in "Volterra integrodifferential equations in Banach spaces and applications", proceedings, Trento 1987, G. Da Prato & M. Iannelli editors, Pitman Res. Notes in Math. Series n°190, Longman Scientific & Technical, Harlow 1989, p. 1-20

QUASILINEAR PARABOLIC INTEGRODIFFERENTIAL SYSTEMS WITH FULLY NONLINEAR BOUNDARY CONDITIONS

P. ACQUISTAPACE-B. TERRENI

UNIVERSITÀ DEGLI STUDI DI PISA

DIPARTIMENTO DI MATEMATICA

GIUGNO 1987
NUMERO 205

# QUASILINEAR PARABOLIC INTEGRODIFFERENTIAL SYSTEMS WITH FULLY NONLINEAR BOUNDARY CONDITIONS

Paolo Acquistapace
Scuola Normale Superiore, I - 56100 PISA

Brunello Terreni
Dipartimento di Matematica, Università di Pisa, I - 56100 PISA

## 0. INTRODUCTION

This paper concerns the study of local existence of continuously differentiable solutions $u: [0,T]\times\overline{\Omega} \to \mathbb{C}^N$ of quasilinear parabolic integrodifferential systems under fully nonlinear boundary conditions. We also prove some results on continuity of solutions with respect to the initial data. Although our method works in the general case of systems of order 2m, we just consider here, for the sake of simplicity, second order systems of the following kind:

$$(0.1)\quad \begin{cases} \dfrac{\partial u^h}{\partial t}(t,x) - \displaystyle\sum_{i,j=1}^{n}\sum_{k=1}^{n} A_{ij}^{hk}(t,x,u(t,x),Du(t,x))\cdot D_jD_ju^k(t,x) = \\ \qquad = F^h(t,x,u(t,x),Du(t,x))+I^h(t,x,u) \quad \text{in } [0,T]\times\Omega, \quad h=1,\dots,N, \\ u^h(0,x)=\varphi^h(x) \quad \text{in } \Omega, \quad h=1,\dots,N, \\ B^h(t,x,u(t,x),Du(t,x))=J^h(t,x,u) \quad \text{in } [0,T]\times\partial\Omega, \quad h=1,\dots,N, \end{cases}$$

where $\Omega$ is a bounded open set of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^2$ and $I(t,x,u),J(t,x,u)$ are functionals of integral type:

$$(0.2)\quad I(t,x,u):=\int_0^t\int_\Omega H(t,s,x,y;u(t,y),Du(t,y);u(s,x),Du(s,x),D^2u(s,x);u(s,y),Du(s,y),D^2u(s,y))\,dyds$$

$$(0.3)\quad J(t,x,u):=\int_0^t\int_{\partial\Omega} K(t,s,x,y;u(t,y),Du(t,y);u(s,x),Du(s,x);u(s,y),Du(s,y))\,dyds$$

We list our assumptions.

(0.4) Regularity. The $N\times N$-matrices $A_{ij}$ and the N-vectors B,F are sufficiently smooth: for instance, $A_{ij}\in C^2(\Lambda,\mathbb{C}^{N^2})$, $B\in C^3(\Lambda,\mathbb{C}^N)$, $F\in C^2(\Lambda,\mathbb{C}^N)$, where $\Lambda:=[0,T]\times\overline{\Omega}\times\mathbb{C}^N\times\mathbb{C}^{nN}$. Concerning the N-vectors H,K, we require that $H\in C^2(\Lambda',\mathbb{C}^N)$, $K\in C^3(\Lambda'',\mathbb{C}^N)$, where $\Lambda':=\Delta\times\overline{\Omega}^2\times\mathbb{C}^N\times\mathbb{C}^{nN}\times(\mathbb{C}^N\times\mathbb{C}^{nN}\times\mathbb{C}^{n^2N})^2$, $\Lambda'':=\Delta\times\overline{\Omega}\times\partial\Omega\times(\mathbb{C}^N\times\mathbb{C}^{nN})^3$ and $\Delta:=\{(t,s)\in[0,T]^2 : t>s\}$; moreover, we need the following growth assumptions on the functions $D^\gamma H(t,s,x,y;u_1,p_1;u_2,p_2,q_2;u_3,p_3,q_3)$, $D^\gamma K(t,s,x,y;u_1,p_1;u_2,p_2;u_3,p_3)$:

$$|D^\gamma H| \le \begin{cases} c_0(M)\eta(t-s)\cdot(t-s)^{\alpha-1}\,[1+|q_2|+|q_3|] & \text{if there are no derivatives in } (t,q_2,q_3), \\ c_0(M)\eta(t-s)\cdot(t-s)^{\alpha-2}\,[1+|q_2|+|q_3|] & \text{if there are no derivatives in } (q_2,q_3) \text{ and just one derivative in } t, \\ c_0(M)\eta(t-s)\cdot(t-s)^{\alpha-1} & \text{if there are no derivatives in } t \text{ and just one in } (q_2,q_3), \\ c_0(M)\eta(t-s)\cdot(t-s)^{\alpha-2} & \text{if there are just one derivative in } t \text{ and just one in } (q_2,q_3), \\ \text{no assumptions} & \text{otherwise,} \end{cases}$$

provided $\sum_{i=1}^{3}[|u_i|+|p_i|]\le M$ and $|\gamma|\le 2$;

$$|D^\gamma K| \le \begin{cases} c_0(M)\eta(t-s)\cdot(t-s)^{\alpha-1/2} & \text{if there are no derivatives in } t, \\ c_0(M)\eta(t-s)\cdot(t-s)^{\alpha-3/2} & \text{if there is just one derivative in } t, \\ \text{no assumptions} & \text{otherwise,} \end{cases}$$

provided $\sum_{i=1}^{3}[|u_i|+|p_i|]\le M$ and $|\gamma|\le 3$.

Here $\alpha$ is a fixed number in $]0,1/2[$ and $\eta(t)$, $t\ge 0$ is a continuous, non-decreasing function such that $\eta(0)=0$.

(0.5) Ellipticity. The pair $\left(\sum_{j=1}^{n} A_{ij}(t,\cdot,u,p)D_iD_j,\ \sum_{i=1}^{n}\frac{\partial B}{\partial p_i}(t,\cdot,u,p)D_i\right)$, where $\frac{\partial B}{\partial p_i}:=\left\{\frac{\partial B^h}{\partial p_i^k}\right\}_{h,k=1,\dots,N}$, is elliptic in the sense of [4,6], uniformly on bounded subsets of $\Lambda$. More precisely, consider the matrices



$$A(\theta;t,x,u,p;\xi,\rho):=\sum_{s,j=1}^{n}A_{sj}(t,x,u,p)\xi_s\xi_j+e^{i\theta}\rho^2 I,\quad \Gamma(t,x,u,p;\xi):=\sum_{j=1}^{n}\frac{\partial B}{\partial p_j}(t,x,u,p)\xi_j,$$

where $\theta\in\mathbb{R}$, $\xi\in\mathbb{R}^n$, $\rho\in\mathbb{R}$. Then we assume that for each $M>0$ the following is true, provided $t\in[0,T]$, $x\in\overline{\Omega}$, $|u|+|p|\le M$:

