.3)} \quad f \in C^{\alpha,2\alpha}\left([0,T] \times \overline{\Omega}, \mathbf{C}^N\right) \quad , \quad g \in C^{\frac{1}{2}+\alpha,1+2\alpha}\left([0,T] \times \overline{\Omega}, \mathbf{C}^N\right),$$

$$\text{(1.4)} \quad B_j^{hk}(x) D_j \varphi^k(x) = g^h(0,x) \qquad \forall x \in \partial\Omega.$$

Due to Lemma 0.5, it is not restrictive for our purposes to suppose moreover that:

$$\text{(1.5)} \quad f(0,x) \equiv \varphi(x) \equiv 0 \quad \text{on } \overline{\Omega}\,, \qquad g(0,x) \equiv 0 \qquad \text{on } \partial\Omega.$$

THEOREM 1.1 *Assume* (1.2),..., (1.5). *Then for any* $T > 0$ *problem* (1.1) *has a unique solution* $u \in E_T$, *and the following estimate holds:*

$$\text{(1.6)} \quad \|u\|_{E_T} \le c_2(\alpha, \Omega, A, B)\Big\{[f]_{C^\alpha(C)} + \sup_{t\in[0,T]} [f(t,\cdot)]_{C^{2\alpha}} + [g]_{C^{\alpha+1/2}(C)} + \sup_{t\in[0,T]} [D_x g(t,\cdot)]_{C^{2\alpha}}\Big\},$$

where the constant c_2 *does not depend on* T.

PROOF. Existence and uniqueness of the solution in E_T were proved in [TERRENI, 1987, Theorem 4.1]. The estimate (1.6) is a consequence of that result and of (1.5). Finally, in order to verify that c_2 is independent of T, we need to revisit the proof of [TERRENI, 1987, Lemmas 2.4–2.5 and Theorem 4.1], taking into account that we may assume, by Lemma 0.5, $u(0,x) \equiv D_t u(0,x) \equiv 0$ in $\overline{\Omega}$. ◇

2. The non-linear problem. Local existence.

Consider now problem (0.1). The main result of this Section is:

THEOREM 2.1 *Suppose that hypotheses* 0.1, 0.2 *and* 0.3 *hold. Then there exists* $\tau_* > 0$ *such that problem* (0.1) *has a unique solution* $u \in E_{\tau_*}$.

PROOF. It consists of two steps: linearization and use of the contraction principle.

Step 1 Let h be the function defined in (0.11) and let z be the function constructed in Lemma 0.5. For each $M > 0$ consider the (not empty) set

$$\text{(2.1)} \quad B_{M,T} := \{w \in E_T : \|w - z\|_{E_T} \le M\,,\ w(0,\cdot) = \varphi, D_t w(0,\cdot) = h\}$$

which is closed in E_T. This is the set where we will linearize problem (0.1). For fixed $w \in B_{M,T}$ consider the linear autonomous problem

$$\text{(2.2)} \quad \begin{cases} D_t v^h - \dfrac{\partial f^h}{\partial q_{sj}^k}(0,x,\varphi,D\varphi,D^2\varphi) D_s D_j v^k = f^h(t,x,w,Dw,D^2w) \\ -\dfrac{\partial f^h}{\partial q_{sj}^k}(0,x,\varphi,D\varphi,D^2\varphi) D_s D_j w^k, & (t,x) \in [0,T] \times \overline{\Omega}, \\ \dfrac{\partial g^h}{\partial p_j^k}(0,x,\varphi,D\varphi) D_j v^k = -g^h(t,x,w,Dw) \\ +\dfrac{\partial g^h}{\partial p_j^k}(0,x,\varphi,D\varphi) D_j v^k\,, \ (t,x) \in [0,T] \times \partial\Omega, \\ v^h(0,x) = \varphi^h(x), & x \in \overline{\Omega} \qquad (h = 1,\dots,N). \end{cases}$$

The data of problem (2.2) satisfy, for every $w \in B_{M,T}$, the hypotheses of [TERRENI, 1987, Theorem 4.1]. Therefore we can find a unique solution $v \in E_T$ of (2.1), so that we have a well defined map

$$\text{(2.3)} \quad S : B_{M,T} \to E_T\,, \qquad S(w) := v.$$

Step 2. The main properties of the map S are expressed in the following lemma (which is an easy consequence of Lemma 3.5 below).

LEMMA 2.2 *For each* $M > 0$ *we have:*

$$\text{(2.4)} \quad \|S(w) - z\|_{E_T} \le c_3(\alpha,\Omega,f,g,\varphi) + c_4(\alpha,\Omega,f,g,\varphi,M)T^\alpha \quad \forall w \in B_{M,T}\,;$$

$$\text{(2.5)} \quad \|S(w_1) - S(w_2)\|_{E_T} \le c_5(\alpha,\Omega,f,g,\varphi,M)T^\alpha \|w_1 - w_2\|_{E_T} \quad \forall w_1, w_2 \in B_{M,T}.$$

The inequalities (2.4) and (2.5) show that the map S satisfies

$$S(w) \in B_{M,\tau_*} \qquad \forall w \in B_{M,\tau^*},$$

$$\|S(w_1) - S(w_2)\|_{E_{\tau_*}} \le \frac{1}{2}\|w_1 - w_2\|_{E_{\tau_*}} \qquad \forall w_1, w_2 \in B_{M,\tau_*}$$

provided we fix in advance M and τ_* such that

$$M > C_3(\alpha,\Omega,f,g,\varphi)$$
$$T \le \left[\min\left\{[2C_5(\alpha,\Omega,f,g,\varphi,M)]^{-1}, \frac{M-c_3(\alpha,f,g,\varphi)}{c_4(\alpha,\Omega,f,g,\varphi,M)}\right\}\right]^{\frac{1}{\alpha}}.$$

Hence we get a unique fixed point $u \in B_{M,\tau_*}$ of the map S, i.e. a unique solution in E_{τ_*} of problem (0.1). ◇

3. Continuous dependence on data.

Consider again problem (0.1), replacing Hypotheses 0.1 and 0.2 by the following stronger ones:

HYPOTHESIS 3.1 $\partial\Omega \in C^{2+2\alpha}$, $\varphi \in C^{2+2\alpha}$, $f \in C^3(\Lambda, \mathbf{C}^N)$, $g \in C^4(\Lambda', \mathbf{C}^N)$ $(\alpha \in]0,1/2[)$, where Λ, Λ' are defined by (0.3).

HYPOTHESIS 3.2 The complex-valued functions

$$A_{sj}^{hk}(x) := \frac{\partial f^h}{\partial q_{sj}^k}(t,x,u,p,q)\,, \qquad B_j^{hk}(x) := \frac{\partial g^h}{\partial p_j^k}(t,x,u,p)$$

satisfy Hypothesis 0.2 uniformly in (t,u,p,q) on compact subsets of $[0,\infty[\,\times\mathbf{C}^N \times \mathbf{C}^{nN} \times \mathbf{C}^{n^2N}$.

