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ON FUNDAMENTAL SOLUTIONS FOR ABSTRACT PARABOLIC EQUATIONS

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0. INTRODUCTION

We consider the linear Cauchy problem

$$(0.1)\quad \begin{cases} u'(t) - A(t)u(t) = f(t), & t\in[0,T] \\ u(0) = x \end{cases}$$

in a Banach space E. Here $x \in E$, $f \in C([0,T],E)$ and for each $t\in[0,T]$ the operator A(t) generates an analytic semigroup $\{e^{sA(t)}\}_{s\geq 0}$ in E; its domain $D_{A(t)}$ may depend on t and be not dense in E.

We make the following assumptions, already introduced in [4]:

HYPOTHESIS I For each $t\in[0,T]$ $A(t):D_{A(t)}\subseteq E \to E$ is a closed linear operator and there exist $\theta_0 \in]\pi/2,\pi[$, $M>0$ such that:

(i) $\rho(A(t)) \supseteq S_{\theta_0} = \{z \in \mathbb{C} : |\arg z| \leq \theta_0\} \cup \{0\}$,

(ii) $\|R(\lambda,A(t))\|_{\mathcal{L}(E)} \leq \frac{M}{1+|\lambda|}$ $\forall\ \lambda \in S_{\theta_0}$, $\forall\ t\in[0,T]$.

HYPOTHESIS II There exist $B>0$, $k\in\mathbb{N}$, $\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k\in[0,2]$ with $\delta = \min_{1\leq i\leq k}(\alpha_i-\beta_i) > 0$, such that:

$$\|A(t)R(\lambda,A(t))[A(t)^{-1}-A(s)^{-1}]\|_{\mathcal{L}(E)} \leq B\sum_{i=1}^{k}(t-s)^{\alpha_i}|\lambda|^{\beta_i-1}$$

$$\forall\ \lambda \in S_{\theta_0}-\{0\},\ \forall\ 0\leq s\leq t\leq T.$$

Hypotheses I-II are generally weaker than those known in the literature (see [5, Section 7] for detailed comparisons); in particular, they allow a unified treatment of problem (0.1) in which neither the constancy of the domains $D_{A(t)}$ [9], [8], [2], [3], nor the strong differentiability of the resolvents $t \to R(\lambda, A(t))$ [6], [10], [11], [1] is required. Actually, under these assumptions and provided the data x, f are sufficiently regular, we have proved in [5] existence, uniqueness and sharp regularity results for strict and classical solutions u of (0.1), as well as a representation formula for $A(\cdot)u(\cdot)$ which is obtained without using fundamental solutions. As a consequence, we derived an a-priori estimate for strict solutions of (0.1) of the following kind:

$$(0.2) \qquad \|u'\|_{C([0,T],E)} + \|Au\|_{C([0,T],E)} \le C\{\|x\|_{D_{A(0)}} + \|f\|_{C^{\delta}([0,T],E)}\}.$$

However this estimate, although interesting of its own, does not seem to be very useful in applications, because (roughly speaking) it involves too strong norms. Thus the aim of this paper is the proof of a better a-priori estimate (i.e. in terms of u, rather than Au) for strict solutions of (0.1). We will also express the solution by the usual variation of parameters formula, finding in particular an explicit representation of the fundamental solution of (0.1).

1. THE A-PRIORI ESTIMATE

A strict (resp. a classical) solution of problem (0.1) is a function u such that u', Au belong to $C([0,T],E)$ (resp $C(]0,T],E)$), $u(0)=x$ and the equation $u'-Au=f$ holds in $[0,T]$ (resp. $]0,T]$).

We prove here the following result:

THEOREM 1.1 *Let $f \in C([0,T],E)$, $x \in D_{A(0)}$ and suppose that $A(0)x+f(0) \in \overline{D_{A(0)}}$. There exists $C>0$ such that if u is a strict solution, then*

$$(1.1) \qquad \|u(t)\|_E \le C\Big(\|x\|_E + \int_0^t \|f(s)\|_E ds\Big) \qquad \forall\, t \in [0,T].$$

Proof First of all we remark that the condition $A(0)x+f(0) \in \overline{D_{A(0)}}$ is necessary for existence of strict solutions and, in fact, if u is a strict solution we must have

$$(1.2) \qquad A(t)u(t)+f(t) \in \overline{D_{A(t)}} \qquad \forall\, t \in [0,T];$$

this can be proved by the argument used in [5, Proposition 3.7(i)].