(i) there exist $\theta_M\in]\pi/2,\pi[$, $c_M>0$ such that

$$|\det A(\theta;t,x,u,p;\xi,\rho)|\ge c_M(|\xi|^2+\rho^2)^N\quad \forall\theta\in[-\theta_M,\theta_M],\ \forall\xi\in\mathbb{R}^n,\ \forall\rho\in\mathbb{R};$$

(ii) for each $\theta\in[-\theta_M,\theta_M]$, $\xi\in\mathbb{R}^n$, $\rho\in\mathbb{R}$ with $|\xi|^2+\rho^2>0$ and $\xi\cdot\nu(x)=0$ the polynomial $\tau\to\det A(\theta;t,x,u,p;\xi+\tau\nu(x),\rho)$ has precisely N roots $\tau_j^+(\theta;t,x,u,p;\xi,\rho)$ with positive imaginary part. Here $\nu(x)$ is the unit outward normal vector at x (which is supposed now to belong to $\partial\Omega$);

(iii) for each $\theta\in[-\theta_M,\theta_M]$, $\xi\in\mathbb{R}^n$, $\rho\in\mathbb{R}$ with $|\xi|^2+\rho^2>0$ and $\xi\cdot\nu(x)=0$ (here, again, $x\in\partial\Omega$) the rows of the matrix $\Gamma(t,x,u,p;\xi+\tau\nu(x))\cdot[A(\theta;t,x,u,p;\xi+\tau\nu(x),\rho)]^{-1}$ are linearly independent modulo the polynomial $\tau\to\prod_{j=1}^{n}(\tau-\tau_j^+(\theta;t,x,u,p;\xi,\rho))$.

(0.6) Compatibility. The initial datum $\varphi$ belongs to $C^1(\overline{\Omega},\mathbb{C}^N)$ and

$$B(0,x,\varphi(x),D\varphi(x))=0\quad \forall x\in\partial\Omega.$$

Here are a few examples where the above hypotheses (neglecting the integro-differential terms) are fulfilled.

EXAMPLE 0.1. Let $A_{ij}\in C^1(\Lambda,\mathbb{C}^{N^2})$ and suppose that for each $M>0$ there exists $\nu_M>0$ such that if $|u|+|p|\le M$

$$\mathrm{Re}\sum_{i,j=1}^{n}(A_{ij}(t,x,u,p)\cdot\eta,\eta)_{\mathbb{C}^N}\xi_i\xi_j\ge\nu_M|\xi|^2|\eta|^2\quad \forall\xi\in\mathbb{R}^n,\ \forall\eta\in\mathbb{R}^N,\ \forall(t,x)\in[0,T]\times\overline{\Omega};$$

moreover take $B(t,x,u,p):=\left\{\sum_{i=1}^{n}b_i(t,x,u)p_i^h\right\}_{h=1,\dots,N}$, with $b_i\in C^1([0,T]\times\overline{\Omega}\times\mathbb{C}^N,\mathbb{R})$ satisfying

$$\sum_{i=1}^{n}b_i(t,x,u)\nu_i(x)\neq 0\quad \forall(t,x,u)\in[0,T]\times\partial\Omega\times\mathbb{C}^N.$$

This is the situation of [2].

EXAMPLE 0.2. Fix $N=1$. For $i=1,\dots,n$ take $a_i \in C^3(\Lambda,\mathbb{R})$ such that for each $M>0$

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial p_j}(t,x,u,p)\xi_i\xi_j \geq \nu_M|\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ \forall (t,x)\in[0,T]\times\overline{\Omega},$$

provided $|u|+|p| \leq M$. Then, setting $A_{ij} := \frac{\partial a_i}{\partial p_j}$, $B := \sum_{i=1}^{n} a_i\nu_i$, we see that variational problems may be written in the form (0.1).

## 1. MAIN THEOREM

Let us fix any $p\in\,]n,\infty[$ and let $\varphi_0$ be a fixed element in $W^{2,p}(\Omega,\mathbb{C}^N)$. For each $r_0>0$, $N_0>0$ we set:

(1.1) $$\mathcal{B}(\varphi_0,N_0,r_0):=\{\varphi\in W^{2,p}(\Omega,\mathbb{C}^N): \|\varphi-\varphi_0\|_{W^{2,p}} \leq r_0,\ B(0,\cdot,\varphi,D\varphi)=0 \text{ on } \partial\Omega,$$
$$Q(0,\varphi)\in B^{2\alpha,p}_{\infty}(\Omega,\mathbb{C}^N) \text{ and } \|Q(0,\varphi)\|_{B^{2\alpha,p}_{\infty}} \leq N_0\},$$

where $B^{2\alpha,p}_{\infty}$ is the Besov-Nikolskii space ([7, Définition 5.8]), and

(1.2) $$Q(0,\varphi):=\sum_{i,j=1}^{n} A_{ij}(0,\cdot,\varphi,D\varphi)\cdot D_iD_j\varphi+F(0,\cdot,\varphi,D\varphi), \qquad x\in\Omega;$$

we also recall that the number $\alpha\in\,]0,1/2[$ was introduced in (0.4).
It is easy to check that $B(\varphi_0,N_0,r_0)$ is a closed subset of $W^{2,p}(\Omega,\mathbb{C}^N)$.
Let us state our main result:

THEOREM 1.1. Assume (0.2),...,(0.5). There exists $\tau\in\,]0,T]$ such that for each $\varphi\in B(\varphi_0,N_0,r_0)$, problem (0.1) has a unique solution $u_\varphi \in C^{1+\alpha}([0,\tau],L^p(\Omega,\mathbb{C}^N)) \cap C^{\alpha}([0,\tau],W^{2,p}(\Omega,\mathbb{C}^N))$; moreover the map $\varphi \to u_\varphi$ is continuous, in the sense that there exists $c_2>0$, depending on $p,\alpha,\varphi_0,N_0,r_0$, such that

(1.3) $$\|u_\varphi-u_\psi\|_{C^{1+\alpha}(L^p)}+\|u_\varphi-u_\psi\|_{C^{\alpha}(W^{2,p})} \leq$$
$$\leq c_2(p,\alpha,\varphi_0,N_0,r_0)\{\|\varphi-\psi\|_{W^{2,p}}+\|Q(0,\varphi)-Q(0,\psi)\|_{B^{2\alpha,p}_{\infty}}\}.$$

If, in addition, $\varphi\in C^2(\overline{\Omega},\mathbb{C}^N)$ and $Q(0,\varphi)\in C^{2,\alpha}(\overline{\Omega},\mathbb{C}^N)$, then



(1.4) $$\frac{\partial u}{\partial t},\ \sum_{i,j=1}^{n} A_{ij}(\cdot,\cdot,u,Du)\cdot D_iD_ju\in C^{\delta}([0,\tau],C(\overline{\Omega},\mathbb{C}^N)) \quad \forall\delta\in\,]0,\alpha[.$$

The proof will be outlined in the next sections; here we want to make a few remarks.

REMARK 1.2. We are forced to take $p\in\,]n,\infty[$ in order to guarantee that $u(t,\cdot),Du(t,\cdot)$ are continuous functions in $\overline{\Omega}$: this allows us to avoid any growth assumption on the nonlinearities $A_{ij}$, $f$, $B$.