Moreover we reformulate Hypothesis 0.3 as follows:

HYPOTHESIS 3.3 Consider for $\tau > 0$ the closed submanifold of $C^{2+2\alpha}(\overline{\Omega}, \mathbf{C}^N)$ defined by

$$V_g(\tau) := \{\varphi \in C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N) : g(\tau,x,\varphi(x),D\varphi(x)) = 0 \quad \forall x \in \partial\Omega\}; \tag{3.2}$$

we assume that $V_g(0)$ is not empty.

If we now choose any $\varphi \in V_g(0)$, Theorem 2.1 yields a unique solution $u \in E_{\tau_*}$ of (0.1); hence we have

$$u(\tau_*,\cdot) \in V_g(\tau_*).$$

This condition and Hypotheses 3.1 and 3.2 allow us to extend the solution of (0.1) to an interval $[\tau_*, \tau_* + \tau_1]$. At the point τ_1 we can start again, and so on. In short, we can consider the maximal interval of existence $[0,\tau_\varphi[$, by defining:

(3.3) $\tau_\varphi := \sup\{\tau > 0$: problem (0.1) has a unique solution in $E_\tau\}$,

(3.4) $u_\varphi :=$ the unique solution of (0.1) in $[0,\tau_\varphi[$.

Consider now the set $D \subset V_g(0) \times [0,\infty[$ defined by:

$$D := \{(\varphi,t) \in V_g(0) \times [0,\infty[: \quad t < \tau_\varphi\}. \tag{3.5}$$

THEOREM 3.3 *Under Hypotheses* 3.1, 3.2 *and* 3.3 *we have:*

(i) D is open in $V_g(0) \times [0,\infty[$ in the topology of $C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N) \times [0,\infty[$;
(ii) the map

$$U : D \to C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N), \qquad U(\varphi,t) = u_\varphi(t,\cdot) \tag{3.6}$$

is continuous as a function from D into $C^{2+2\delta}(\overline{\Omega},\mathbf{C}^N)$ for each $\delta \in [0,\alpha[$.

PROOF. We need a more general version of Theorem 2.1. First of all, we remark that

$$\|U_\varphi\|_{E_{\tau_\varphi-\epsilon}} \le c_6(\alpha,\Omega,f,g,\varphi,\epsilon) \qquad \forall\epsilon \in]0,\tau_\varphi[. \tag{3.7}$$

Next, for $t_0 \ge 0$, $r_0 > 0$, $\varphi_0 \in C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)$ we set:

$$Q(\varphi_0,t_0,r_0) := \{\psi \in V_g(t_0) : \|\psi - \varphi_0\|_{C^{2+2\alpha}} \le r_0\}. \tag{3.8}$$

Of course if $\varphi \in V_g(0)$ and $t_0 \in [0,\tau_\varphi[$ then

$$u_\varphi(t_0,\cdot) \in Q(u_\varphi(t_0,\cdot),t_0,r_0) \qquad \forall r_0 > 0.$$

Now for $0 \le t_0 < t_0 + \sigma =: \tau$ define (analogously to (0.8), (0.9)) the Banach space

$$E_{t_0,\tau} := C^{1+\alpha,2+2\alpha}([t_0,\tau]\times\overline{\Omega},\mathbf{C}^N).$$

PROPOSITION 3.4 *Let $\varphi \in V_g(0)$. For each $\epsilon \in]0,\tau_\varphi[$ and $r_0 > 0$ there exists $\sigma \in]0,\epsilon]$ depending on $\alpha,\Omega,f,g,\varphi,r_0,\epsilon$, such that for every $t_0 \in [0,\tau_\varphi - \epsilon]$ and $\psi \in Q(u_\varphi(t_0,\cdot),t_0,r_0)$ the problem*

$$\begin{cases} D_t v^h = f^h(t,x,v,Dv,D^2v) & , \ (t,x) \in [t_0,\tau]\times\overline{\Omega}\,,\\ g^h(t,x,v,Dv) = 0 & , \ (t,x) \in [t_0,\tau]\times\partial\Omega\,,\\ v^h(t_0,x) = \psi^h(x) & , \ x \in \overline{\Omega} \quad (h = 1,\dots,N) \end{cases} \tag{3.9}$$

where $\tau := t_0 + \sigma$, has a unique solution $v_\psi \in E_{t_0,\tau}$; moreover there exists $c_7(\alpha,\Omega,f,g,\varphi,r_0,\epsilon) \ge 1$ such that

$$\|v_\psi - v_\chi\|_{E_{t_0,\tau}} \le c_7(\alpha,\Omega,f,g,\varphi,r_0,\epsilon)\|\psi - \chi\|_{C^{2+2\alpha}} \qquad \forall\psi,\chi \in Q(u_\varphi(t,\cdot),t_0,r_0). \tag{3.10}$$

PROOF. We repeat, "mutatis mutandis", the proof of Theorem 2.1.

Fix $\epsilon \in]0, \tau_\varphi[$, $r_0 > 0$, $t_0 \in [0, \tau_\varphi - \epsilon]$, $\psi \in Q(u_\varphi(t_0, \cdot), t_0, r_0)$ and let $\sigma \in]0, \epsilon]$. Choose a function $z_\psi \in E_{t_0,\tau}$ such that

$$(3.11) \qquad z_\psi(t_0, \cdot) = \psi\,, \qquad D_t z_\psi(t_0, \cdot) = h := f(t_0, \cdot, \psi, D\psi, D^2\psi) \text{ in } \overline{\Omega}$$

(this can be done as in Lemma 0.5). Finally set

$$(3.12) \qquad B_{M,t_0,\tau}(\psi) := \{v \in E_{t_0,\tau} : \|v - z_\psi\|_{E_{t_0,\tau}} \le M,\ v(t_0, \cdot) = \psi,\ D_t v(t_0, \cdot) = h\}\,.$$

For fixed $w \in B_{M,T_0,\tau}(\psi)$ we linearize problem (3.9) in the following way:

$$(3.13) \qquad \begin{cases} D_t v^h - \dfrac{\partial f^h}{\partial q^k_{sj}}(t_0, x, \psi, D\psi, D^2\psi) D_s D_j v^k \\ \qquad\qquad = F_w(t,x)\,, & (t,x) \in [t_0, \tau] \times \overline{\Omega}\,, \\ \dfrac{\partial g^h}{\partial p^k_j}(t_0, x, \psi, D\psi) D_j v^k = G_w(t,x)\,, & (t,x) \in [t_0, \tau] \times \partial\Omega\,, \\ v^h(t_0, x) = \psi^h(x)\,, & x \in \overline{\Omega} \qquad (h = 1, \dots, N)\,, \end{cases}$$

where

$$(3.14) \qquad \begin{cases} F_w(t,x) := \left\{ f^h(t, x, w, Dw, D^2w) - \frac{\partial f^h}{\partial q^k_{sj}}(t_0, x, w, Dw, D^2w) D_s D_j w \right\}_{h=1,\dots,N}\,, \\ G_w(t,x) := \left\{ -g^h(t, x, w, Dw) + \frac{\partial g^h}{\partial p^k_j}(t_0, x, \psi, D\psi,) D_j w \right\}_{h=1,\dots,N}\,. \end{cases}$$

