ZYGMUND CLASSES WITH BOUNDARY CONDITIONS AS INTERPOLATION SPACES

Paolo ACQUISTAPACE

UNIVERSITÀ DEGLI STUDI DI PISA

DIPARTIMENTO DI MATEMATICA

GENNAIO 1988
NUMERO 220



# ZYGMUND CLASSES WITH BOUNDARY CONDITIONS AS INTERPOLATION SPACES

Paolo Acquistapace

## 0. INTRODUCTION

This paper is concerned with the characterization of the real interpolation spaces $(D_A, E)_{\alpha,\infty}$ [Lions-Peetre, 1964] and $(D_A, E)_\alpha$ [Da Prato-Grisvard, 1979], where A is an elliptic differential operator of order 2m, with general boundary conditions, and E is the Banach space of continuous functions on a bounded open set $\Omega \subset \mathbb{R}^n$; following [Grisvard, 1969], we denote such spaces respectively by $D_A(\theta,\infty)$ and $D_A(\theta)$, where $\theta = 1-\alpha$.

In an earlier paper [Acquistapace-Terreni, 1987] we studied the case $2m\theta \notin \mathbb{N}$; the purpose here is to study the "critical" cases $2m\theta = q \in \mathbb{N}$. All notations here are the same as in [Acquistapace-Terreni, 1987].

## 1. ASSUMPTIONS

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$, $n \geq 1$, with $C^{2m}$ boundary, $m \geq 1$. We introduce the differential operators

(1.1) $$A(x,D) := \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha , \quad x \in \overline{\Omega}$$

(1.2) $$B_j(x,D) := \sum_{|\beta| = m_j} b_{j\beta}(x) D^\beta , \quad x \in \partial\Omega , \quad j = 1,\ldots,m,$$

under the following assumptions:

(1.3) $$a_\alpha \in C(\overline{\Omega}, \mathbb{C}) , \ |\alpha| \leq 2m ; \ b_{j\beta} \in C^{2m-m_j}(\partial\Omega, \mathbb{C}) , \ |\beta| \leq m_j , \ j=1,\ldots,m$$

(1.4) (ellipticity) There exist $\eta \in [0, 2\pi[$ , $\nu > 0$ such that

$$\nu(|\xi|^{2m} + t^{2m}) \leq \Big| \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha - (-1)^m e^{i\eta} t^{2m} \Big| \quad \forall x \in \overline{\Omega}, \ \forall \xi \in \mathbb{R}^n, \ \forall t \in \mathbb{R}.$$

(1.5) (root condition) If $x\in\partial\Omega$, $\xi\in\mathbb{R}^n$, $t\in\mathbb{R}$ and $|\xi|+|t|>0$, $(\xi|v(x))=0$, then the polynomial

$$\zeta \mapsto \sum_{|\alpha|=2m} a_\alpha(x)(\xi+\zeta v(x))^\alpha-(-1)^m e^{i\eta} t^{2m}$$

has exactly m roots $\zeta_j^+(x,\xi,t)$ with positive imaginary part.

(1.6) (complementing condition) If $x\in\partial\Omega$, $\xi\in\mathbb{R}^n$, $t\in\mathbb{R}$ and $|\xi|+|t|>0$, $(\xi|v(x))=0$, then the m polynomials

$$\zeta \mapsto \sum_{|\beta|=m_j} b_{j\beta}(x)(\xi+\zeta v(x))^\beta$$

are linearly independent modulo the polynomial

$$\zeta \mapsto \prod_{j=1}^m (\zeta-\zeta_j^+(x,\xi,t)).$$

(1.7) $0 \le m_j \le m_i \le 2m-1$ if $1 \le j < i \le m$.

REMARK 1.1. Condition (1.7) replaces here the normality condition assumed in [Acquistapace-Terreni, 1987].
Indeed, it is easily seen that the transversality condition ((1.9) of that paper), i.e.

$$\sum_{|\beta|=m_j} b_{j\beta}(x)v(x)^\beta \neq 0 , \quad x\in\partial\Omega , \; j=1,\dots,m$$

is implied by the complementing condition (1.6); thus the only difference here with respect to [Acquistapace-Terreni, 1987] is that we just require the orders of the boundary operators to be less than 2m, without forcing them to be different from one another.
We remark that this weakened form of the normality condition is sufficient to prove all results of [Acquistapace-Terreni, 1987] : actually, the proof of the main result there [Theorem 2.3] depends only on the



basic elliptic existence theory and spectral estimates [Theorems 1.1-1.2 of that paper] which in turn still hold under these assumptions, as shown in [Geymorat-Grisvard, 1967, Theorem 4.1] .

Under hypotheses (1.1),...,(1.7) the abstract operator A, defined in the space $E := C(\overline{\Omega})$ by

$$(1.8)\quad \begin{cases} D_A := \{u\in \bigcap_{p\in[1,\infty[} W^{2m,p}(\Omega): A(\cdot,D)u\in C(\overline{\Omega}),\ B_j(\cdot,D)u=0 \text{ on } \partial\Omega \\ \qquad\qquad \text{for } j=1,\dots,m\} , \\ Au := A(\cdot,D)u \end{cases}$$

is the infinitesimal generator of an analytic semigroup in E [Stewart, 1980]; in particular, possibly replacing $A(\cdot,D)$ by $A(\cdot,D)-\omega I$ $(\omega>0)$ we may assume that

$$(1.9)\quad \rho(A)\supset\{\lambda\in\mathbb{C}:\text{Re }\lambda>0\} , \quad \|R(\lambda,A)\|_{L(E)} \le \frac{M}{|\lambda|} \text{ if } \text{Re }\lambda>0.$$

Hence the spaces $D_A(\theta,\infty)$ and $D_A(\theta)$ can be characterized [Grisvard, 1969] by:

$$(1.10)\quad D_A(\theta,\infty)=\{x\in E : [x]_{D_A(\theta,\infty)} := \sup_{s\ge 1} s^\theta \|AR(s,A)x\|_E <\infty\},$$

$$(1.11)\quad D_A(\theta) = \{x\in D_A(\theta,\infty): \lim_{s\to\infty} s^\theta \|AR(s,A)x\|_E = 0\} .$$

## 2. PRELIMINARIES AND THE MAIN RESULT.

We list here some preliminary results which are necessary in order to state our main theorem. First of all we define the Zygmund classes: if $\Omega\subset\mathbb{R}^n$ is an open set, we define for $q\in\mathbb{N}^+$ :

$$(2.1)\quad \Lambda^q(\overline{\Omega}) := \{f\in C^{q-1}(\overline{\Omega}): [f]_{\Lambda^q(\overline{\Omega})}$$

$$:= \sum_{|\alpha|=q-1} \sup\left\{\frac{|D^\alpha f(x)+D^\alpha f(y)-2D^\alpha f((x+y)/2)|}{|x-y|} : x,y,\frac{x+y}{2}\in\overline{\Omega}\right\}<\infty\},$$

(2.2) $$\lambda^q(\overline{\Omega}):=\{f\in\Lambda^q(\overline{\Omega}) : \lim_{r\downarrow 0}\ \sup_{x_0\in\overline{\Omega}}\ [f]_{\Lambda^q(\overline{\Omega\cap B(x_0,r)})}=0\}$$

where the space $C^{q-1}(\overline{\Omega})$ consists of functions whose derivatives of order $|\alpha|\leq q-1$ are uniformly continuous and bounded in $\overline{\Omega}$.