REMARK 1.3. The compatibility conditions $B(0,\cdot,\varphi,D\varphi)=0$ on $\partial\Omega$ and $Q(0,\varphi)\in B^{2\alpha,p}_{\infty}(\Omega,\mathbb{C}^N)$ are necessary for the validity of the first assertion of Theorem 1.1, which is optimal in this sense (and we might also weaken (0.4) somewhat). On the other hand, we cannot replace, in (1.4), $\delta$ by $\alpha$, because of the "bad" behaviour of the space $C(\overline{\Omega},\mathbb{C}^N)$ with respect to maximal time regularity in parabolic evolution problems: compare with [1, Remark 6.4].

REMARK 1.4. Theorem 1.1 is a local existence result; however, due to the fact that the compatibility conditions are preserved in time, it is clear that the local solution may be continued, by standard arguments, up to a maximal time interval $[0,T(\varphi)[$.

REMARK 1.5. We believe that the extension of Theorem 1.1 to the case of quasilinear parabolic systems of arbitrary order, with the elliptic part satisfying the assumptions of [4,6], is straightforward.

Our proof proceeds essentially as in [2], i.e. it uses two main ingredients: linearization of the problem, and the contraction principle. However, there are some differences with respect to [2], due to the presence of the integrodifferential terms and - which is more important - of the fully nonlinear boundary conditions; the latter generalization requires a slight refinement of the classical results of [3,4] from the point of view of the smoothness of the boundary coefficients.
It is to be noted that in [2] we had very sharp assumptions from the point of view of smoothness, and consequently it was necessary there to apply a further regularization procedure of the solution, since the fixed point had been obtained in a space strictly larger than the space of data. The same regularization technique is not applicable here, because of the full nonlinearity at the boundary: this forces us

to take stronger smoothness assumptions on the data, which allow us to get the fixed point directly in the optimal class.

## 2. AUXILIARY RESULTS

We collect here some propositions which shall be used later on.

Consider the linear differential operators

$$(2.1)\qquad A(x,D):=\sum_{i,j=1}^{n} a_{ij}(x)D_iD_j\ ,\qquad a_{ij}\in C(\overline{\Omega},\mathbb{C}^{N^2}),$$

$$(2.2)\qquad B(x,D):=\sum_{i=1}^{n} b_i(x)D_i\ ,\qquad b_i\in W^{1,p}(\Omega,\mathbb{C}^{N^2}),\ p>n,$$

and suppose that (0.5) is satisfied by the pair $(A(\cdot,D),B(\cdot,D))$ uniformly in $\overline{\Omega}$.

Then we have:

THEOREM 2.1. *Under the above assumptions, there exist* $\lambda_0>0$, $\theta_0\in\mathbb{R}$, $c_3>0$ *(depending on p,A,B) such that for each* $f\in L^p(\Omega,\mathbb{C}^N)$, $g\in W^{1,p}(\Omega,\mathbb{C}^N)$ *and* $\lambda\in S_{\lambda_0,\theta_0}:=\{z\in\mathbb{C}:|\arg(z-\lambda_0)|\le\theta_0\}\cup\{\lambda_0\}$, *the problem*

$$(2.3)\qquad \begin{cases}\lambda u-A(\cdot,D)u=f & \text{in } \Omega,\\ B(\cdot,D)u=g & \text{on } \partial\Omega,\\ u\in W^{2,p}(\Omega,\mathbb{C}^N)\end{cases}$$

*has a unique solution, satisfying in addition, for each* $\lambda\in S_{\lambda_0,\theta_0}$ :

$$(2.4)\qquad \sum_{r=0}^{2}[1+|\lambda-\lambda_0|]^{1-r/2}\|D^r u\|_{L^p}\le$$

$$\le c_3(p,A,B)[\|f\|_{L^p}+\inf_{\substack{\psi\in W^{1,p}(\Omega,\mathbb{C}^N)\\ \psi=g \text{ on } \partial\Omega}}\{[1+|\lambda-\lambda_0|]^{1/2}\|\psi\|_{L^p}+\|D\psi\|_{L^p}\}].$$

*Proof.* This result is proved in [6, Teorema 5.3] and [5, Theorems 12.2-13.1], under the stronger assumption $b_i\in C^1(\overline{\Omega},\mathbb{C}^{N^2})$.

Such assumption was necessary in order to apply the elliptic estimates of [3,4] which hold for all $p\in]1,\infty[$. But if we restrict p to be greater than n, it is easily seen that the argument of [3,4] still works when $b_i$ is just in $W^{1,p}$: indeed, it is enough to make suitable modifications in [3, page 703] (see also [4, page 77]) as shown in the Appendix below.



With the generalized version of the elliptic estimates at our disposal, in order to get (2.4) and, consequently, uniqueness, we just need to repeat the argument of [5, Theorem 12.2]. Concerning existence, we first approximate the coefficients of $B(\cdot,D)$ in the $W^{1,p}(\Omega,\mathbb{C}^{N^2})$-norm by more regular ones, and next, we find (by [6, Teorema 5.3]) the corresponding solutions; finally, we note that, due to (2.4), such solutions converge in $W^{2,p}(\Omega,\mathbb{C}^N)$ to a solution of (2.3), as it is easily seen. ■

Consider now the evolution problem

$$(2.5)\qquad \begin{cases}\dfrac{\partial u}{\partial t}-A(\cdot,D)u=f & \text{in } [0,T]\times\Omega,\\ B(\cdot,D)u=g & \text{in } [0,T]\times\partial\Omega,\\ u(0,\cdot)=\varphi & \text{in } \Omega\end{cases}$$

with A,B defined by (2.1),(2.2) and satisfying (0.5) uniformly in $\overline{\Omega}$. We have:

THEOREM 2.2. *Fix* $p>n$, $\alpha\in]0,1/2[$. *If* $f\in C^\alpha([0,T],L^p(\Omega,\mathbb{C}^N))$, $g\in C^\alpha([0,T],W^{1,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1/2}([0,T],L^p(\Omega,\mathbb{C}^N))$, $\varphi\in W^{2,p}(\Omega,\mathbb{C}^N)$ *and in addition the compatibility conditions*

$$(2.6)\qquad B(\cdot,D)\varphi=g(0,\cdot)\ \text{on}\ \partial\Omega,\qquad A(\cdot,D)\varphi+f(0,\cdot)\in B^{2\alpha,p}_{\infty}(\Omega,\mathbb{C}^N)$$

*are fulfilled, then problem* (2.5) *has a unique solution* $u\in C^\alpha([0,T],W^{2,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1}([0,T],L^p(\Omega,\mathbb{C}^N))$; *moreover there exists* $c_4>0$, *depending on* p,α,A,B, *such that:*

$$(2.7)\qquad \|u\|_{C^\alpha(L^p)}+\|u'\|_{C^\alpha(L^p)}\le c_4(p,\alpha,A,B)[\|\varphi\|_{W^{2,p}}+\|f(0,\cdot)\|_{L^p}+\|A(\cdot,D)\varphi+f(0)\|_{B^{2\alpha,p}_\infty}$$

$$+[f]_{C^\alpha(L^p)}+[g]_{C^\alpha(W^{1,p})}+[g]_{C^{\alpha+1/2}(L^p)}].$$

*Proof.* One can repeat the arguments of [8] using Theorem 2.1 instead of [8, Theorem 1.4]. ■

Next, let us recall the following imbedding property:

PROPOSITION 2.3. Let $p>n$, $\alpha\in\,]0,1/2[$. Then we have the continuous imbeddings:

$$C^{\alpha}([0,T],W^{2,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1}([0,T],L^p(\Omega,\mathbb{C}^N))\hookrightarrow\begin{cases}C^{\alpha+1/2}([0,T],W^{1,p}(\Omega,\mathbb{C}^N)),\\ C^{\alpha+\sigma}([0,T],C^1(\overline{\Omega},\mathbb{C}^N))\quad\forall\sigma\in\,]0,1/2-n/2p[\,,\\ C^{\alpha+\theta}([0,T],C(\overline{\Omega},\mathbb{C}^N))\quad\forall\theta\in\,]0,1-n/2p[\end{cases}$$

Proof. The first inclusion follows by interpolation. Concerning the others, interpolation yields

$$C^{\alpha}([0,T],W^{2,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1}([0,T],L^p(\Omega,\mathbb{C}^N))\hookrightarrow C^{\alpha+\gamma}([0,T],B_{\infty}^{2(1-\gamma),p}(\Omega,\mathbb{C}^N))$$
$$\forall\gamma\in\,]0,1[\,,\ \gamma\neq 1-\alpha,$$

and the results follow by Sobolev's Theorem. ∎

We conclude this section with an estimate of the growth of the nonlinearities appearing in problem (0.1) in terms of the growth of the unknown function u.

LEMMA 2.4. Assume (0.4). Let $v\in C^{\alpha}([0,T],W^{2,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1}([0,T],L^p(\Omega,\mathbb{C}^N))$ with $\|v\|_{C^{\alpha}(W^{2,p})}+\|v\|_{C^{\alpha+1}(L^p)}\leq M$, and for any function $f:\Lambda\to\mathbb{C}^h$ (where $h\in\mathbb{N}^+$ and $\Lambda$ is defined in (0.4)) set

$$\tilde{f}(t,x):=f(t,x,v(t,x),Dv(t,x)),\quad t\in[0,T],\ x\in\overline{\Omega}.$$

Then if $D^{\gamma}$ is any derivative with respect to the variables $(t,x,u,p)\in\Lambda$, there exists $c_5>0$, depending on $M$, such that

$$\sum_{|\gamma|\leq 2}\|\widetilde{D^{\gamma}F}\|_{C([0,T]\times\overline{\Omega})}+\sum_{|\gamma|\leq 2}\sum_{i,j=1}^{n}\|\widetilde{D^{\gamma}A_{ij}}\|_{C([0,T]\times\overline{\Omega})}+\sum_{|\gamma|\leq 3}\|\widetilde{D^{\gamma}B}\|_{C([0,T]\times\overline{\Omega})}$$
$$+\sum_{|\gamma|\leq 1}\|\widetilde{D^{\gamma}B}\|_{C([0,T],W^{1,p})}\leq c_5(M).$$

Proof. Quite easy. ∎

A similar result holds for the integrands H,K of the functionals I,J defined in (0.2), (0.3):



LEMMA 2.5. Assume (0.4). Let $v\in C^{\alpha}([0,T],W^{2,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1}([0,T],L^p(\Omega,\mathbb{C}^N))$ with $\|v\|_{C^{\alpha}(W^{2,p})}+\|v\|_{C^{\alpha+1}(L^p)}\leq M$, and introduce the notations:

$$\tilde{H}(t,s,x,y):=H(t,s,x,y;v(t,y),Dv(t,y);v(s,x),Dv(s,x),D^2v(s,x);v(s,y),Dv(s,y),D^2(s,y)),$$

$$\tilde{K}(t,s,x,y):=K(t,s,x,y;v(t,y),Dv(t,y);v(s,x),Dv(s,x);v(s,y),Dv(s,y)).$$

Then there exists $c_6>0$, depending on $M,\alpha$, such that:

$$|D^{\gamma}H(t,s,x,y)|\leq\begin{cases}c_6(M,\alpha)\eta(t-s)\cdot(t-s)^{\alpha-1} & \text{if there are no derivatives in } t,\\ c_6(M,\alpha)\eta(t-s)\cdot(t-s)^{\alpha-2} & \text{if there is one derivative in } t \text{ at most.}\end{cases}$$

provided $|\gamma|\leq 2$, and

$$|D^{\gamma}K(t,s,x,y)|\leq\begin{cases}c_6(M,\alpha)\eta(t-s)\cdot(t-s)^{\alpha-1/2} & \text{if there are no derivatives in } t,\\ c_6(M,\alpha)\eta(t-s)\cdot(t-s)^{\alpha-3/2} & \text{if there is one derivative in } t \text{ at most,}\end{cases}$$

provided $|\gamma|\leq 3$.

Proof. Completely straightforward. ∎

## 3. PROOF OF THEOREM 1.1.

### 3.A. LOCAL EXISTENCE

Fix $p\in\,]n,\infty[$. Consider the Banach space

(3.1) $$E_{\alpha,p}(T):=C^{\alpha}([0,T],W^{2,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1}([0,T],L^p(\Omega,\mathbb{C}^N))$$

with its obvious norm; we also introduce

(3.2) $$[u]_{E_{\alpha,p}(T)} := [u']_{C^\alpha(L^p)} + [D^2u]_{C^\alpha(L^p)} .$$

For each $\varphi\in B(\varphi_0,N_0,r_0)$ (see (1.1)) and $M>0$ set:

(3.3) $$B_{M,\alpha,p,T,\varphi} := \{v\in E_{\alpha,p}(T) : v(0,\cdot)=\varphi,\ \|v-\varphi\|_{E_{\alpha,p}(T)} \le M\}.$$

We linearize problem (0.1) by considering, for fixed $v\in B_{M,\alpha,p,T,\varphi}$, the problem:

(3.4) $$\begin{cases} \dfrac{\partial u^h}{\partial t}(t,x) - \displaystyle\sum_{i,j=1}^{n}\sum_{k=1}^{N} A_{ij}^{hk}(0,x,\varphi(x),D\varphi(x))\cdot D_iD_ju^k(t,x) = F^h_{v,\varphi}(t,x) & \text{in } [0,T]\times\Omega,\ h=1,\dots,N, \\ u^h(0,x)=\varphi^h(x) \quad \text{in } \Omega,\quad h=1,\dots,N, & \\ \displaystyle\sum_{i=1}^{n}\sum_{k=1}^{N} \frac{\partial B^h}{\partial p_i^k}(0,x,\varphi(x),D\varphi(x))D_iu^k(t,x) = G^h_{v,\varphi}(t,x) & \text{in } [0,T]\times\partial\Omega,\ h=1,\dots,N, \end{cases}$$

where:

(3.5) $$\begin{aligned} F_{v,\varphi}(t,x) :=& F(t,x,v(t,x),Dv(t,x)) \\ &- \sum_{i,j=1}^{n} [A_{ij}(0,x,\varphi(x),D\varphi(x)) - A_{ij}(t,x,v(t,x),Dv(t,x))]\cdot D_iD_jv(t,x) \\ &+ I(t,x,v), \end{aligned}$$

(3.6) $$G_{v,\varphi}(t,x) := \sum_{i=1}^{n} \frac{\partial B}{\partial p_i}(0,x,\varphi(x),D\varphi(x))\cdot D_iv(t,x) - B(t,x,v(t,x),Dv(t,x)) + J(t,x,v).$$

In order to apply Theorem 2.2, we need the following lemmas, whose proofs will be given in Section 4.