We remark that by (3.8), (3.7) we have

$$(3.15) \qquad \begin{aligned} \|\psi\|_{C^{2+2\alpha}} &\le \|\psi - u_\varphi(t_0, \cdot)\|_{C^{2+2\alpha}} + \|u_\varphi(t_0, \cdot)\|_{C^{2+2\alpha}} \le \\ &\le r_0 + c_6(\alpha, \Omega, f, g, \varphi, \epsilon) \qquad \forall \psi \in Q(u_\varphi(t_0, \cdot), t_0, r_0). \end{aligned}$$

Next, by Hypothesis 3.2 we see that the coefficients of the linear problem (3.13), as well as their derivatives, are bounded by constants depending on $f, g, \varphi, r_0, \epsilon$ but not on t_0; the same holds for the constants involved by ellipticity and by the complementing condition. Moreover it is easy to see that

$$F_w \in C^{\alpha,2\alpha}\left([t_0, \tau] \times \overline{\Omega}, \mathbf{C}^N\right),\ G_w \in C^{\frac{1}{2}+\alpha, 1+2\alpha}\left([t_0, \tau] \times \overline{\Omega}, \mathbf{C}^N\right), \\ \psi \in C^{2+2\alpha}(\overline{\Omega}, \mathbf{C}^N)$$

and that the compatibility condition

$$\frac{\partial g}{\partial p^k_j}(t_0, x, \psi(x), D\psi(x))\, D_j \psi^k(x) = G_w(0, x) \qquad \forall x \in \partial\Omega$$

holds. Henc by revisiting the proof of [TERRENI, 1987, Theorem 4.1] we deduce the existence of a unique solution $v \in E_{t_0,\tau}$ of Problem (3.13). Thus the map

$$(3.16) \qquad S_\psi : B_{M,t_0,\tau}(\psi) \to E_{t_0,\tau}\,, \qquad S_\psi(w) = v$$

is well defined; in addition the analogue of Lemma 2.2 holds (for a proof see the Appendix):

LEMMA 3.5 *For each $M > 0$ we have:*

$$(3.17) \qquad \begin{aligned} &\|S_\psi(w) - z_\psi\|_{E_{t_0,\tau}} \le \\ &\qquad c_8(\alpha, \Omega, f, g, \varphi, r_0, \epsilon) + c_9(\alpha, \Omega, f, g, \varphi, r_0, \epsilon, M)(\tau - t_0)^\alpha \\ &\qquad\qquad \forall \psi \in Q(u_\varphi(t_0, \cdot), t_0, r_0)\,, \qquad \forall w \in B_{M,t_0,\tau}(\psi)\,; \end{aligned}$$

$$(3.18) \qquad \begin{aligned} &\|S_\psi(v) - S_\chi(w)\|_{E_{t_0,\tau}} \le \\ &\qquad c_10(\alpha, \Omega, f, g, \varphi, r_0, \epsilon, M)\left\{\|\psi - \chi\|_{C^{2+2\alpha}} + (\tau - t_0)^\alpha \|v - w\|_{E_{t_0,\tau}}\right\} \\ &\qquad\qquad \forall \psi, \chi \in Q(u_\varphi(t_0, \cdot), t_0, r_0)\,, \quad \forall v \in B_{M,t_0,\tau}(\psi)\,, \quad \forall w \in B_{M,t_0,\tau}(\chi)\,. \end{aligned}$$

As in Section 2, the above lemma implies that there exists $\sigma \in]0, \epsilon]$ depending on $(\alpha, \Omega, f, g, \varphi, r_0, \epsilon)$ such that Problem (3.9) has a unique solution $v_\psi \equiv S_\psi(v_\psi) \in E_{t_0,\tau}$.

Finally (3.10) follows by (3.18). This completes the proof of Proposition 3.4. ◇

Let us prove now the first part of Theorem 3.3. Fix $(\varphi, t_*) \in D$, which means $\varphi \in V_g(0)$ and $t_* \in [0, \tau_\varphi[$. Fix $r_0 > 0$ and $\epsilon \in]0, \frac{1}{2}(\tau_\varphi - t_*)[$, and let $\sigma \in]0, \epsilon]$ be the number determined by Proposition 3.4. Define

$$s := \left[\frac{t_* + \epsilon}{\sigma}\right], \qquad \tau_k := k\sigma, \qquad k = 0, 1, \dots, s+1;$$

then in particular

$$t_* + \epsilon < \tau_{s+1} = (s+1)\sigma \le \left(\frac{t_* + \epsilon}{\sigma} + 1\right)\sigma = t_* + \epsilon + \sigma \le t_* + 2\epsilon < \tau_\varphi.$$

Now let $\psi \in V_g(0)$ be such that

$$\|\psi - \varphi\|_{C^{2+2\alpha}} < r_0 \left[c_7(\alpha, \Omega, f, g, \varphi, r_0, \epsilon)\right]^{-1-s} \le r_0;$$

according to (3.4), denote the maximal solutions of (0.1) with initial data φ, ψ by u_φ, u_ψ. By applying Proposition 3.4 with $t_0 = \tau_0 = 0$, we get, since $\psi \in Q(\varphi, 0, r_0)$:

$$\|u_\varphi - u_\psi\|_{E_{0,\tau_1}} \le c_7 \|\varphi - \psi\|_{C^{2+2\alpha}} \le r_0 c_7^{-s} \le r_0,$$

which implies in particular $u_\psi(\tau_1,\cdot) \in Q(u_\varphi(\tau_1,\cdot),\tau_1,r_0)$. An iteration of this argument shows, since the solution of (3.9) is unique, that

$$\text{(3.19)} \qquad \|u_\varphi - u_\psi\|_{E_{\tau_k,\tau_{k+1}}} \le r_0 c_7^{-s+k} \le r_0\,, \qquad k = 0,1,\dots,s.$$

The last step proves that u_ψ is defined in the whole interval $[0,\tau_{s+1}]$ and hence in $[0,t_*+\epsilon]$; consequently, by definition, $t_* + \epsilon < \tau_\psi$. We have shown that $(\psi,t) \in D$ provided $(\varphi,t_*) \in D$ and

$$\psi \in V_g(0), \|\psi-\varphi\|_{C^{2+2\alpha}} < r_0\left[c_7(\alpha,\Omega,f,g,\varphi,r_0,\epsilon)\right]^{-\left[\frac{t_*+\epsilon}{\sigma}\right]-1},$$
$$|t-t_*| < \epsilon, \qquad \epsilon \in]0,\frac{1}{2}(\tau_\varphi - t_*)[.$$

This shows that D is open in $V_g(0) \times [0,\infty[$.