It is well known that $\Lambda^q(\mathbb{R}^n)$ can be described in an alternative way [Triebel, 1978, Theorem 2.7.2/2] :

(2.3) $$\Lambda^q(\mathbb{R}^n) = \{f\in C(\mathbb{R}^n) : \sup\{|h|^{-q}|(\Delta_h^{q+1}f)(x)|\,|\,h\in\mathbb{R}^n-\{0\},\ x\in\mathbb{R}^n\}<\infty\},$$

where

(2.4) $$(\Delta_h^{q+1}f)(x) := \sum_{j=0}^{q+1}(-1)^{q+1-j}\binom{q+1}{j}f(x+jh).$$

Moreover we have the proper inclusions

$$C^q(\overline{\Omega})\hookrightarrow Lip^q(\overline{\Omega})\hookrightarrow\Lambda^q(\overline{\Omega})\hookrightarrow C^\alpha(\overline{\Omega})\quad\text{if}\quad 0\leq\alpha<q\quad\text{and}\quad q\in\mathbb{N}^+,$$

provided $\partial\Omega$ is of class $C^1$.

It will be useful the following extension property:

PROPOSITION 2.1. If $\Omega$ is bounded with $\partial\Omega\in C^q$, then there exists an extension operator $E:\Lambda^q(\overline{\Omega})\to\Lambda^q(\mathbb{R}^n)$ such that:

(i) $Ef|_{\overline{\Omega}}\equiv f$,

(ii) $Ef$ has compact support in $\mathbb{R}^n$,

(iii) $\|Ef\|_{\Lambda^q(\mathbb{R}^n)}\leq C(\Omega,n,q)\,\|f\|_{\Lambda^q(\overline{\Omega})}$. ∎

This result holds under much more general assumption [Jonsson-Wallin,



1979, Theorem 5.1] , but when $\partial\Omega\in C^q$ it is possible to give an easier proof, which we omit for brevity.

We now turn to some interpolation properties.

PROPOSITION 2.2. If $\Omega$ is bounded with $\partial\Omega\in C^q$, then

$$\Lambda^q(\overline{\Omega}) = (C^{q+\varepsilon}(\overline{\Omega}),C^{q-\varepsilon}(\overline{\Omega}))_{1/2,\infty}\qquad\forall\varepsilon\in]0,1[$$

Proof. The case $\partial\Omega\in C^\infty$ is proved in [Triebel, 1978, Theorem 4.5.2/1]. Let $f\in\Lambda^q(\overline{\Omega})$. By Proposition 2.1, we extend $f$ to a function $F\in\Lambda^q(\mathbb{R}^n)$. As, by [Triebel, 1978, Theorem 2.7.2/1] ,

$$\Lambda^q(\mathbb{R}^n) = (C^{q+\varepsilon}(\mathbb{R}^n)\ ,\ C^{q-\varepsilon}(\mathbb{R}^n))_{1/2,\infty},$$

we have, by definition, $F=U(0,\cdot)$, where $U:]0,1]\to C^{q+\varepsilon}(\mathbb{R}^n)$ satisfies

$$\sup_{t\in]0,1]} t^{1/2}\{\|U(t)\|_{C^{q+\varepsilon}(\mathbb{R}^n)}+\|U'(t)\|_{C^{q-\varepsilon}(\mathbb{R}^n)}\}\leq K.$$

Hence $u(t):=U(t)|_{\overline{\Omega}}$ satisfies the same inequality with $\mathbb{R}^n$ replaced by $\overline{\Omega}$, and $u(0)=f$. Thus $f\in(C^{q+\varepsilon}(\overline{\Omega}),C^{q-\varepsilon}(\overline{\Omega}))_{1/2,\infty}$. Conversely, let $f\in(C^{q+\varepsilon}(\overline{\Omega}),C^{q-\varepsilon}(\overline{\Omega}))_{1/2,\infty}$. Then $f=u(0)$, with $u(t)$ satisfying

(2.5) $$\sup_{t\in]0,1]} t^{1/2}\{\|u(t)\|_{C^{q+\varepsilon}(\overline{\Omega})}+\|u'(t)\|_{C^{q-\varepsilon}(\overline{\Omega})}\}\leq K_f$$

Let now $|\alpha|=q-1$ and fix $x,y,\frac{x+y}{2}\in\overline{\Omega}$. If $\Omega$ is convex, we write

$$|D^\alpha f(x)+D^\alpha f(y)-2D^\alpha f((x+y)/2)|\leq|[\Delta^2_{\frac{y-x}{2}}D^\alpha(f-u(t,\cdot))](x)|$$

$$+|(\Delta^2_{\frac{y-x}{2}}D^\alpha u(t,\cdot))(x)| = |\int_0^t[\Delta^2_{\frac{y-x}{2}}D^\alpha u_s(s,\cdot)](x)ds|$$

$$+|\int_0^1(\nabla D^\alpha u(t,\tfrac{x+y}{2}+s\tfrac{x-y}{2})-\nabla D^\alpha u(t,\tfrac{x+y}{2}-s\tfrac{x-y}{2})|\ \tfrac{x-y}{2})ds|$$

$$\leq c\,K_f|x-y|^{1-\varepsilon}t^{1/2}+c\,K_f|x-y|^{1+\varepsilon}t^{-1/2},$$

and choosing $t=\left(\frac{|x-y|}{\operatorname{diam}\Omega}\right)^{2\varepsilon}$ we obtain $f\in\Lambda^q(\overline{\Omega})$. If $\Omega$ is not convex we have to construct two $C^1$ curves contained in $\overline{\Omega}$, joining $\frac{x+y}{2}$ to x and y, and whose lengths do not exceed $M|x-y|$, where M depends only on $\Omega$; this can be done since $\partial\Omega$ is of class $C^1$ at least, as shown in [Acquistapace-Terreni, 1984, Lemma 1.16]. Then we can repeat the preceding argument, by integrating along such curves, and the result follows as above. ∎

PROPOSITION 2.3. Let $\Omega$ be a bounded open set with $\partial\Omega \in C^{k+1}$, $k \in \mathbb{N}^+$. Then

$$\|f\|_{C^k(\partial\Omega)} \le M_\Omega \|f\|^{1/2}_{C^{k+1}(\partial\Omega)} \|f\|^{1/2}_{C^{k-1}(\partial\Omega)} \quad \forall f \in C^{k+1}(\partial\Omega).$$

This estimate is also well known. ∎

Let us go back now to the situation described in Section 1. As $\partial\Omega \in C^{2m}$, the distance function

$$(2.6) \qquad d(x) := \inf\{|x-y| : y \in \partial\Omega\}, \quad x \in \overline{\Omega}$$

belongs to $C^{2m}((\partial\Omega)_r)$, when $(\partial\Omega)_r := \{x \in \overline{\Omega} : d(x) \le r\}$ and $r$ is a suitable positive number; in addition $\nabla d = -\nu$ on $\partial\Omega$, where $\nu(x)$ is the unit outward normal vector at $x \in \partial\Omega$ (see [Gilbarg-Trudinger, 1977, Appendix]): hence $\nu \in C^{2m-1}(\partial\Omega, \mathbb{R}^n)$.