LEMMA 3.1. *Assume* (0.4), (0.5), *and let* $\varphi\in B(\varphi_0,N_0,r_0)$. *There exists a continuous, non-decreasing function* $\omega_0(t)$, $t\ge 0$, *with* $\omega_0(0)=0$, *such that for each* $K>0$ *and* $v\in B_{K,\alpha,p,T,\varphi}$ *with* $\|v\|_{E_{\alpha,p}(T)} \le K$ *we have*:

$$[F_{v,\varphi}]_{C^\alpha(L^p)} + [G_{v,\varphi}]_{C^\alpha(W^{1,p})} + [G_{v,\varphi}]_{C^{\alpha+1/2}(L^p)} \le c_7(K)\omega_0(T).$$

LEMMA 3.2. *Assume* (0.4), (0.5), *and let* $\varphi,\psi\in B(\varphi_0,N_0,r_0)$. *There exists a continuous, non-decreasing function* $\omega_1(t)$, $t\ge 0$, *with* $\omega_1(0)=0$, *such that for each* $K>0$ *and*



$v\in B_{K,\alpha,p,T,\varphi}$, $w\in B_{K,\alpha,p,T,\psi}$ *with* $\|v\|_{E_{\alpha,p}(T)}, \|w\|_{E_{\alpha,p}(T)} \le K$ *we have*

$$\begin{aligned} &[F_{v,\varphi}-F_{w,\psi}]_{C^\alpha(L^p)} + [G_{v,\varphi}-G_{w,\psi}]_{C^\alpha(W^{1,p})} + [G_{v,\varphi}-G_{w,\psi}]_{C^{\alpha+1/2}(L^p)} \le \\ &\qquad \le c_8(K)\{\|\varphi-\psi\|_{W^{2,p}} + \omega_1(T)\,[v-w]_{E_{\alpha,p}(T)}\}. \end{aligned}$$

By Lemma 3.1, the data of problem (3.4) satisfy:

$$A_{ij}(0,\cdot,\varphi,D\varphi)\in C(\overline{\Omega},\mathbb{C}^{N^2}),\quad \frac{\partial B}{\partial p_i}(0,\cdot,\varphi,D\varphi)\in W^{1,p}(\Omega,\mathbb{C}^{N^2}),$$

$$F_{v,\varphi}\in C^\alpha([0,T],L^p(\Omega,\mathbb{C}^N)),\quad G_{v,\varphi}\in C^\alpha([0,T],W^{1,p}(\Omega,\mathbb{C}^N))\cap C^{\alpha+1/2}([0,T],L^p(\Omega,\mathbb{C}^N)),$$

$$\varphi\in W^{2,p}(\Omega,\mathbb{C}^N),$$

and the compatibility conditions

(3.7) $$B(0,\cdot,\varphi,D\varphi) = 0 \text{ on } \partial\Omega,\qquad Q(0,\varphi)\in B^{2\alpha,p}(\Omega,\mathbb{C}^N)$$

hold. Hence, Theorem 2.2 implies the existence of a unique solution $u=:S(v)$ of problem (3.4), beloging to the space $E_{\alpha,p}(T)$.

Consequently $w:=u-\varphi$ is the unique solution of the problem

(3.8) $$\begin{cases} \dfrac{\partial w}{\partial t} - \displaystyle\sum_{i,j=1}^{n} A_{ij}(0,\cdot,\varphi,D\varphi)\cdot D_iD_jw = F_{v,\varphi} + \sum_{i,j=1}^{n} A_{ij}(0,\cdot,\varphi,D\varphi)\cdot D_iD_j\varphi & \text{in } [0,T]\times\Omega, \\ w(0,\cdot) = 0 \quad \text{in } \Omega, & \\ \displaystyle\sum_{i=1}^{n} \frac{\partial B}{\partial p_i}(0,\cdot,\varphi,D\varphi)\cdot D_iw = G_{v,\varphi} - \sum_{i=1}^{n} \frac{\partial B}{\partial p_i}(0,\cdot,\varphi,D\varphi)\cdot D_i\varphi & \text{in } [0,T]\times\partial\Omega, \end{cases}$$

and by (2.7), (3.5), (3.6), (1.2) the following estimate holds:

(3.9) $$\begin{aligned} \|u-\varphi\|_{E_{\alpha,p}(T)} \le\ & c_7(p,\alpha,\|\varphi\|_{W^{2,p}}) \\ & \{\|Q(0,\varphi)\|_{B^{2\alpha,p}_\infty} + [F_{v,\varphi}]_{C^\alpha(L^p)} + [G_{v,\varphi}]_{C^\alpha(W^{1,p})} + [G_{v,\varphi}]_{C^{\alpha+1/2}(L^p)}\}. \end{aligned}$$

Hence, if we set $R:=c_7(p,\alpha,\|\varphi\|_{W^{2,p}})\,\|Q(0,\varphi)\|_{B^{2\alpha,p}_\infty}$ and choose $K>R$, we get by Lemma 3.1:

$$\|S(v)-\varphi\|_{E_{\alpha,p}(T)} \le R + c_8(K)\omega_0(T), \tag{3.10}$$

where $\omega_0(T)\downarrow 0$ as $T\downarrow 0$.

Next, if $v,w\in B_{M,\alpha,p,\varphi,T}$ and $h:=S(v)-S(w)$, then $h$ solves the problem:

$$\left\{\begin{array}{l} \dfrac{\partial h}{\partial t} - \sum\limits_{i,j=1}^{n} A_{ij}(0,\cdot,\varphi,D\varphi)\cdot D_iD_jh = F_{v,\varphi}-F_{w,\varphi} \quad \text{in } [0,T]\times\Omega , \\ h(0,\cdot) = 0 \quad \text{in } \Omega, \\ \sum\limits_{i=1}^{n} \dfrac{\partial B}{\partial p_i}(0,\cdot,\varphi,D\varphi)\cdot D_iD_jh = G_{v,\varphi}-G_{w,\varphi} \quad \text{in } [0,T]\times\Omega , \end{array}\right. \tag{3.11}$$

and again (2.7), (3.5), (3.6) yield:

$$\|S(v)-S(w)\|_{E_{\alpha,p}(T)} \le c_9(p,\alpha,\|\varphi\|_{W^{2,p}}) \tag{3.12}$$

$$\{[F_{v,\varphi}-F_{w,\varphi}]_{C^\alpha(L^p)} + [G_{v,\varphi}-G_{w,\varphi}]_{C^\alpha(W^{1,p})} + [G_{v,\varphi}-G_{w,\varphi}]_{C^{\alpha+1/2}(L^p)}\};$$

thus by Lemma 3.2 we obtain:

$$\|S(v)-S(w)\|_{E_{\alpha,p}(T)} \le c_{10}(K)\omega_1(T)\ \|v-w\|_{E_{\alpha,p}(T)} \tag{3.13}$$

where $\omega_1(T)\downarrow 0$ as $T\downarrow 0$.

Hence if $T=:\tau$ is small enough, the contraction principle is applicable and we get a unique solution $u_\varphi$ of problem (0.1) in $[0,\tau]$, belonging to the class $E_{\alpha,p}(\tau)$. This proves the first assertion of Theorem 1.1; note that $\tau$ depends on $\varphi_0,N_0,r_0,p,\alpha$, but not on $\varphi\in B(\varphi_0,N_0,r_0)$.