Let us prove now the second part of Theorem 3.3. Fix $r_0 > 0$, $(\varphi,t) \in D$ and let

$$(\psi,s) \in I := \{\psi \in V_g(0) : \|\psi - \varphi\|_{C^{2+2\alpha}} < \epsilon\} \times]t-\epsilon, t+\epsilon[.$$

where $\epsilon \in]0, \min\{t, \tau_\varphi - t\}[$. Then (3.10) yields for each $s \in]0,t+\epsilon[$:

$$\text{(3.20)} \qquad \|u_\varphi(s,\cdot) - u_\psi(s,\cdot)\|_{C^{2+2\alpha}} \le c_7(\alpha,\Omega,f,g,\varphi,r_0,\epsilon)\|\varphi-\psi\|_{C^{2+2\alpha}}.$$

On the other hand, we recall that, by interpolation (see [TERRENI, 1988, Lemma 2.5])

$$\text{(3.21)} \qquad E_{0,\tau_\varphi-\epsilon} \hookrightarrow C^{\alpha-\delta}\left([0,\tau_\varphi-\epsilon], C^{2+2\delta}(\overline{\Omega},\mathbf{C}^N)\right) \qquad \forall\delta \in]0,\alpha[.$$

By (3.21) and (3.20) we conclude that

$$\begin{aligned} &\|u_\varphi(t,\cdot) - u_\psi(s,\cdot)\|_{C^{2+2\delta}} \\ &\quad \le \|u_\varphi(t,\cdot) - u_\varphi(s,\cdot)\|_{C^{2+2\delta}} + \|u_\varphi(s,\cdot) - u_\psi(s,\cdot)\|_{C^{2+2\alpha}} \\ &\quad \le c_{11}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon)\left\{|t-s|^{\alpha-\delta} + \|\varphi-\psi\|_{C^{2+2\alpha}}\right\}. \end{aligned}$$

The proof of Theorem 3.3 is complete. ◇

REMARKS 3.6 (a) Hypothesis 3.1 is required just in the proof of (3.18); in order to get (3.17) Hypothesis 0.1 (together with Hypotheses 3.2, 3.3) would be sufficient.
(b) In order to have continuity of the map U (see (3.6)) in the topology of $C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)$ we have to take $\varphi \in h^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)$ and to repeat the proof in the Banach space

$$\begin{aligned} E_{t_0,\tau} &:= h^{1+\alpha,2+2\alpha}\left([t_0,\tau]\times\overline{\Omega},\mathbf{C}^N\right) \\ &\equiv h^{1+\alpha}\left([t_0,\tau], C(\overline{\Omega},\mathbf{C}^N)\right) \cap C\left([t_0,\tau], h^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)\right); \end{aligned}$$

here h^α means "little α-Hölder continuous", i.e.

$$F \in h^\alpha \Leftrightarrow |F(x)-F(y)| = o\left(|x-y|^\alpha\right) \qquad \text{as} \qquad |x-y|^\alpha \downarrow 0.$$

(c) If the boundary condition is autonomous, i.e.

$$\frac{\partial g}{\partial t}(t,x,u,p) = 0 \qquad \forall (t,x,u,p) \in [0,\infty[\times\partial\Omega\times\mathbf{C}^N\times\mathbf{C}^{nN}$$

then the map U generates a semiflow (see [AMANN, 1986]), which is bounded in $C^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)$ and continuous in $C^{2+2\delta}(\overline{\Omega},\mathbf{C}^N)$ $\quad\forall\delta \in]0,\alpha[$ (or in $h^{2+2\alpha}(\overline{\Omega},\mathbf{C}^N)$).

APPENDIX: PROOF OF LEMMAS 2.2 AND 3.5

We need some auxiliary result.

LEMMA A.1 *Let Ω be a bounded open set of $\mathbf{R}^n$ with $\partial\Omega \in C^{2+2\alpha}$. There exists $c_{12}(\alpha, \Omega, T) \geq 1$ such that for each $v \in E_{t_0,\tau}$ with $0 \leq t_0 < \tau \leq T$ we have:*

(A.1) $$\|D^j v\|_{C(C)} \leq c_{12}\left\{(\tau - t_0)^{\alpha+1-j/2} H_\alpha + (\tau - t_0)^{1-j/2}\|D_t v(t_0, \cdot)\|_C + \right.$$
$$\left. + (\tau - t_0)^{-j/2}\|v(t_0, \cdot)\|_C\right\}, \quad j = 0, 1, 2;$$

(A.2) $$[D^j v]_{C^\alpha(C)} \leq c_{12}\left\{(\tau - t_0)^{1-j/2} H_\alpha + (\tau - t_0)^{1-\alpha-j/2}\|D_t v(t_0, \cdot)\|_C\right\}, \quad j = 0, 1, 2;$$

(A.3) $$\sup_{t\in[t_0,\tau]} [D^j v(t, \cdot)]_{C^{2\alpha}} \leq c_{12}\left\{(\tau - t_0)^{1-j/2} H_\alpha + (2 - j)(\tau - t_0)^{1-\alpha-j/2}\|D_t v(t_0, \cdot)\|_C + \right.$$
$$\left. + (2 - j)(1 - j)(\tau - t_0)^{-\alpha}\|v(t_0, \cdot)\|_C\right\}, \quad j = 0, 1, 2;$$

(A.4) $$[D^j v]_{C^{\alpha+1/2}(C)} \leq c_{12}\left\{(\tau - t_0)^{1/2-j/2} H_\alpha + (\tau - t_0)^{1/2-\alpha-j/2}\|D_t v(t_0, \cdot)\|_C\right\}, \quad j = 0, 1,$$

where

$$H_\alpha := [D_t v]_{C^\alpha(C)} + \sup_{t\in[t_0,\tau]} \left[D_x^2 v(t, \cdot)\right]_{C^{2\alpha}}.$$

PROOF. It is a standard computation. ◇

Similarly to Lemma 0.5 we can easily prove:

LEMMA A.2. *Fix $\psi \in C^{2+2\alpha}(\overline{\Omega}, \mathbf{C}^N)$, and $h \in C^{2\alpha}(\overline{\Omega}, \mathbf{C}^N)$; let H be an extension of h to $\mathbf{R}^n$ such that (0.13) holds. Then if $0 \leq t_0 < \tau \leq T$ the function*

(A.5) $$z(t, x) := \psi(x) + (t - t_0)\left(\eta_{\sqrt{t-t_0}} * H\right)(x) \quad , \quad (t, x) \in [t_0, \tau] \times \overline{\Omega},$$

satisfies:

(A.6) $$z(t_0, \cdot) = \psi, \qquad D_t z(t_0, \cdot) = h \qquad \text{in} \quad \overline{\Omega},$$

(A.7) $$\|z\|_{C^{1+\alpha,2+2\alpha}} + \|D_t z\|_{C^{\alpha,2\alpha}} + \|D_x z\|_{C^{\frac{1}{2}+\alpha,1+2\alpha}} + \|D_x^2 z\|_{C^{\alpha,2\alpha}} \leq$$
$$\leq c_{13}(\alpha, \Omega)\left\{\|\psi\|_{C^{2+2\alpha}} + \|h\|_{C^{2\alpha}}\right\},$$

(A.8) $$\|D_t z - h\|_{L^\infty(C^{2\alpha})} + \|z - \psi\|_{L^\infty(C^{2+2\alpha})} \leq c_{13}(\alpha, \Omega)(\tau - t_0)^\alpha \|h\|_{C^{2\alpha}}.$$ ◇