We want now to define suitable subspaces of $\Lambda^q(\overline{\Omega})$ determined by some kind of boundary conditions. We start with considering, for fixed j, the boundary operator $B_j(x,D)$ defined in (1.2). For each $\gamma \in \mathbb{N}^n$ with $|\gamma| = m_j$ we set

$N(\gamma)$ := number of non-zero components of $\gamma$ =
= cardinality of $\{\beta \in \mathbb{N}^n : |\beta| = m_j - 1,\ \beta + e^i = \gamma$ for some $i = 1,\dots,m\}$.

Hence we have, denoting by $\overline{B_j}$ the principal part of $B_j$:

$$(2.7) \qquad \overline{B}_j(x,D)u = \sum_{|\gamma|=m_j} b_{j\gamma}(x) D^\gamma u(x) =$$



$$= \sum_{|\beta|=m_j-1} \sum_{i=1}^{n} [N(\beta+e^i)]^{-1} b_{j,\beta+e^i}(x) D^\beta D_i u(x)$$

Set now for $|\beta| = m_j - 1$ and $i = 1,\dots,n$

$$(2.8) \qquad c^{j,\beta}(x) := \{c_i^{j,\beta}(x)\}_{1\le i\le n}; \quad c_i^{j,\beta}(x) := [N(\beta+e^i)]^{-1} b_{j,\beta+e^i}(x),$$

$$Tc^{j,\beta}(x) := c^{j,\beta}(x) - (c^{j,\beta}(x)|\nu(x))\nu(x).$$

Then we can rewrite $\overline{B}_j(x,D)u$ as:

$$(2.9) \qquad \overline{B}_j(x,D)u = \sum_{|\beta|=m_j-1} \sum_{i=1}^{n} c_i^{j,\beta}(x) D^\beta D_i u(x) =$$

$$= \sum_{|\beta|=m_j-1} (c^{j,\beta}(x)|\nabla D^\beta u(x)) =$$

$$= \sum_{|\beta|=m_j-1} \{(c^{j,\beta}(x)|\nu(x)) \frac{\partial D^\beta u(x)}{\partial\nu(x)}$$

$$+ (\mathrm{Re}\, Tc^{j,\beta}(x)|\nabla D^\beta u(x)) + i(\mathrm{Im}\, Tc^{j,\beta}(x)|\nabla D^\beta u(x))\}.$$

We now introduce the integral curves associated to the (real) vector fields $\nabla d$, $\mathrm{Re}\, Tc^{j,\beta}$, $\mathrm{Im}\, Tc^{j,\beta}$, namely

$$(2.10) \qquad \begin{cases} \frac{d}{d\sigma}\mu(\sigma,x) = \nabla d(\mu(\sigma,x)), \quad \sigma>0 \\ \mu(0,x) = x \in \partial\Omega, \end{cases}$$

$$(2.11) \qquad \begin{cases} \frac{d}{d\sigma}\lambda^{j,\beta}(\sigma,x) = \mathrm{Re}\, Tc^{j,\beta}(\lambda^{j,\beta}(\sigma,x)) \quad \sigma>0, \\ \lambda^{j,\beta}(0,x) = x \in \partial\Omega, \end{cases}$$

$$(2.12) \qquad \begin{cases} \frac{d}{d\sigma}\eta^{j,\beta}(\sigma,x) = \mathrm{Im}\, Tc^{j,\beta}(\eta^{j,\beta}(\sigma,x)), \quad \sigma>0 \\ \eta^{j,\beta}(0,x) = x \in \partial\Omega. \end{cases}$$

$$= -\frac{1}{\sigma}\left[\frac{\partial f}{\partial\nu(X)}(X-\sigma X) - \frac{\partial f}{\partial\nu(X)}(X)\right],$$

and finally

$$D_A(\tfrac{1}{4},\infty)=\{f\in\Lambda^1(\bar\Omega): f=0 \text{ on } \partial\Omega\},\quad D_A(\tfrac{1}{4}) = \lambda^1(\bar\Omega)\cap D_A(\tfrac{1}{4},\infty);$$

$$D_A(\tfrac{1}{2},\infty)=\{f\in\Lambda^2(\bar\Omega): f=0 \text{ on } \partial\Omega,$$

$$\sup_{\sigma\in]0,\sigma_0]}\ \sup_{X\in\partial\Omega}\frac{1}{\sigma}\left|\frac{\partial f}{\partial\nu(X)}(X-\sigma\nu(X)) - \frac{\partial f}{\partial\nu(X)}(X)\right|<\infty\},$$

$$D_A(\tfrac{1}{2}) = \{f\in\Lambda^2(\bar\Omega): f=0 \text{ on } \partial\Omega,\ \lim_{\sigma\downarrow 0}\frac{1}{\sigma}\left[\frac{\partial f}{\partial\nu(X)}(X-\sigma\nu(X)) - \frac{\partial f}{\partial\nu(X)}(X)\right]$$

$$= b_1(X)f_x(X)+b_2(X)f_y(X)+b_0(X)f(X) \quad \forall X\in\partial\Omega\},$$

$$D_A(\tfrac{3}{4},\infty)=\{f\in\Lambda^3(\bar\Omega): f=0 \text{ and } B_2(X,D)f=0 \text{ on } \partial\Omega\},\ D_A(\tfrac{3}{4})=\lambda^3(\bar\Omega)\cap D_A(\tfrac{3}{4},\infty).$$

REMARK 2.6. In (2.13) we may replace the ratio $\frac{D^\beta f(\mu(\sigma,x))-D^\beta f(x)}{\sigma}$ by the simpler one

$$\frac{D^\beta f(x-\sigma\nu(x))-D^\beta f(x)}{\sigma};$$

indeed for $f\in\Lambda^q(\bar\Omega)$ the difference between such terms is $o(1)$ as $\sigma\downarrow 0$ (see Remark 5.3 below).