### 3.B CONTINUOUS DEPENDENCE ON THE INITIAL DATUM

Next, fix $\varphi,\psi\in B(\varphi_0,N_0,r_0)$ and consider the difference $v:=u_\varphi-u_\psi$. Redefine by $F_\varphi,G_\varphi$ the functions (3.5), (3.6) (which depend in fact on $u_\varphi$ and $\varphi$): then $v$ solves:

$$\left\{\begin{array}{l} \dfrac{\partial v}{\partial t} - \sum\limits_{i,j=1}^{n} A_{ij}(0,\cdot,\varphi,D\varphi)\cdot D_iD_jv = F^{\varphi,\psi} \quad \text{in } [0,\tau]\times\Omega, \quad v(0,\cdot)=\varphi-\psi \quad \text{in } \Omega, \\ \sum\limits_{i=1}^{n} \dfrac{\partial B}{\partial p_i}(0,\cdot,\varphi,D\varphi)\cdot D_iv = G^{\varphi,\psi} \quad \text{in } [0,\tau]\times\partial\Omega , \end{array}\right. \tag{3.14}$$



where

$$F^{\varphi,\psi} := F_\varphi-F_\psi + \sum_{i,j=1}^{n} [A_{ij}(0,\cdot,\varphi,D\varphi)-A_{ij}(0,\cdot,\psi,D\psi)]\cdot D_iD_ju_\psi , \tag{3.15}$$

$$G^{\varphi,\psi} := G_\varphi-G_\psi - \sum_{i=1}^{n} [\frac{\partial B}{\partial p_i}(0,\cdot,\varphi,D\varphi)-\frac{\partial B}{\partial p_i}(0,\cdot,\psi,D\psi)]\cdot D_iu_\psi ; \tag{3.16}$$

it is easy to verify that the compatibility conditions (3.7) hold. Concerning the regularity of $F^{\varphi,\psi}$, $G^{\varphi,\psi}$, we have:

<u>LEMMA</u> 3.3. <u>Assume</u> (0.4), (0.5) <u>and let</u> $\varphi,\psi\in B(\varphi_0,N_0,r_0)$. <u>Then we have</u>:

$$\|F^{\varphi,\psi}(0,\cdot)\|_{L^p} + \|G^{\varphi,\psi}(0,\cdot)\|_{W^{1,p}} \le c_{11}(p,M,\varphi_0,r_0)\|\varphi-\psi\|_{W^{2,p}} ,$$

$$[F^{\varphi,\psi}]_{C^\alpha(L^p)} + [G^{\varphi,\psi}]_{C^\alpha(W^{1,p})} + [G^{\varphi,\psi}]_{C^{\alpha+1/2}(L^p)}$$

$$\le c_{12}(p,\alpha,M,\varphi_0,r_0)\{\|\varphi-\psi\|_{W^{2,p}} + [u_\varphi-u_\psi]_{E_{\alpha,p}(\tau)}\cdot\omega_2(\tau)\} ,$$

<u>where</u> $\omega_2(t)$, $t\ge 0$ , <u>is a non-decreasing continuous function with</u> $\omega_2(0)=0$.

<u>Proof</u>. Easy consequences of (3.15), (3.16), (3.5), (3.6) and Lemma 3.2. ∎

By (2.7) and lemma 3.3 we easily get:

$$\|u_\varphi-u_\psi\|_{E_{\alpha,p}(\tau)} \le c_{12}(p,\alpha,M,\varphi_0,r_0)$$

$$\cdot\{\|\varphi-\psi\|_{W^{2,p}} + \|Q(0,\varphi)-Q(0,\psi)\|_{B^{2\alpha,p}_\infty} + \omega_2(\tau)\cdot[u_\varphi-u_\psi]_{E_{\alpha,p}(\tau)}\} ,$$

so that, possibly taking a smaller $\tau$, we deduce (1.3). Thus we have shown continuous dependence on $\varphi$ of the local solution $u_\varphi$ of problem (0.1).

### 3.C HIGHER REGULARITY

Suppose finally that $\varphi\in C^2(\bar\Omega,\mathbb{C}^N)$ and $Q(0,\varphi)\in C^{2\alpha}(\bar\Omega,\mathbb{C}^N)$, and fix $\delta\in]0,\alpha[$. Then for each $p\in]\frac{n}{2(\alpha-\delta)},\infty[$ we can apply the preceding theory, obtaining a local solution $u:=u_\varphi\in E_{\alpha,p}(\tau)$, where $\tau$ depends on $p$. Now we have by Proposition 2.3 (with $\sigma = 1/2+\delta-\alpha$, $\theta = 1+\delta-\alpha$):

$$u\in E_{\alpha,p}(\tau)\hookrightarrow C^{\delta+1/2}([0,\tau],C^1(\overline{\Omega},\mathbb{C}^N))\cap C^{\delta+1}([0,\tau],C(\overline{\Omega},\mathbb{C}^N)).$$

Consequently we have $\frac{\partial u}{\partial t}\in C^\delta([0,T],C(\overline{\Omega},\mathbb{C}^N))$; moreover it is easy to see that

$$(t,x)\to F(t,x,u(t,x),Du(t,x)) + I(t,x,u)\in C^\delta([0,T],C(\overline{\Omega},\mathbb{C}^N)),$$

so that, by difference, we get $(t,x)\to \sum_{i,j=1}^n A_{ij}(t,x,u(t,x),Du(t,x))\cdot D_iD_ju(t,x)$ $\in C^\delta([0,T],C(\overline{\Omega},\mathbb{C}^N))$.

The proof of Theorem 1.1 is complete. ∎

## 4. PROOF OF LEMMAS 3.1-3.2.

Proof of Lemma 3.1. Fix $0\le\tau<t\le T$ and set for $\xi\in]0,1]$:

$$P_\xi := (\tau+\xi(t-\tau),x,v(\tau,x)+\xi(v(t,x)-v(\tau,x)),Dv(\tau,x)+\xi(Dv(t,x)-Dv(\tau,x))),$$

$$\overline{P} := (0,x,\varphi(x),D\varphi(x))$$

$$Q_\xi := (\tau+\xi(t-\tau),s,x,y;v(\tau,y)+\xi(v(t,y)-v(\tau,y)),Dv(\tau,y)+\xi(Dv(t,y)-Dv(\tau,y)); v(s,x),Dv(s,x),D^2v(s,x);u(s,y),Dv(s,y),D^2v(s,y)).$$

The recalling (3.5) we can write:

$$(4.1)\quad \|F_{v,\varphi}(t,\cdot)-F_{v,\varphi}(\tau,\cdot)\|_{L^p}\le\Big\|\int_0^1\frac{d}{d\xi}F(P_\xi)d\xi\Big\|_{L^p}$$
$$+\Big\|\int_0^1\frac{d}{d\xi}\Big[\sum_{i,j=1}^n(A_{ij}(\overline{P})-A_{ij}(P_\xi))\cdot(D_iD_jv(\tau,\cdot)+\xi(D_iD_jv(t,\cdot)-D_iD_jv(\tau,\cdot)))\Big]d\xi\Big\|_{L^p}$$
$$+\Big\|\int_\tau^t\int_\Omega H(Q_1)dyds\Big\|_{L^p}$$
$$+\Big\|\int_0^\tau\int_\Omega\int_0^1\frac{d}{d\xi}H(Q_\xi)d\xi dyds\Big\|_{L^p}=:I_1+I_2+I_3+I_4.$$