For $P : \Lambda \to \mathbf{C}^N$, $Q : \Lambda' \to \mathbf{C}^N$ and $v \in C^{1+\alpha,2+2\alpha}\left([t_0, \tau] \times \overline{\Omega}, \mathbf{C}^N\right)$ introduce the notation

(A.9) $$\begin{cases} \tilde{P}(t, x; v) := P(t, x, v(t, x), Dv(t, x), D^2 v(t, x)), \\ \tilde{Q}(t, x; v) := Q(t, x, v(t, x), Dv(t, x)). \end{cases}$$

LEMMA A.3. *Fix $0 \leq t_0 < \tau \leq T$ and let $\eta \in E_{t_0,\tau}$ with $\|\eta\|_{E_{t_0,\tau}} \leq L$. Then:*

(i) for all $P \in C(\Lambda, \mathbf{C}^N)$, $\quad Q \in C(\Lambda', \mathbf{C}^N)$ we have:

(A.10) $$\|\tilde{P}(\cdot, \cdot; \eta\|_{C(C)} \leq c_{14}(\Omega, P, L, T)$$
(A.11) $$\|\tilde{Q}(\cdot, \cdot; \eta)\|_{C(C)} \leq c_{15}(\Omega, Q, L, T).$$

Assume in addition that $\eta(t_0, \cdot) = \psi$, $D_t\eta(t_0, \cdot) = h$ with $\psi \in C^{2+2\alpha}(\overline{\Omega}, \mathbf{C}^N)$, $\quad h \in C^{2\alpha}(\overline{\Omega}, \mathbf{C}^N)$. Then:

(ii) for all $P \in C^1(\Lambda, \mathbf{C}^N)$, $Q \in C^1(\Lambda', \mathbf{C}^N)$ we have

(A.12) $$\|\tilde{P}(\cdot, \cdot; \eta)\|_{C^{\alpha,2\alpha}} \leq c_{16}(\alpha, \Omega, P, T, L, \psi, h),$$
(A.13) $$\|\tilde{Q}(\cdot, \cdot; \eta)\|_{L^\infty(C^{2\alpha})} + \|\tilde{Q}(\cdot, \cdot; \eta)\|_{C^{\alpha+1/2}(C)} \leq c_{17}(\alpha, \Omega, Q, T, L, \psi, h),$$
(A.14) $$\|\tilde{P}(\cdot, \cdot; \eta) - \tilde{P}(t_0, \cdot; \psi)\|_{C(C)} \leq c_{18}(\alpha, \Omega, P, T, L, \psi, h) \cdot (\tau - t_0)^\alpha,$$
(A.15) $$\|\tilde{Q}(\cdot, \cdot; \eta) - \tilde{Q}(t_0, \cdot; \psi)\|_{C(C)} \leq c_{19}(\alpha, \Omega, Q, L, T, \psi, h)(\tau - t_0)^{\alpha+1/2};$$

(iii) for all $Q \in C^2(\Lambda', \mathbf{C}^N)$ we have:

(A.16) $$\|\tilde{Q}(\cdot, \cdot; \eta)\|_{C^{1/2+\alpha,1+2\alpha}} \leq c_{20}(\alpha, \Omega, Q, L, T, \psi, h)$$
(A.17) $$\|\tilde{Q}(\cdot, \cdot; \eta) - \tilde{Q}(t_0, \cdot; \psi)\|_{L^\infty(C^1)} \leq c_{21}(\alpha, \Omega, Q, L, T, \psi, h)(\tau - t_0)^\alpha.$$

PROOF. (A.10) and (A.11) are evident. Let us prove (A.12): due to (A.10) we have to estimate

$$\left[\tilde{P}(\cdot, \cdot; \eta)\right]_{C^\alpha(C)} + \sup_{t\in[t_0,\tau]} \left[\tilde{P}(\cdot, \cdot; \eta)\right]_{C^{2\alpha}}.$$

By introducing the point

$$\xi_\sigma := \Big(s+\sigma(t-s), x, \eta(s,x)+\sigma\left(\eta(t,x)-\eta(s,x)\right), D\eta(s,x)+\sigma\left(D\eta(t,x)-D\eta(s,x)\right), D^2\eta(s,x)+\sigma\left(D^2\eta(t,x)-D^2\eta(s,x)\right)\Big) \in \Lambda,$$

we have for all $t, s \in [t_0, \tau]$:

$$\tilde P(t,x;\eta) - \tilde P(s,x;\eta) = \int_0^1 \Big\{ \frac{\partial P}{\partial t}(\xi_\sigma)(t-s) + \frac{\partial P}{\partial u}(\xi_\sigma)\cdot(\eta(t,x)-\eta(s,x)) + \frac{\partial P}{\partial p_j}(\xi_\sigma)\cdot(D_j\eta(t,x)-D_j\eta(s,x)) + \frac{\partial P}{\partial q_{ij}}(\xi_\sigma)\cdot(D_iD_j\eta(t,x)-D_iD_j\eta(s,x)) \Big\} d\sigma.$$

Now let z be the function given by Lemma A.2. Writing $\eta = (\eta - z) + z$ and noting that $\eta(t_0,\cdot) = z(t_0,\cdot)$, $D_t\eta(t_0,\cdot) = D_t z(t_0,\cdot)$, by (A.10), (A.2) and (A.7) we easily get

$$\text{(A.18)} \qquad |\tilde P(t,x;\eta) - \tilde P(s,x;\eta)| \le c_{22}(\alpha,\Omega,P,L,T)(t-s)^\alpha \{1 + L + \|\psi\|_{C^{2+2\alpha}} + \|h\|_{C^{2\alpha}}\}.$$

Next, setting now

$$\xi_\sigma := \Big(t, y+\sigma(x-y), \eta(t,y)+\sigma\left(\eta(t,x)-\eta(t,y)\right), D\eta(t,y)+\sigma\left(D\eta(t,x)-D\eta(t,y)\right), D^2\eta(t,y)+\sigma\left(D^2\eta(t,x)-D^2\eta(t,y)\right)\Big) \in \Lambda,$$

and writing again $\eta = (\eta - z) + z$, we get by (A.10), (A.3) and (A.7):

$$\text{(A.19)} \quad \Big|\tilde P(t,x;\eta) - \tilde P(t,y;\eta)\Big| = \Big|\int_0^1 \Big\{ \frac{\partial P}{\partial x_j}(\xi_\sigma)(x_j-y_j) + \frac{\partial P}{\partial u}(\xi_\sigma)\cdot(\eta(t,x)-\eta(t,y)) + \frac{\partial P}{\partial p_j}(\xi_\sigma)\cdot(D_j\eta(t,x)-D_j\eta(t,y)) + \frac{\partial P}{\partial q_{ij}}(\xi_\sigma)\cdot(D_iD_j\eta(t,x)-D_iD_j\eta(t,y)) \Big\} d\sigma\Big| \le c_{23}(\alpha,\Omega,P,L,T)|x-y|^{2\alpha}\{1 + L + \|\psi\|_{C^{2+2\alpha}} + \|h\|_{C^{2\alpha}}\}.$$

By (A.18) and (A.19) we obtain (A.12).