## 3. THE FIRST INCLUSION

Let $A$ be defined by (1.8) and suppose that (1.9) holds. We want to prove the following

THEOREM 3.1. If $\theta\in]0,1[$ and $q := 2m\theta\in\mathbb{N}$, then

$$\Lambda^q_B(\bar\Omega)\hookrightarrow D_A(\theta,\infty)$$

Proof. According to (1.10) it is enough to show that



$$= \sum_{|\beta|=m_j-1}\ \sum_{i=1}^{n}[N(\beta+e^i)]^{-1}\, b_{j,\beta+e^i}(x)D^\beta D_i u(x)$$

Set now for $|\beta|=m_j-1$ and $i=1,\dots,n$

(2.8) $$c^{j,\beta}(x):=\{c_i^{j,\beta}(x)\}_{1\le i\le n};\quad c_i^{j,\beta}(x):=[N(\beta+e^i)]^{-1}b_{j,\beta+e^i}(x),$$

$$Tc^{j,\beta}(x):=c^{j,\beta}(x) - (c^{j,\beta}(x)|\nu(x))\nu(x).$$

Then we can rewrite $\bar B_j(x,D)u$ as:

(2.9) $$\bar B_j(x,D)u = \sum_{|\beta|=m_j-1}\ \sum_{i=1}^{n} c_i^{j,\beta}(x)D^\beta D_i u(x) =$$

$$= \sum_{|\beta|=m_j-1}(c^{j,\beta}(x)|\nabla D^\beta u(x)) =$$

$$= \sum_{|\beta|=m_j-1}\{(c^{j,\beta}(x)|\nu(x))\frac{\partial D^\beta u(x)}{\partial\nu(x)}$$

$$+ (\operatorname{Re} Tc^{j,\beta}(x)|\nabla D^\beta u(x)) + i(\operatorname{Im} Tc^{j,\beta}(x)|\nabla D^\beta u(x))\}.$$

We now introduce the integral curves associated to the (real) vector fields $\nabla d$, $\operatorname{Re} Tc^{j,\beta}$, $\operatorname{Im} Tc^{j,\beta}$, namely

(2.10) $$\begin{cases}\frac{d}{d\sigma}\mu(\sigma,x) = \nabla d(\mu(\sigma,x)),\ \sigma>0\\ \mu(0,x) = x\in\partial\Omega,\end{cases}$$

(2.11) $$\begin{cases}\frac{d}{d\sigma}\lambda^{j,\beta}(\sigma,x)=\operatorname{Re} Tc^{j,\beta}(\lambda^{j,\beta}(\sigma,x))\ \ \sigma>0,\\ \lambda^{j,\beta}(0,x) = x\in\partial\Omega,\end{cases}$$

(2.12) $$\begin{cases}\frac{d}{d\sigma}\eta^{j,\beta}(\sigma,x) = \operatorname{Im} Tc^{j,\beta}(\eta^{j,\beta}(\sigma,x)),\ \sigma>0\\ \eta^{j,\beta}(0,x) = x\in\partial\Omega.\end{cases}$$

$$= -\frac{1}{\sigma}\left[\frac{\partial f}{\partial\nu(X)}(X-\sigma X) - \frac{\partial f}{\partial\nu(X)}(X)\right],$$

and finally

$$D_A(\tfrac{1}{4},\infty) = \{f\in\Lambda^1(\overline{\Omega}) : f=0 \text{ on } \partial\Omega\}, \quad D_A(\tfrac{1}{4}) = \lambda^1(\overline{\Omega})\cap D_A(\tfrac{1}{4},\infty);$$

$$D_A(\tfrac{1}{2},\infty) = \{f\in\Lambda^2(\overline{\Omega}) : f=0 \text{ on } \partial\Omega,$$

$$\sup_{\sigma\in]0,\sigma_0]}\ \sup_{X\in\partial\Omega} \frac{1}{\sigma}\left|\frac{\partial f}{\partial\nu(X)}(X-\sigma\nu(X)) - \frac{\partial f}{\partial\nu(X)}(X)\right| < \infty\},$$

$$D_A(\tfrac{1}{2}) = \{f\in\Lambda^2(\overline{\Omega}) : f=0 \text{ on } \partial\Omega,\ \lim_{\sigma\downarrow 0}\frac{1}{\sigma}\left[\frac{\partial f}{\partial\nu(X)}(X-\sigma\nu(X)) - \frac{\partial f}{\partial\nu(X)}(X)\right]$$

$$= b_1(X)f_x(X)+b_2(X)f_y(X)+b_0(X)f(X) \quad \forall X\in\partial\Omega\},$$

$$D_A(\tfrac{3}{4},\infty) = \{f\in\Lambda^3(\overline{\Omega}) : f=0 \text{ and } B_2(X,D)f=0 \text{ on } \partial\Omega\},\ D_A(\tfrac{3}{4}) = \lambda^3(\overline{\Omega})\cap D_A(\tfrac{3}{4},\infty).$$

<u>REMARK</u> 2.6. In (2.13) we may replace the ratio $\frac{D^\beta f(\mu(\sigma,x))-D^\beta f(x)}{\sigma}$ by the simpler one

$$\frac{D^\beta f(x-\sigma\nu(x))-D^\beta f(x)}{\sigma};$$

indeed for $f\in\Lambda^q(\overline{\Omega})$ the difference between such terms is $o(1)$ as $\sigma\downarrow 0$ (see Remark 5.3 below).

## 3. THE FIRST INCLUSION

Let $A$ be defined by (1.8) and suppose that (1.9) holds. We want to prove the following

<u>THEOREM</u> 3.1. If $\theta\in]0,1[$ and $q := 2m\theta\in\mathbb{N}$, then

$$\Lambda^q_B(\overline{\Omega}) \hookrightarrow D_A(\theta,\infty)$$

<u>Proof</u>. According to (1.10) it is enough to show that



$$(3.1)\qquad \sup_{s\geq 1} s^\theta \|AR(s,A)f\|_{C(\overline{\Omega})} \leq C_\theta \|f\|_{\Lambda^q_B(\overline{\Omega})} \qquad \forall f\in\Lambda^q_B(\overline{\Omega}).$$

As in [Acquistapace-Terreni, 1987], this will be done by constructing, for each fixed $f\in\Lambda^q_B(\overline{\Omega})$, a function $w : [1,\infty[\ \to C(\overline{\Omega})$ such that

$$(3.2)\qquad \|w(s)-f\|_{C(\overline{\Omega})} \leq C_\theta\, s^{-\theta}\|f\|_{\Lambda^q(\overline{\Omega})} \qquad \forall s\geq 1$$

$$(3.3)\qquad \|AR(s,A)w(s)\|_{C(\overline{\Omega})} \leq c_\theta\, s^{-\theta}\|f\|_{\Lambda^q_B(\overline{\Omega})} \qquad \forall s\geq 1.$$

Obviously (3.2) and (3.3) imply (3.1).