Now recalling Proposition 2.3 and Lemmas 2.4-2.5, we have (provided $\sigma\in]0,1/2-n/2p[$):

$$(4.2)\quad I_1=\Big\|\int_0^1\{\frac{\partial F}{\partial t}(P_\xi)\cdot(t-\tau)+\frac{\partial F}{\partial u}(P_\xi)\cdot(v(t,\cdot)-v(\tau,\cdot))+\frac{\partial F}{\partial p}(P_\xi)\cdot(Dv(t,\cdot)-Dv(\tau,\cdot))\}d\xi\Big\|_{L^p}$$
$$\le c(K)\,(t-\tau)^{\alpha+1/2},$$



$$(4.3)\quad I_2\le\sum_{i,j=1}^n\Big\|\int_0^1\{\frac{\partial A_{ij}}{\partial t}(P_\xi)(t-\tau)+\frac{\partial A_{ij}}{\partial u}(P_\xi)\cdot(v(t,\cdot)-v(\tau,\cdot))+\frac{\partial A_{ij}}{\partial p}(P_\xi)$$
$$\cdot(Dv(t,\cdot)-Dv(\tau,\cdot))\}\cdot(D_iD_jv(\tau,\cdot)+\xi(D_iD_jv(t,\cdot)-D_iD_jv(\tau,\cdot)))\}d\xi\Big\|_{L^p}$$
$$+\sum_{i,j=1}^n\Big\|\int_0^1(A_{ij}(\overline{P})-A_{ij}(P_\xi)):(D_iD_jv(t,\cdot)-D_iD_jv(\tau,\cdot))d\xi\Big\|_{L^p}$$
$$\le c(K)\,[(t-\tau)^{\alpha+\sigma}+(t-\tau)^\alpha\,\omega_0(T)],$$

$$(4.4)\quad I_3\le c(K)\int_\tau^t\Big\|\int_\Omega\eta(t-s)(t-s)^{\alpha-1}\{1+|D^2v(s,\cdot)|+|D^2v(s,y)|\}\Big\|_{L^p}dyds$$
$$\le c(K)(t-\tau)^\alpha\,\omega_0(T);$$

finally concerning $I_4$ we write

$$(4.5)\quad I_4=\Big\|\int_0^\tau\int_\Omega\int_0^1\{\frac{\partial H}{\partial t}(Q_\xi)(t-\tau)+\frac{\partial H}{\partial u_1}(Q_\xi)\cdot(v(t,y)-v(\tau,y))+\frac{\partial H}{\partial p_1}(Q_\xi)$$
$$\cdot(Dv(t,y)-Dv(\tau,y))\}d\xi dyds\Big\|_{L^p}$$

and all integrals but the first are easily estimated by $c(K)(t-\tau)^{\alpha+1/2}$. The first one needs some care: it does not exceed

$$(4.6)\quad \eta(T)c(K)\int_0^\tau\int_0^1(\tau-s+\xi(t-\tau))^{\alpha-2}(t-\tau)d\xi ds=\eta(T)\,c(K)\int_0^\tau\int_0^{t-\tau}(\beta+\sigma)^{\alpha-2}d\beta d\sigma$$
$$\le\eta(T)\,c(K)(t-\tau)^\alpha.$$

Hence recalling (4.1),...,(4.5) we get the estimate for $[F_{v,\varphi}]_{C^\alpha(L^p)}$.

The estimates for $G_v$ can be obtained similarly: indeed, setting

$$R_\xi:=(\tau+\xi(t-\tau),s,x,y;v(\tau,y)+\xi(v(t,y)-v(\tau,y));Dv(\tau,y)+\xi(Dv(t,y)-Dv(\tau,y)); v(s,x),Dv(s,x);v(s,y),Dv(s,y)),$$

we have:

$$(4.7)\quad G_{v,\varphi}(t,x)-G_{v,\varphi}(\tau,x)=\sum_{i=1}^n\frac{\partial B}{\partial p_i}(\overline{P})\cdot(D_iv(t,x)-D_iv(\tau,x))$$

$$- \int_0^1 \{\frac{\partial B}{\partial t}(P_\xi)(t-\tau) + \frac{\partial B}{\partial u}(P_\xi)\cdot(v(t,x)-v(\tau,x)) + \frac{\partial B}{\partial p}(P_\xi)\cdot(Dv(t,x)-Dv(\tau,x))\}d\xi$$

$$+ \int_\tau^t \int_{\partial\Omega} K(R_1)dyds$$

$$+ \int_0^\tau \int_{\partial\Omega} \int_0^1 \{\frac{\partial K}{\partial t}(R_\xi)(t-\tau) + \frac{\partial K}{\partial u_1}(R_\xi)\cdot(v(t,y)-v(\tau,y)) + \frac{\partial K}{\partial p_1}(R_\xi)\cdot(Dv(t,y)-Dv(\tau,y))\}d\xi dyds$$

$$= T_1 + T_2 + T_3 + T_4.$$

Now we need to rewrite $T_1+T_2$ as:

$$(4.8)\quad T_1+T_2 = \int_0^1 (\frac{\partial B}{\partial p}(\bar{P}) - \frac{\partial B}{\partial p}(P_\xi))\cdot(Dv(t,x)-Dv(\tau,x))d\xi$$

$$- \int_0^1 \{\frac{\partial B}{\partial t}(P_\xi)(t-\tau) + \frac{\partial B}{\partial u}(P_\xi)\cdot(v(t,x)-v(\tau,x))\}d\xi ,$$

and it is easy to see that

$$(4.9)\quad \|T_1+T_2\|_{L^p} + \|T_3\|_{L^p} \le c(K)\,\omega_0(T)(t-\tau)^{\alpha+1/2} ;$$

the same holds for the integrals appearing in $T_4$, with the only exception of the first, whose $L^p$-norm does not exceed

$$(4.10)\quad \eta(T)c(K)\int_0^\tau \int_0^1 (\tau-s+\xi(t-\tau))^{\alpha-3/2}(t-\tau)d\xi ds \le \eta(T)\; c(K)(t-\tau)^{\alpha+1/2}.$$

Thus we get the estimate for $[G_{v,\varphi}]_{C^{\alpha+1/2}(L^p)}$.

Finally, in order to estimate $[G_{v,\varphi}]_{C^\alpha(W^{1,p})}$, we evaluate the gradient (with respect to x) of (4.7), and treat each term as before: more precisely, the first integral in $T_4$ generates some terms needing an estimate like (4.6), whereas the two terms containing $D^2v(t,x)-D^2v(\tau,x)$, which arise from $T_1$ and from the last integral in $T_2$, have to be coupled together; the result is

$$\int_0^1 [\frac{\partial B}{\partial p}(\bar{P}) - \frac{\partial B}{\partial p}(P_\xi)]\cdot(D^2v(t,x)-D^2v(\tau,x))d\xi,$$



whose $L^p$-norm can be easily estimated by $c(K)\omega_0(T)(t-\tau)^\alpha$. The remaining terms can also be estimated, tediously but easily, by the same quantity.

This shows that the estimate for $[G_{v,\varphi}]_{C^\alpha(W^{1,p})}$ is true. The proof of Lemma 3.1 is now complete.