The proof of (A.13) is quite similar and can be omitted.
Let us verify (A.14). Setting

$$\xi_\sigma := \Big(t_0+\sigma(t-t_0), x, \psi(x)+\sigma\left(\eta(t,x)-\psi(x)\right), D\psi(x)+\sigma\left(D\eta(t,x)-D\psi(x)\right), D^2\psi(x)+\sigma\left(D^2\eta(t,x)-D^2\psi(x)\right)\Big) \in \Lambda,$$

as before we get by (A.10), (A.1) and (A.8)

$$\Big|\tilde P(t,x;\eta) - \tilde P(t_0,x;\psi)\Big| = \Big|\int_0^1 \Big\{ \frac{\partial P}{\partial t}(\xi_\sigma)\cdot(t-t_0) + \frac{\partial P}{\partial u}(\xi_\sigma)\cdot(\eta(t,x)-\psi(x)) + \frac{\partial P}{\partial p_j}(\xi_\sigma)\cdot(D_j\eta(t,x)-D_j\psi(x)) + \frac{\partial P}{\partial q_{ij}}(\xi_\sigma)\cdot(D_iD_j\eta(t,x)-D_iD_j\psi(x)) \Big\} d\sigma\Big| \le c_{24}(\alpha,\Omega,P,L,T)(\tau-t_0)^\alpha\{1 + L + \|\psi\|_{C^{2+2\alpha}} + \|h\|_{C^{2\alpha}}\}.$$

The remaining estimates can be proved quite analogously. ◇

Let us prove now Lemmas 2.2 and 3.5. As Lemma 2.2 follows easily by Lemma 3.5, we just prove the latter. We remark however that Lemma 2.2 holds under weaker regularity assumptions than Lemma 3.5 (Hypothesis 0.1 instead of 3.1: see Remark A.4 below).

Proof of (3.17).

Set $u := S_\psi(w) - z_\psi$. Then u solves the linear problem

$$\begin{cases} D_t u - A_{sj}(x)\cdot D_sD_j u = F_{\psi,w}(t,x), & (t,x) \in [t_0,\tau]\times\overline\Omega, \\ B_j(x)\cdot D_j u = G_{\psi,w}(t,x), & (t,x)\in[t_0,\tau]\times\partial\Omega, \\ u(t_0,x) = 0, & x \in \overline\Omega, \end{cases}$$

where:

$$A_{sj}(x) := \frac{\partial f}{\partial q_{sj}}(t_0, x, \psi(x), D\psi(x), D^2\psi(x)),$$

$$B_j(x) := \frac{\partial g}{\partial p_j}(t_0, x, \psi(x), D\psi(x)),$$

$$F_{\psi,w}(t,x) := f(t,x,w,Dw,D^2w) - D_t z_\psi - A_{sj}(x)\cdot D_sD_j(w - z_\psi),$$

$$G_{\psi,w}(t,x) := -g(t,x,w,Dw) + B_j(x)\cdot D_j(w - z_\psi).$$

Note that $F_{\psi,w}(t_0,x) \equiv 0$ in $\overline\Omega$ and $G_{\psi,w}(t_0,x) \equiv 0$ on $\partial\Omega$; moreover all assumptions of Theorem 1.1 hold uniformly with respect to $t_0 \in [0, t_\varphi - \epsilon]$.

Hence by (1.6) we have, recalling (3.15):

$$\text{(A.20)} \qquad \|u\|_{E_{t_0,\tau}} \le c_{25}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon)\Big\{[F_{\psi,w}]_{C^\alpha(C)} + \sup_{t\in[t_0,\tau]}[F_{\psi,w}(t,\cdot)]_{C^{2\alpha}} +$$

$$+ [G_{\psi,w}]_{C^{\alpha+1/2}(C^0)} + \sup_{t\in[t_0,\tau]} [DG_{\psi,w}]_{C^{2\alpha}} \Big\},$$

and all what we have to do is to estimate the right member of (A.20).

Setting

$$\xi_\sigma := \Big(t, x, z_\psi(t,x) + \sigma\,(w(t,x) - z_\psi(t,x))\,, Dz_\psi(t,x) + \sigma\,(Dw(t,x) - Dz_\psi(t,x))\,, \\ D^2 z_\psi(t,x) + \sigma\,(D^2 w(t,x) - D^2 z_\psi(t,x))\Big) \in \Lambda,$$

we can split $F_{\psi,w}$ in the following way:

$$F_{\psi,w}(t,x) = \int_0^1 \Big\{ \frac{\partial f}{\partial u}(\xi_\sigma)\cdot(w(t,x) - z_\psi(t,x)) + \frac{\partial f}{\partial p_j}(\xi_\sigma)\cdot(D_j w(t,x) - D_j z_\psi(t,x)) + \\ + \left[\frac{\partial f}{\partial q_{ij}}(\xi_\sigma) - \frac{\partial f}{\partial q_{ij}}(\xi_0)\right]\cdot(D_iD_j w(t,x) - D_iD_j z_\psi(t,x))\Big\} d\sigma + [f(\xi_0) - D_t z_\psi(t,x)] + \\ + \left[\frac{\partial f}{\partial q_{ij}}(\xi_0) - \frac{\partial f}{\partial q_{ij}}(t_0, x, \psi(x), D\psi(x), D^2\psi(x))\right]\cdot(D_iD_j w(t,x) - D_iD_j z_\psi(t,x)) = \\ =: I_1 + I_2 + I_3.$$

By (A.12), (A.14), (A.1) and (A.2) we find

$$[I_1]_{C^\alpha(C)} \le c_{26}(\alpha,\Omega,f,\varphi,r_0,\epsilon,M) \quad (\tau - t_0)^\alpha,$$

and similary, since $\frac{\partial f}{\partial q_{ij}}(t_0, x, \psi(x), D\psi(x), D^2\psi(x))$ does not depend on t, we get

$$[I_3]_{C^\alpha(C)} \le c_{26}(\alpha,\Omega,f,\varphi,r_0,\epsilon,M) \quad (\tau - t_0)^\alpha;$$

on the other hand by (A.12) and (A.7) we have

$$[I_2]_{C^\alpha(C)} \le c_{27}(\alpha,\Omega,f,\varphi,r_0,\epsilon).$$

Similary, using (A.12), (A.14), (A.3) and (A.7) we obtain

$$\sup_{t\in[t_0,\tau]} [I_1(t)]_{C^{2\alpha}} + \sup_{t\in[t_0,\tau]} [I_3(t)]_{C^{2\alpha}} \le c_{28}(\alpha,\Omega,f,\varphi,r_0,\epsilon,M) \quad (\tau - t_0)^\alpha,$$
$$\sup_{t\in[t_0,\tau]} [I_2(t)]_{C^{2\alpha}} \le c_{29}(\alpha,\Omega,f,\varphi,r_0,\epsilon).$$