Let $f\in\Lambda^q_B(\overline{\Omega})$ and consider an extension $F\in\Lambda^q_0(\mathbb{R}^n)$, which exists by Proposition 2.1. Fix a real-valued function $\phi\in C_0^\infty(\mathbb{R}^n)$ such that $0\leq\phi\leq 1$, $\phi\equiv 0$ outside $B(0,1)$, $\int_{\mathbb{R}^n}\phi(z)dz=1$ and $\phi$ is even in each variable, and set

$$\phi_t(z) := t^{-n}\phi(\tfrac{z}{t}), \quad z\in\mathbb{R}^n, \quad t\in]0,1].$$

Define finally (compare with (2.4)):

$$v(t,x) := \int_{\mathbb{R}^n}[(-1)^q(\Delta_z^{q+1}F)(x)+F(x)]\,\phi_t(z)dz, \quad x\in\mathbb{R}^n, \quad t\in]0,1];$$

then clearly

$$v(t,x) - F(x) = \int_{\mathbb{R}^n}\sum_{h=0}^{q+1}(-1)^{h+1}\binom{q+1}{h}F(x+hz)\phi_t(z)dz.$$

Hence if $|\beta|\leq q-1$ we have by (2.3) (since $D^\beta F\in\Lambda^{q-|\beta|}(\mathbb{R}^n)$):

$$(3.5)\qquad |D^\beta v(t,x)-D^\beta F(x)| = \left|\int_{\mathbb{R}^n}(-1)^q(\Delta_z^{q+1}D^\beta F)(x)\phi_t(z)dz\right| \leq$$

$$\leq C_q[F]_{\Lambda^q(\mathbb{R}^n)}\int_{\mathbb{R}^n}|z|^{q-|\beta|}\phi_t(z)dz \leq$$

$$\leq c_q[f]_{\Lambda^q(\overline{\Omega})}\, t^{q-|\beta|}.$$

Next, if $|\alpha|=q-1$ and $|\gamma|$ is even and larger than 1, we get:

$$|D^{\alpha+\gamma}v(t,x)|=|D^{\gamma}\int_{\mathbb{R}^n}(-1)^q(\Delta_z^{q+1}D^{\alpha}F)(x)\phi_t(z)dz|\le$$

$$\le\sum_{h=1}^{q+1}\binom{q+1}{h}\left|\int_{\mathbb{R}^n}D^{\alpha}F(y)D^{\gamma}\phi_t\left(\frac{y-x}{h}\right)dy\right|h^{-|\gamma|}=$$

$$=\sum_{h=1}^{q+1}\binom{q+1}{h}\frac{h^{-|\gamma|}}{2}\left|\int_{\mathbb{R}^n}[D^{\alpha}F(x+hz)+D^{\alpha}F(x-hz)-\right.$$

$$\left.-2D^{\alpha}F(x)]\,D^{\gamma}\phi_t(z)dz\right|\le c_q[F]_{\Lambda^q(\mathbb{R}^n)}t^{1-|\gamma|}\le$$

$$\le c_q[f]_{\Lambda^q(\overline{\Omega})}t^{q-|\alpha+\gamma|};$$

thus if $|\beta|>q$ and $|\beta|-q$ is even we obtain

$$|D^{\beta}v(t,x)|\le c_q[f]_{\Lambda^q(\overline{\Omega})}t^{q-|\beta|}. \tag{3.6}$$

On the other hand if $|\beta|>q$ and $|\beta|-q$ is odd, by interpolation (3.6) yields

$$\|D^{\beta}v(t,\cdot)\|_{C(\mathbb{R}^n)}\le c\|\nabla^{|\beta|+1}v(t,\cdot)\|^{1/2}_{C(\mathbb{R}^n)}\|\nabla^{|\beta|-1}v(t,\cdot)\|^{1/2}_{C(\mathbb{R}^n)}\le \tag{3.7}$$

$$\le c_q[f]_{\Lambda^q(\overline{\Omega})}t^{q-|\beta|}.$$

By (3.6) and (3.7) we conclude that

$$\|\nabla^{|\beta|}v(t,\cdot)\|_{C(\mathbb{R}^n)}\le c_q[f]_{\Lambda^q(\overline{\Omega})}t^{q-|\beta|}. \tag{3.8}$$

Finally if $|\beta|=q$ we do not have the boundedness of $\|\nabla^q v(t,\cdot)\|_{C(\mathbb{R}^n)}$, but the weaker estimate (due to the fact that $F\in C^{q-\varepsilon}(\mathbb{R}^n)$ $\forall\varepsilon\in]0,q[$)

$$\|\nabla^q v(t,\cdot)\|_{C(\mathbb{R}^n)}\le c_{q,\varepsilon}[f]_{C^{q-\varepsilon}(\overline{\Omega})}t^{-\varepsilon}\quad\forall\varepsilon\in]0,q[. \tag{3.9}$$

Hence if we set

$$w(s)(x)=w(s,x):=v(s^{-1/2m},x),\quad x\in\overline{\Omega},\quad s\ge 1$$



we easily obtain:

PROPOSITION 3.2. We have $w\in C^{\infty}([1,\infty[\times\overline{\Omega})$ and:

(i) $\|\nabla^k w(s)-\nabla^k f\|_{C(\overline{\Omega})}\le c_q[f]_{\Lambda^q(\overline{\Omega})}s^{-(q-k)/2m}$ if $k<q$,

(ii) $\|\nabla^q w(s)\|_{C(\overline{\Omega})}\le c_{q,\varepsilon}[f]_{C^{2m(\theta-\varepsilon)}(\overline{\Omega})}s^{\varepsilon}$ $\forall\varepsilon\in]0,q/2m[$,

(iii) $\|\nabla^k w(s)\|_{C(\overline{\Omega})}\le c_q[f]_{\Lambda^q(\overline{\Omega})}s^{(k-q)/2m}$ if $k>q$. ∎

In particular, part (i) implies (3.2).

Let us prove (3.3). Set $u(s):=s\,R(s,A)w(s)$, $s\ge 1$, so that

$$AR(s,A)w(s)=u(s)-w(s),\quad s\ge 1. \tag{3.10}$$

Looking at the elliptic problem solved by $u(s)-w(s)$, and applying the spectral estimate in $C^0(\overline{\Omega})$ (see in particular [Acquistapace-Terreni, 1987, (3.13)] we have:

$$\|u(s)-w(s)\|_{C(\overline{\Omega})}\le c\{s^{-1}\|A(\cdot,D)w(s)\|_{C(\overline{\Omega})}+ \tag{3.11}$$

$$+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}s^{-(k+m_j)/2m}\|B_j(\cdot,D)w(s)\|_{C^k(\partial\Omega)}\}=:J_1+J_2.$$