Proof of Lemma 3.2. Fix $0\le\tau<t\le T$ and set for $\xi,\eta\in]0,1]$:

$$v^\eta(x):=v(\tau,x)+\eta[v(t,x)-v(\tau,x)],\quad w^\eta(x):=w(\tau,x)+\eta[w(t,x)-w(\tau,x)] ,$$

$$z_{\eta\xi}(x):=w^\eta(x)+\xi[v^\eta(x)-w^\eta(x)]$$

$$\beta_\xi(s,x):=w(s,x)+\xi[v(s,x)-w(s,x)],$$

$$\bar{P}_\xi:=(0,x,\psi(x)+\xi(\varphi(x)-\psi(x)),D\psi(x)+\xi(D\varphi(x)-D\psi(x))),$$

$$P_{\eta\xi}:=(\tau+\eta(t-\tau),x,z_{\eta\xi}(x),Dz_{\eta\xi}(x))$$

$$Q_{\eta\xi}:=(\tau+\eta(t-\tau),s,x,y;\beta_\xi(\tau+\eta(t-\tau),y),D\beta_\xi(\tau+\eta(t-\tau),y);\beta_\xi(s,x),D\beta_\xi(s,x),D^2\beta_\xi(s,x);$$
$$\beta_\xi(s,y),D\beta_\xi(s,y),D^2\beta_\xi(s,y)).$$

Then a very tedious calculation shows that:

$$(4.11)\quad \|F_{v,\varphi}(t,\cdot)-F_{w,\psi}(t,\cdot)-F_{v,\varphi}(\tau,\cdot)+F_{w,\psi}(\tau,\cdot)\|_{L^p}$$

$$= \|\int_0^1 (\frac{d}{d\xi}F_{w+\xi(v-w),\psi+\xi(\varphi-\psi)}(t,\cdot) - \frac{d}{d\xi}F_{w+\xi(v-w),\psi+\xi(\varphi-\psi)}(\tau,\cdot))d\xi\|_{L^p}$$

$$\le \|\int_0^1 \frac{d}{d\eta}\int_0^1 \{\frac{\partial F}{\partial u}(P_{\eta\xi})\cdot(v^\eta-w^\eta) + \frac{\partial F}{\partial p}(P_{\eta\xi})\cdot(Dv^\eta-Dw^\eta)\}d\xi d\eta\|_{L^p}$$

$$+ \|\int_0^1 \frac{d}{d\eta}\int_0^1 \sum_{i,j=1}^n \{\frac{\partial A_{ij}}{\partial u}(\bar{P}_\xi)\cdot(\varphi-\psi) - \frac{\partial A_{ij}}{\partial u}(P_{\eta\xi})(v^\eta-w^\eta)$$

$$+ \frac{\partial A_{ij}}{\partial p}(\bar{P}_\xi)\cdot(D\varphi-D\psi) - \frac{\partial A_{ij}}{\partial p}(P_{\eta\xi})\cdot(Dv^\eta-Dw^\eta)\}\cdot D_iD_jz_{\eta\xi}\; d\xi d\eta\|_{L^p}$$

$$+ \|\int_0^1 \frac{d}{d\eta}\int_0^1 \sum_{i,j=1}^n (A_{ij}(\bar{P}_\xi)-A_{ij}(P_{\eta\xi}))\cdot(D_iD_jv^\eta-D_iD_jw^\eta)d\xi d\eta\|_{L^p}$$

$$+ \|\int_0^1 \int_\tau^t \int_\Omega \frac{d}{d\xi}H(Q_{1\xi})dydsd\xi\|_{L^p}$$

$$+ \| \int_0^1 \frac{d}{d\eta} \int_0^1 \int_0^\tau \int_\Omega \frac{d}{d\xi} H(Q_{\eta\xi})dyds d\xi d\eta \|_{L^p} ;$$

no particolar problems (except for lenght and bore!) arise in estimating each term; we just need some care for the last one, since it generates, among others, several terms containing second order derivatives of H, with one derivative with respect to t: for instance, the first of such terms is

$$\int_0^1 \int_0^1 \int_0^\tau \int_\Omega \frac{\partial^2 H}{\partial t \partial u_1} (Q_{\eta\xi}) \cdot (t-\tau)(v^\eta - w^\eta) dyds d\xi d\eta ;$$

all such terms can be estimated as in (4.6).

Thus we easily arrive to the estimate for $[F_{v,\varphi} - F_{w,\psi}]_{C^\alpha(L^p)}$.

The estimates for $G_{v,\varphi} - G_{w,\psi}$ are similar in nature and we omit their proof for the sake of brevity. We only remark that, analogously to the proofs above, we need to couple together suitably the terms which contain derivatives of $v,w$ "of maximal order" (i.e. terms containing $Dv(t,x)-Dw(t,x)-Dv(\tau,x)+Dw(\tau,x)$ for the estimate in the $W^{1,p}$-norm, and terms containing $D^2v(t,x)-D^2w(t,x)-D^2v(\tau,x)+D^2w(\tau,x)$ for the estimatê in the $L^p$-norm); moreover some other terms of integrodifferential type, containing one time derivative of the integrand K, have to be treated as in (4.6), (4.10).

This concludes the proof of Lemma 3.2. ∎

## APPENDIX

This short appendix is devoted to the proof of the slight refinement of [3], which we referred to in the proof of Theorem 2.1.

We follow the notations of [3, pages 702-703]. Let $p>n$ and assume that the coefficients of the boundary operators $B_j$ belong to $W^{\ell-m_j,p}(\Sigma)$, with norms bounded by k. Due to Sobolev's Theorem, the coefficients of $B_j$ also belong to $C^{\ell-m_j-1}(\bar{\Sigma})$, so that the oscillation in $\Sigma_r$ of their $(\ell-m_j-1)$-th order derivatives does not exceed $\omega(r)$, where $\omega(R)\downarrow 0$ as $R\downarrow 0$. Our goal is the proof of [3, Theorem 15.1] under the above weaker assumptions.

It is sufficient to remark that



$$\| (B_j'(0;D)-B_j'(x;D)-B_j''(x;D))u(x,t)\|_{W^{\ell-m_j,p}} \leq c\{k\|u\|_{W^{\ell-1,p}} + \omega(r)\|u\|_{W^{\ell,p}}\} ;$$

hence we obtain that [3, formula (15,3)'] still holds with $r$ replaced by $\omega(r)$. The remaining part of the proof does not need any change. ∎

## REFERENCES


[1] P. ACQUISTAPACE, Evolution operators and strong solutions of abstract linear parabolic equations, preprint Sc. Norm. Sup. Pisa (1987).

[2] P. ACQUISTAPACE, B. TERRENI, On quasilinear parabolic systems, preprint Sc. Norm. Sup. Pisa (1987).

[3] S. AGMON, A. DOUGLIS, L. NIRENBERG, Estimates near the boundary of solutions of elliptic partial differential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 12 (1959) 623-727.

[4] S. AGMON, A. DOUGLIS, L. NIRENBERG, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, Comm. Pure Appl. Math. 17 (1964) 35-92.

[5] H. AMANN, Existence and regularity for semilinear parabolic evolution equations, Ann. Sc. Norm. Sup. Pisa (4) 11 (1984) 593-676.

[6] G. GEYMONAT, P. GRISVARD, Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici, Rend. Sem. Mat. Univ. Padova 38 (1967) 121-173.

[7] P. GRISVARD, Equations différentielles abstraites, Ann. Sci. Ec. Norm. Sup, (4) 2 (1969) 311-395.

[8] B. TERRENI, Non-homogeneous initial-boundary value problems for linear parabolic systems, preprint Dip. Mat. Univ. Pisa (1987).