This shows that

$$\text{(A.21)} \qquad [F_{\psi,w}]_{C^{\alpha,2\alpha}} \le c_{30}(\alpha,\Omega,f,\varphi,r_0,\epsilon) + c_{31}(\alpha,\Omega,f,\varphi,r_0,\epsilon,M) \quad (\tau - t_0)^\alpha.$$

Now we split $G_{\psi,w}$. Setting

$$\xi_\sigma := (t, x, z_\psi(t,x) + \sigma\,(w(t,x) - z_\psi(t,x))\,, Dz_\psi(t,x) + \sigma\,(Dw(t,x) - Dz_\psi(t,x))) \in \Lambda',$$

we can write:

$$G_{\psi,w}(t,x) = -\int_0^1 \Big\{ \frac{\partial g}{\partial u}(\xi_\sigma)\cdot(w(t,x) - z_\psi(t,x)) + \\ + \left[\frac{\partial g}{\partial p_j}(\xi_\sigma) - \frac{\partial g}{\partial p_j}(\xi_0)\right]\cdot D_j(w(t,x) - z_\psi(t,x))\Big\} d\sigma - \\ - g(\xi_0) - \left[\frac{\partial g}{\partial p_j}(\xi_0) - \frac{\partial g}{\partial p_j}(t_0, x, \psi(x), D\psi(x))\right]\cdot D_j(w(t,x) - z_\psi(t,x)) = \\ =: I_1 + I_2 + I_3.$$

Proceeding as above, using (A.1), (A.4), (A.3), (A.7), (A.11), (A.13), (A.15), it is just routine to check that

$$\text{(A.22)} \qquad [G_{\psi,w}]_{C^{\alpha+1/2}(C)} + \sup_{t\in[t_0,\tau]} [DG_{\psi,w}]_{C^{2\alpha}} \le \\ \le c_{32}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon) + c_{33}(\alpha,\Omega,f,\varphi,r_0,\epsilon,M) \ (\tau - t_0)^\alpha;$$

by (A.20), (A.21) and (A.22), we get (3.17).

Proof of (3.18).

We rewrite $S_\psi(v) - S_\chi(w)$ as:

$$S_\psi(v) - S_\chi(w) = [S_\psi(v) - z_\psi] - [S_\chi(w) - z_\chi] + [z_\psi - z_\chi] =: u_1 - u_2 + [z_\psi - z_\chi].$$

The difference $u := u_1 - u_2$ solves the linear problem

$$\begin{cases} D_t u - \dfrac{\partial f}{\partial q_{ij}}(t_0, x, \psi, D\psi, D^2\psi)\cdot D_iD_j u = F_{\psi,\chi,v,w}(t,x), & (t,x) \in [t_0,\tau]\times\overline{\Omega}, \\ \dfrac{\partial g}{\partial p_j}(t_0, x, \psi, D\psi)\cdot D_j u = G_{\psi,\chi,v,w}(t,x), & (t,x)\in[t_0,\tau]\times\partial\Omega, \\ u(t_0,x) = 0, & x\in\overline{\Omega}, \end{cases}$$

where:

$$\begin{aligned}
F_{\psi,\chi,v,w}(t,x) := {} & \left[f(t,x,v,Dv,D^2v) - f(t,x,w,Dw,D^2w)\right] - \\
& - \left[\frac{\partial f}{\partial q_{sj}}(t_0,x,\psi,D\psi,D^2\psi)\cdot D_sD_jv - \frac{\partial f}{\partial q_{sj}}(t_0,x,\chi,D\chi,D^2\chi)\cdot D_sD_jw\right] + \\
& + \left[\frac{\partial f}{\partial q_{sj}}(t_0,x,\psi,D\psi,D^2\psi) - \frac{\partial f}{\partial q_{sj}}(t_0,x,\chi,D\chi,D^2\chi)\right]\cdot D_sD_jS_\chi(w) - \\
& - \left[D_t \frac{\partial f}{\partial q_{sj}}(t_0,x,\psi,D\psi,D^2\psi)\cdot D_sD_j\right](z_\psi - z_\chi), \\
G_{\psi,\chi,v,w}(t,x) := {} & \left[g(t,x,v,Dv) - g(t,x,w,Dw)\right] + \\
& + \left[\frac{\partial g}{\partial p_j}(t_0,x,\psi,D\psi)\cdot D_jv - \frac{\partial g}{\partial p_j}(t_0,x,\chi,D\chi)\cdot D_jw\right] - \\
& - \left[\frac{\partial g}{\partial p_j}(t_0,x,\psi,D\psi) - \frac{\partial g}{\partial p_j}(t_0,x,\chi,D\chi)\right]\cdot D_jS_\chi(w) - \\
& - \left[\frac{\partial g}{\partial p_j}(t_0,x,\psi,D\psi)\cdot D_j\right](z_\psi - z_\chi).
\end{aligned}$$

It is easy to check that

$$F_{\psi,\chi,v,w}(t_0,x) \equiv 0 \quad \text{in} \quad \Omega, \qquad G_{\psi,\chi,v,w}(t_0,x) \equiv 0 \quad \text{on} \quad \partial\Omega,$$

so that by the linear estimate (1.6) we have:

$$\begin{aligned}
\text{(A.23)} \quad & \|S_\psi(v) - S_\chi(w)\|_{E_{t_0,\tau}} \le \|z_\psi - z_\chi\|_{E_{t_0,\tau}} + c_{34}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon)\Big\{[F_{\psi,\chi,v,w}]_{C^\alpha(C)} + \\
& + \sup_{t\in[t_0,\tau]}[F_{\psi,\chi,v,w}]_{C^{2\alpha}} + [G_{\psi,\chi,v,w}]_{C^{\alpha+\frac12}(C)} + \sup_{t\in[t_0,\tau]}[D_xG_{\psi,\chi,v,w}]_{C^{2\alpha}}\Big\}.
\end{aligned}$$

Firstly we remark that by (A.7), (A.10), (A.12) it is not difficult to deduce that

$$\begin{aligned}
\text{(A.24)} \quad & \|z_\psi - z_\chi\|_{C^{1+\alpha,2+2\alpha}} + \|D_t(z_\psi - z_\chi)\|_{C^{\alpha,2\alpha}} + \|D_x(z_\psi - z_\chi)\|_{C^{1/2+\alpha,1+2\alpha}} + \\
& + \|D_x^2(z_\psi - z_\chi)\|_{C^{\alpha,2\alpha}} \le\le c_{35}(\alpha,\Omega,f,\varphi,r_0,\epsilon)\|\psi - \chi\|_{C^{2+2\alpha}}.
\end{aligned}$$