Clearly by Proposition 3.2

$$J_1\le c\,s^{-1}\sum_{|\beta|\le 2m}\|D^{\beta}w(s)\|_{C(\overline{\Omega})}\le c\,s^{-1}\{1+s^{(2m-q)/2m}\}[f]_{\Lambda^q(\overline{\Omega})}\le \tag{3.12}$$

$$\le c\,s^{-q/2m}[f]_{\Lambda^q(\overline{\Omega})}.$$

In order to estimate $J_2$, we split it as follows:

$$J_2=c\sum_{j=1}^{m}\{\sum_{k=0}^{q-1-m_j}+\sum_{k=q-m_j}+\sum_{k=q+1-m_j}^{2m-m_j}\}\|B_j(\cdot,D)w(s)\|_{C^k(\partial\Omega)}. \tag{3.13}$$

$$\cdot\, s^{-(m_j+k)/2m} =: I_1 + I_2 + I_3 ,$$

where some of the sums may be empty (e.g. $I_2$ has no addenda when $m_j>q$). Again Proposition 3.2 yields

$$(3.14) \quad I_1 = c\sum_{j=1}^{m}\sum_{k=0}^{q-1-m_j} \|B_j(\cdot,D)(w(s)-f)\|_{C^k(\partial\Omega)} \cdot$$

$$\cdot\, s^{-(m_j+k)/2m} \le c_q[f]_{\Lambda^q(\bar\Omega)}\, s^{-q/2m} ,$$

$$(3.25) \quad I_3 \le c\sum_{j=1}^{m}\sum_{k=q+1-m_j}^{2m-m_j}\sum_{h=0}^{k+m_j} \|\nabla^h w(s)\|_{C(\bar\Omega)} \cdot s^{-(m_j+k)/2m}$$

$$\le c_q[f]_{\Lambda^q(\bar\Omega)}\{1+s^{-(q-k-m_j)/2m}\}\, s^{-(m_j+k)/2m} \le c_q[f]_{\Lambda^q(\bar\Omega)}\, s^{-q/2m};$$

concerning $I_2$, we will show that

$$(3.16) \quad \|B_j(\cdot,D)w(s)\|_{C^{q-m_j}(\partial\Omega)} \le c_q\{[f]_{\Lambda^q(\bar\Omega)} + [f]_{\Lambda^q_B(\bar\Omega)}\} \quad \forall s\ge 1, \forall j=1,\dots,m:$$

this will imply, by (3.13), (3.14) and (3.15), that

$$J_2 \le c_q\{[f]_{\Lambda^q(\bar\Omega)} + [f]_{\Lambda^q_B(\bar\Omega)}\}\, s^{-q/2m}$$

and, recalling (3.12), (3.10) and (3.11), the proof of (3.3) will be complete.

Let us prove (3.16) (note that this term appears only when $m_j \le q$). If $m_j<q$, by Propositions 2.3 and 3.2 we get:

$$(3.17) \quad \|B_j(\cdot,D)w(s)\|_{C^{q-m_j}(\partial\Omega)} \le$$

$$\le c\,\{\|B_j(\cdot,D)w(s)\|^{1/2}_{C^{q-m_j+1}(\partial\Omega)} \|B_j(\cdot,D)(w(s)-f)\|^{1/2}_{C^{q-m_j-1}(\partial\Omega)}\} \le$$

$$\le c_q\,\{s^{1/4m}s^{-1/4m}\}\,[f]_{\Lambda^q(\bar\Omega)} = c_q\,[f]_{\Lambda^q(\bar\Omega)} ,$$



and it remains to consider the case $m_j=q$, i.e. to estimate the quantity

$$B_j(x,D)w(s,x) = \bar B_j(x,D)w(s,x) + \mathring{B}(x,D)w(s,x) , \quad x\in\partial\Omega , \; s\ge 1,$$

where we have denoted by $\bar B_j$ the principal part of $B_j$ and by $\mathring{B}_j$ its lower order part. Obviously

$$(3.18) \quad |\mathring{B}_j(x,D)w(s,x)| \le c\|w(s)\|_{C^{m_j-1}(\bar\Omega)} \le c_q[f]_{\Lambda^q(\bar\Omega)} ;$$

on the other hand, recalling (2.13) we can write:

$$(3.19) \quad |\bar B_j(x,D)w(s,x)| \le |\bar B_j(x,D)w(s,x) - [\Delta_\sigma^{B_j} w(s,\cdot)](x)| +$$

$$+ |[\Delta_\sigma^{B_j}(w(s,\cdot)-f)](x)| + |(\Delta_\sigma^{B_j} f)(x)| =$$

$$=: T_1 + T_2 + T_3 ,$$

where $\sigma\in]0,\sigma_0]$. Now, taking into account (2.9) and (2.13) it can be checked that

$$T_1 \le c\,\sigma\{\|\nabla^{q+1}w(s)\|_{C(\bar\Omega)} + \|\nabla^q w(s)\|_{C(\bar\Omega)}\} \le c_q\sigma\, s^{1/2m}\,[f]_{\Lambda^q(\bar\Omega)} ,$$

$$T_2 \le c\,\sigma^{-1}\,\|\nabla^{q-1}(w(s)-f)\|_{C(\bar\Omega)} \le c_q\sigma^{-1}s^{-1/2m}\,[f]_{\Lambda^q(\bar\Omega)} ,$$

whereas, by assumption, $T_3$ does not exceed $[f]_{\Lambda^q_B(\bar\Omega)}$. Hence, choosing $\sigma := \sigma_0\cdot s^{-1/2m}$, we deduce

$$(3.20) \quad |\bar B_j(x,D)w(s,x)| \le c_q\{[f]_{\Lambda^q_B(\bar\Omega)}\} \quad \forall s\ge 1, \; \forall x\in\partial\Omega, \; \forall j=1,\dots,m.$$

By (3.17), (3.18) and (3.20) we finally obtain (3.16). The proof is complete. ∎

## 4. THE SECOND INCLUSION.

Let A be defined again by (1.8) and assume (1.9). We want to prove:

THEOREM 4.1. If $\theta\in]0,1[$ and $q := 2m\theta \in \mathbb{N}$, then

$$D_A(\theta,\infty) \hookrightarrow \Lambda^q_B(\overline{\Omega}).$$

Proof. Let $f\in D_A(\theta,\infty)$. By the Reiteration Theorem [Triebel, 1978, Theorem 1.10.2] and by [Acquistapace-Terreni, 987, Theorem 2.3] we have (with equivalence of norms)

$$D_A(\theta,\infty)=(D_A(\tfrac{q+\varepsilon}{2m},\infty)\ ,\ D_A(\tfrac{q-\varepsilon}{2m},\infty))_{\frac{1}{2},\infty} = (C_B^{q+\varepsilon}(\overline{\Omega}),C_B^{q-\varepsilon}(\overline{\Omega}))_{\frac{1}{2},\infty}$$

where $\varepsilon\in]0,1[$. Then, by definition [Lions-Peetre, 1964], there exists a function $u : ]0,1] \to C_B^{q+\varepsilon}(\overline{\Omega})$ such that:

$$\begin{cases} \sup_{t\in]0,1]} t^{1/2}\|u(t)\|_{C^{q+\varepsilon}(\overline{\Omega})} + \sup_{t\in]0,1]} t^{1/2}\|u'(t)\|_{C^{q-\varepsilon}(\overline{\Omega})} \le K_q[f]_{D_A(\theta,\infty)} \\ B_j(\cdot,D)u(t)=0 \text{ on } \partial\Omega \quad \forall t\in]0,1] \quad \text{if} \quad m_j\le q \\ u(0) = f. \end{cases}$$

Thus, in particular, $f\in (C^{q+\varepsilon}(\overline{\Omega}),C^{q-\varepsilon}(\overline{\Omega}))_{\frac{1}{2},\infty}$, i.e., by Proposition 2.1, $f\in\Lambda^q(\overline{\Omega})$; in addition $B_j(\cdot,D)f=0$ on $\partial\Omega$ if $m_j<q$, since $u(t) \to f$ in $C^{q-\varepsilon}(\overline{\Omega})$ as $t\downarrow 0$.

We have to show that $[f]_{\Lambda^q_B(\overline{\Omega})}$ is finite. The following lemma is crucial for this purpose:

LEMMA 4.2. Fix $|\gamma|=q-1$. We have:

(i) $|D^\gamma f(x)-D^\gamma f(y)-D^\gamma u(t,x)+D^\gamma u(t,y)| \le$
$\le c_q [f]_{D_A(\theta,\infty)} t^{1/2}|x-y|^{1-\varepsilon} \quad \forall t\in]0,1]\ ,\ \forall x,y\in\overline{\Omega};$



(ii) If $v\in \mathrm{Lip}(\overline{\Omega},\mathbb{R}^n)$ and $\beta:[0,\sigma_0] \to \overline{\Omega}$ solves

$$\begin{cases} \beta'(\sigma) = v(\beta(\sigma))\ ,\quad \sigma\in[0,\sigma_0] \\ \beta(0) = x\in\overline{\Omega} \end{cases}$$

then

$$\left|\frac{D^\gamma u(t,\beta(\sigma))-D^\gamma u(t,x)}{\sigma} - (\nabla D^\gamma u(t,x)|v(x))\right| \le c_q [f]_{D_A(\theta,\infty)} t^{-1/2}\sigma^\varepsilon$$

$$\forall t\in]0,1]\ ,\ \forall\sigma\in]0,\sigma_0].$$

Proof. (i) We have:

$$|D^\gamma f(x)-D^\gamma f(y)-D^\gamma u(t,x)+D^\gamma u(t,y)| = \left|\int_0^t [D^\gamma u_s(s,x)-D^\gamma u_s(s,y)]ds\right| \le$$
$$\le \int_0^t [u'(s)]_{C^{q-\varepsilon}(\overline{\Omega})} ds\ |x-y|^{1-\varepsilon} \le K_1 [f]_{D_A(\theta,\infty)} t^{1/2}|x-y|^{1-\varepsilon}.$$

(ii) We have:

$$\left|\frac{D^\gamma u(t,\beta(\sigma))-D^\gamma u(t,x)}{\sigma} - (\nabla D^\gamma u(t,x)|v(x))\right| =$$

$$= \left|\frac{1}{\sigma}\int_0^\sigma \frac{d}{d\rho} D^\gamma u(t,\beta(\rho))d\rho - (\nabla D^\gamma u(t,x)|v(x))\right| =$$

$$= \left|\frac{1}{\sigma}\int_0^\sigma [(\nabla D^\gamma u(t,\beta(\rho))|v(\beta(\rho))-(\nabla D^\gamma u(t,x)|v(x))]d\rho\right| \le$$

$$\le \frac{1}{\sigma}\int_0^\sigma \{[u(t)]_{C^{q+\varepsilon}(\overline{\Omega})} \|v\|^{1+\varepsilon}_{C(\overline{\Omega})}\rho^\varepsilon + \|\nabla^q u(t)\|_{C(\overline{\Omega})}[v]_{\mathrm{Lip}(\overline{\Omega})}\|v\|_{C(\overline{\Omega})}\rho\}d\rho \le$$

$$\le c\,K_q [f]_{D_A(\theta,\infty)} t^{-1/2}\sigma^\varepsilon. \ ∎$$

Let us show that $[f]_{\Lambda^q_B(\overline{\Omega})}$ is finite. Indeed, let $m_j=q$ and fix $x\in\partial\Omega$; by Lemma 4.2, recalling that $B_j(x,D)u(t,x)=0$, we get:

$$|(\Delta_\sigma^{B_j}f)(x)| \le |(\Delta_\sigma^{B_j}[f-u(t,\cdot)])(x)|+|[\Delta_\sigma^{B_j}u(t,\cdot)](x)-\overline{B}_j(x,D)u(t,x)|+$$

$$+ |\overset{\circ}{B}_j(x,D)[u(t,x)-f(x)]| + |\overset{\circ}{B}_j(x,D)f(x)| =: S_1+S_2+S_3+S_4.$$

Now by Lemma 4.2(i)

$$S_1 \le c_q \sigma^{-1} [f]_{D_A(\theta,\infty)} t^{1/2} \sum_{|\gamma|=q-1} \{ |\mu(\sigma,x)-x|^{1-\varepsilon} + |\lambda^{j\gamma}(\sigma,x)-x|^{1-\varepsilon} +$$

$$+ |\eta^{j\gamma}(\sigma,x)-x|^{1-\varepsilon} \} \le c_q [f]_{D_A(\theta,\infty)} t^{1/2} \sigma^{-\varepsilon},$$

and by Lemma 4.2(ii)

$$S_2 \le c_q [f]_{D_A(\theta,\infty)} t^{-1/2} \sigma^{\varepsilon} ;$$

on the other hand

$$S_3 \le \int_0^t \|u'(s)\|_{C^{q-1}(\overline{\Omega})} ds \le c_q t^{1/2} [f]_{D_A(\theta,\infty)} ,$$

and finally

$$S_4 \le c\|f\|_{C^{q-1}(\overline{\Omega})} \le c\|f\|_{D_A(\theta,\infty)} .$$

Hence choosing $t = (\frac{\sigma}{\sigma_0})^{2\varepsilon}$ we get

$$[f]_{\Lambda^q_B(\overline{\Omega})} \le c_q \|f\|_{D_A(\theta,\infty)} ,$$

and the proof is complete. ∎

5. IMPROVEMENTS AND REMARKS.

By Theorems 3.1 and 4.1 the first equality of Theorem 2.4 is established. In order to check the second one, just a few remarks are needed. Concerning the first inclusion, repeating the argument of Section 3 we see that the right-hand sides of the inequalities of Proposition 3.2 have to be multiplied by $o(1)$ (as $s\to\infty$), due to the fact that f belongs to $\lambda^q_B(\overline{\Omega})$ now. Consequently we get



$$(5.1) \qquad \lim_{s\to\infty} s^{\theta} \|w(s)-f\|_{C(\overline{\Omega})} = 0,$$

which replaces (3.1). In order to get the analogous of (3.2), i.e.

$$(5.2) \qquad \lim_{s\to\infty} s^{\theta} \|AR(s,A)w(s)\|_{C(\overline{\Omega})} = 0,$$

it is readily seen that the main point is to show that if $m_j \le q$

$$\lim_{s\to\infty} s^{-\theta} \|B_j(\cdot,D)w(s)\|_{C^{q-m_j}(\partial\Omega)} = 0.$$