Next, we split $F_{\psi,\chi,v,w}$: setting

$$\begin{aligned}
\xi_\sigma := {} & \Big(t,x,w(t,x) + \sigma\left(v(t,x) - w(t,x)\right), Dw(t,x) + \sigma\left(Dv(t,x) - Dw(t,x)\right), \\
& D^2w(t,x) + \sigma\left(D^2v(t,x) - D^2w(t,x)\right)\Big) \in A, \\
\xi'_\sigma := {} & \Big(t_0,x,\chi(x) + \sigma\left(\psi(x) - \chi(x)\right), D\chi(x) + \sigma\left(D\psi(x) - D\chi(x)\right), \\
& D^2\chi(x) + \sigma\left(D^2\psi(x) - D^2\chi(x)\right)\Big) \in A,
\end{aligned}$$

we get:

$$\begin{aligned}
F_{\psi,\chi,v,w}(t,x) = {} & \int_0^1 \Big\{\frac{\partial f}{\partial u}(\xi_\sigma)\cdot(v(t,x) - w(t,x)) + \frac{\partial f}{\partial p_j}(\xi_\sigma)\cdot D_j\left(v(t,x) - w(t,x)\right) + \\
& + \int_0^1 \Big[\frac{\partial^2 f}{\partial q_{sj}\partial u}(\xi_{\sigma r})\cdot(v(t,x) - w(t,x)) + \frac{\partial^2 f}{\partial q_{sj}\partial p_i}(\xi_{\sigma r})\cdot(D_iv(t,x) - D_iw(t,x)) + \\
& + \frac{\partial^2 f}{\partial q_{sj}\partial q_{ki}}(\xi_{\sigma r})\cdot(D_kD_iv(t,x) - D_kD_iw(t,x))\Big]dr\cdot(D_sD_jv(t,x) - D_sD_jw(t,x))\Big\}d\sigma + \\
& + \left[\frac{\partial f}{\partial q_{si}}(t,x,w,Dw,D^2w) - \frac{\partial f}{\partial q_{sj}}(t_0,x,\psi,D\psi,D^2\psi))\right]\cdot D_sD_j\left(v(t,x) - w(t,x)\right) - \\
& - \left[\frac{\partial f}{\partial q_{sj}}(t_0,x,\psi,D\psi,D^2\psi) - \frac{\partial f}{\partial q_{sj}}(t_0,x,\chi,D\chi,D^2\chi))\right]\cdot D_sD_j\left(w - S_\chi(w)\right) + \\
& + \left[D_t - \frac{\partial f}{\partial q_{sj}}(t_0,x,\psi,D\psi,D^2\psi)\cdot D_sD_j\right](z_\psi - z_\chi)(t,x) =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

By using once again (A.12), (A.10), (A.14), (A.24), (A.1), (A.2) and (A.3) we find that

$$\begin{aligned}
\text{(A.25)} \quad & \sum_{i=1}^4 [I_i]_{C^\alpha(C)} + \sup_{t\in[t_0,\tau]}\{[I_1(t)]_{C^{2\alpha}} + [I_2(t)]_{C^{2\alpha}} + [I_4(t)]_{C^{2\alpha}}\} \le \\
& \le c_{36}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon,M)\left\{\|\psi - \chi\|_{C^{2+2\alpha}} + (\tau - t_0)^\alpha\|v - w\|_{E_{t_0,\tau}}\right\};
\end{aligned}$$

on the other hand we similarly see that

$$\begin{aligned}
\text{(A.26)} \quad & \sup_{t\in[t_0,\tau]}[I_3(t)]_{C^{2\alpha}} \le c_{37}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon,M)\Big\{\|\psi - \chi\|_{C^2} + \\
& + (\tau - t_0)^\alpha \sum_{s,j=1}^n \left[\frac{\partial f}{\partial q_{sj}}(t_0,\cdot,\psi,D\psi,D^2\psi)\right] - \left[\frac{\partial f}{\partial q_{sj}}(t_0,\cdot,\chi,D\chi,D^2\chi)\right]_{C^{2\alpha}}\Big\} \le \\
& \le c_{38}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon,M)\;\|\psi - \chi\|_{C^{2+2\alpha}},
\end{aligned}$$

where in the last estimate we have used the fact that $f \in C^3(A, \mathbf{C}^N)$.

Our last task is the estimate for $G_{\psi,\chi,v,w}$. With

$$\xi_\sigma := (t,x,w(t,x) + \sigma(v(t,x) - w(t,x)), Dw(t,x) + \sigma(Dv(t,x) - Dw(t,x))) \in A',$$

we split $G_{\psi,\chi,v,w}$ as:

$$
\begin{aligned}
G_{\psi,\chi,v,w}(t,x) = -\int_0^1 \Big\{ & \frac{\partial g}{\partial u}(\xi_\sigma)\cdot(v(t,x)-w(t,x)) + \\
& + \left[\frac{\partial g}{\partial p_j}(\xi_\sigma) - \frac{\partial g}{\partial p_j}(\xi_0)\right]\cdot(D_j v(t,x) - D_j w(t,x))\Big\}\, d\sigma - \\
& - \left[\frac{\partial g}{\partial p_j}(t,x,w,Dw) - \frac{\partial g}{\partial p_j}(t_0,x,\psi,D\psi)\right]\cdot(D_j v(t,x) - D_j w(t,x)) + \\
& + \left[\frac{\partial g}{\partial p_j}(t_0,x,\psi,D\psi) - \frac{\partial g}{\partial p_j}(t_0,x,\chi,D\chi)\right]\cdot D_j\,(w - S_\chi(w)) - \\
& - \frac{\partial g}{\partial p_j}(t_0,x,\psi,D\psi) D_j(z_\psi - z_\chi) =: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

By (A.13), (A.11), (A.15), (A.24), (A.1), (A.3) and (A.4) we get after standard calculations:

(A.27)
$$
\begin{aligned}
\sum_{i=1}^{4} [I_i]_{C^{\alpha+1/2}(C)} + \sup_{t\in[t_0,\tau]} \{[DI_1(t)]_{C^{2\alpha}} + [DI_2(t)]_{C^{2\alpha}} + [DI_4(t)]_{C^{2\alpha}}\} \le \\
\le c_{39}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon,M)\,\{\|\psi-\chi\|_{C^{2+2\alpha}} + (\tau-t_0)^\alpha \|v-w\|_{E_{t_0,\tau}}\},
\end{aligned}
$$

whereas

(A.28)
$$
\begin{aligned}
\sup_{t\in[t_0,\tau]} [DI_3(t)]_{C^{2\alpha}} \le c_{40}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon,M)\Big\{ \|\psi-\chi\|_{C^2} + \\
+ (\tau-t_0)^{\alpha+1/2} \sum_{j=1}^{n} \left[D\left(\frac{\partial g}{\partial p_j}(t_0,\cdot,\psi,D\psi) - \frac{\partial g}{\partial p_j}(t_0,\cdot,\chi,D\chi)\right)\right]_{C^{2\alpha}} \Big\} \le \\
\le c_{41}(\alpha,\Omega,f,g,\varphi,r_0,\epsilon,M)\,\|\psi-\chi\|_{C^{2+2\alpha}},
\end{aligned}
$$

where we have used the fact that $g \in C^4(\Lambda', \mathbf{C}^N)$.

By (A.23), (A.24), (A.25), (A.26), (A.27) and (A.28) we finally get (3.18).

The proof of Lemma 3.5 is complete. ◇

REMARK A.4. Hypothesis 3.1 is needed only in the estimates (A.26), (A.28). For all other calculations Hypothesis 0.1 is sufficient. This is the case, for example, when $\psi = \chi$ in Lemma 3.5.

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