This is easily proved, similarly to (3.17), when $m_j<q$. In the case $m_j=q$ we split

$$B_j(x,D)w(s,x) = \overline{B}_j(x,D)w(s,x) + \overset{\circ}{B}_j(x,D)[w(s,x)-f(x)] + \overset{\circ}{B}_j(x,D)f(x);$$

but

$$|\overset{\circ}{B}_j(x,D)[w(s,x)-f(x)]| \le c\|w(s)-f\|_{C^{m_j-1}(\overline{\Omega})} \le c_q [f]_{\Lambda^q(\overline{\Omega})} s^{-1} = o(1) \text{ as } s\to\infty,$$

whereas, using the notations of (3.19),

$$|\overline{B}_j(x,D)w(s,x) + \overset{\circ}{B}_j(x,D)f(x)| \le T_1 + T_2 + |(\Delta_\sigma^{B_j} f)(x) + \overset{\circ}{B}_j(x,D)f(x)| \le$$

$$\le c_q \sigma s^{1/2m} o(1) + c_q \sigma^{-1} s^{-1/2m} o(1) + o(1) \quad \text{as} \quad s\to+\infty,$$

where in estimating the last term we have used the boundary condition satisfied by f. Taking $\sigma := \sigma_0 s^{-1/2m}$ we finally get (5.2), which together with (5.1) yields

$$\lim_{s\to+\infty} s^{\theta} \|AR(s,A)f\|_{C(\overline{\Omega})} = 0,$$

i.e., by (1.11), $f \in D_A(\theta)$.

The second inclusion is easier: if $f \in D_A(\theta)$, then, as $D_A(\theta)$ is the closure of $D_A$ in $D_A(\theta,\infty)$, we take a sequence $\{u_n\} \subset D_A$ such that $u_n \to f$

in $D_A(\theta,\infty)$: then

$$\begin{cases} u_n \to f \quad \text{in} \quad \Lambda^q(\overline{\Omega}) \\ [u_n - f]_{\Lambda^q_B(\overline{\Omega})} \to 0 . \end{cases}$$

The first condition implies $f\in\lambda^q(\overline{\Omega})$ since $D_A \hookrightarrow \lambda^q(\overline{\Omega})$ and $\lambda^q(\overline{\Omega})$ is a closed subspace of $\Lambda^q(\overline{\Omega})$; the second one easily yields (since $B_j(\cdot,D)u_n=0$ for $j=1,\ldots,m$):

$$\lim_{\sigma\downarrow 0} (\Delta_\sigma^{B_j} f)(x) = -\overset{\circ}{B}_j(x,D)f(x)$$

provided $x\in\partial\Omega$ and $m_j=q$. Thus $f\in\lambda^q_B(\overline{\Omega})$ and Theorem 2.4 is completely proved. ∎

REMARK 5.1. Theorem 2.4 still holds in the case of elliptic systems in a possibly unbounded open set which is uniformly regular of class $C^{2m}$ [Amann, 1984; Geymonat-Grisvard, 1967 ] (compare with [Acquistapace-Terreni, 1987, Remark 5.1] ).

REMARK 5.2. Theorem 2.4 in the case m=1 was proved, with different techniques, in [Acquistapace-Terreni, 1984] .

REMARK 5.3. In the definition (2.13) of $\Delta_\sigma^{B_j} f$ we may replace the integral curve $\mu(\sigma,x)$ of (2.10) by the segment $x-\sigma\nu(x)$, $\sigma\in[0,\sigma_0]$. Indeed, if $f\in\Lambda^q(\overline{\Omega})$ we have (denoting by $u(t,\cdot)$ the function satisfying (2.5) and such that $u(0,\cdot)=f$):

$$\left|\frac{D^\beta f(\mu(\sigma,x))-D^\beta f(x)}{\sigma} - \frac{D^\beta f(x-\sigma\nu(x))-D^\beta f(x)}{\sigma}\right| =$$

$$= \left|-\frac{1}{\sigma}\int_0^t [D^\beta u_s(s,\mu(\sigma,x))-D^\beta u_s(s,x-\sigma\nu(x))]\,ds + \right.$$

$$\left. + \frac{1}{\sigma}\int_0^\sigma [(\nabla D^\beta u(t,\mu(\rho,x))|\nabla d(\mu(\rho,x)))+(\nabla D^\beta u(t,x-\rho\nu(x))|\nu(x))]\,d\rho\right| \le$$



$$\le c\,\sigma^{-1}t^{1/2}\|f\|_{\Lambda^q(\overline{\Omega})} |\mu(\sigma,x)-x+\sigma\nu(x)|^{1-\varepsilon} +$$

$$+ c\,\sigma^{-1}t^{-1/2}\|f\|_{\Lambda^q(\overline{\Omega})} \int_0^\sigma [\,|\mu(\rho,x)-x+\rho\nu(x)|^\varepsilon+|\mu(\rho,x)-x|]\,d\rho \le$$

$$\le c\|f\|_{\Lambda^q(\overline{\Omega})} \{t^{1/2}\sigma^{-\varepsilon}\,o(1) + t^{-1/2}\sigma^\varepsilon\,o(1)\} \le o(1) \quad \text{as} \quad \sigma\downarrow 0$$

provided we choose $t = (\frac{\sigma}{\sigma_0})^{2\varepsilon}$.

REFERENCES.


P. ACQUISTAPACE, B. TERRENI, Characterization of Hölder and Zygmund classes as interpolation spaces, Pubbl. Dip. Mat. Univ. Pisa, n. 61 (1984).

P. ACQUISTAPACE, B. TERRENI, Hölder classes with boundary conditions as interpolation spaces, Math. Z. 195 (1987) 451-471.

H. AMANN, Existence and regularity for semilinear parabolic evolution equations, Ann. Sc. Norm. Sup. Pisa (4) 11 (1984) 593-576.

G. DA PRATO, P. GRISVARD, Équations d'évolution abstraites non linéaires de type parabolique, Ann. Mat. Pura Appl. (4) 120 (1979) 329-396.

G. GEYMONAT, P. GRISVARD, Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici, Rend. Sem. Mat. Univ. Padova, 38 (1967) 121-173.

D. GILBARG, N.S. TRUDINGER, "Elliptic partial differential equations of second order", Springer-Verlag, Berlin/Heidelberg/New York, 1977.

P. GRISVARD, Équations différentielles abstraites, Ann. Scient. Éc. Norm. Sup. (4) 2 (1969) 311-395.

A. JONSSON, H. WALLIN, The trace to closed sets of functions in $\mathbb{R}^n$ with second difference of order O(h), J. Approx. Theory 26 (1979) 159-184.

J.L. LIONS, J. PEETRE, Sur une classe d'espaces d'interpolation, Inst. Hautes Études Sc. Publ. Math. 19 (1964) 5-68.

H.B. STEWART, Generation of analytic semigroups by strongly elliptic operators under general boundary conditions, Trans. Amer. Math. Soc. 259 (1980) 299-310.

H. TRIEBEL, "Interpolation theory, function spaces, differential operators", North-Holland, Amsterdam/New York/Oxford 1978.

Pubblicato in proprio
Dipartimento di Matematica
Università di Pisa
Via Buonarroti, 2
56100 PISA-ITALY

Silvio Rossi 88