Let now $t \in]0,T]$ and define

$$v(s) = e^{(t-s)A(t)}u(s), \quad s \in [0,t].$$

Then for $s \in [0,t[$

$$v'(s) = A(t)e^{(t-s)A(t)}[A(t)^{-1}-A(s)^{-1}]A(s)u(s) + e^{(t-s)A(t)}f(s).$$

Fix $\varepsilon \in]0,1[$, integrate between 0 and $t-\varepsilon t$ and operate with $A(t)$: the result is

$$(1.3) \qquad A(t)u(t) - \int_0^t Q(t,s)A(s)u(s)ds = G_\varepsilon(t), \qquad t \in]0,T],$$

where we have set

$$(1.4) \qquad Q(t,s) = A(t)^2 e^{(t-s)A(t)}[A(t)^{-1}-A(s)^{-1}], \quad 0 \le s < t \le T,$$

and

$$(1.5) \qquad G_\varepsilon(t) = -\int_{t-\varepsilon t}^{t} Q(t,s)A(s)u(s)ds + A(t)u(t) - A(t)e^{t\varepsilon A(t)}u(t-\varepsilon t)$$
$$+ A(t)e^{tA(t)}x + \int_0^{t-\varepsilon t} A(t)e^{(t-s)A(t)}f(s)ds, \qquad t \in]0,T].$$

By [5, Lemma 2.3(i)] it follows that

$$(1.6) \qquad \|Q(t,s)\|_{\mathcal{L}(E)} \le K(t-s)^{\delta-1} \qquad \forall\, 0 \le s < t \le T$$

so that for any fixed $r \in [0,T[$ the Volterra integral operator

$$(1.7) \qquad Q_r g(t) = \int_r^t Q(t,s)g(s)ds, \qquad t \in [r,T]$$

is well defined in $C([r,T],E)$ or in $L^1(r,T,E)$ (and even in more general spaces, see [5, Proposition 2.4]). Moreover, $(1-Q_r)^{-1}$ exists as a bounded operator in the same spaces [5, Proposition 2.6]. However in this section we are only interested to the case r=0 and we denote the operator Q_0 simply by Q.

Now we turn on the function G_ε and split it into several terms:

$$(1.8)\qquad G_\varepsilon(t) = -\int_{t-\varepsilon t}^{t} Q(t,s)A(s)u(s)ds$$
$$-[e^{\varepsilon tA(t)}-1-\varepsilon tA(t)e^{\varepsilon tA(t)}][A(t)u(t)+f(t)] + [e^{\varepsilon tA(t)}-1]f(t)$$
$$-A(t)e^{\varepsilon tA(t)}[u(t-\varepsilon t)-u(t)+\varepsilon tu'(t)] + [A(t)e^{tA(t)}-A(0)e^{tA(0)}]x$$
$$+A(0)e^{tA(0)}x + \int_0^{t-\varepsilon t}[A(t)e^{(t-s)A(t)}-A(s)e^{(t-s)A(s)}]f(s)ds$$
$$+\int_0^{t-\varepsilon t} A(s)e^{(t-s)A(s)}f(s)ds = \sum_{i=1}^{8} I_i\ ;$$

thus by the results of [5] it is easily seen that $G_\varepsilon \in L^\infty(0,T,E)\cap C(]0,T],E)$ for each $\varepsilon\in]0,1[$.

Hence by (1.5) we deduce that

$$A(t)u(t) = [(1-Q)^{-1}G_\varepsilon](t),\qquad t\in]0,T].$$

Integrate between εt and t: by using again the equation of (0.1) we get

$$(1.9)\qquad u(t) = u(\varepsilon t) + \int_{\varepsilon t}^{t} f(s)ds + \int_{\varepsilon t}^{t}[(1-Q)^{-1}G_\varepsilon](\tau)d\tau.$$

In order to get (1.1), we have to estimate the right member of this equality, by using the splitting (1.8), and then pass to the limit as $\varepsilon\to0^+$.

First we show that

$$(1.10)\qquad \lim_{\varepsilon\to0^+}\int_{\varepsilon t}^{t}\|[(1-Q)^{-1}(I_1+I_2+I_4)](\tau)\|_E d\tau = 0 \quad \text{uniformly in } t.$$

Indeed, as in [5, Proposition 2.6] it follows that

$$\|(1-Q)^{-1}\|_{\mathcal{L}(L^1(0,t,E))} \le C \qquad \forall\ t\in]0,T];$$

on the other hand, by (1.6)

$$\|I_1(\tau)\|_E \le C(\varepsilon T)^\delta\|A(\cdot)u(\cdot)\|_{C([0,T],E)}$$

whereas by (1.2)

$$\begin{cases}\lim_{\varepsilon\to0^+}(\|I_2(\tau)\|_E + \|I_4(\tau)\|_E) = 0,\\ \|I_2(\tau)\|_E + \|I_4(\tau)\|_E \le C\|u'\|_{C([0,T],E)} \qquad \forall\ \tau\in]0,T].\end{cases}$$

Thus (1.10) follows by Lebesgue's Theorem.

Next, it is easily seen that

$$(1.11)\qquad \int_{\varepsilon t}^{t}\|[(1-Q)^{-1}I_3](\tau)\|_E d\tau \le C\int_0^t\|f(s)\|_E ds \quad \forall\varepsilon\in]0,1[,\ \forall\ t\in]0,T],$$

and by [5, Lemma 1.10(i)]

$$(1.12)\qquad \int_{\varepsilon t}^{t}\|[(1-Q)^{-1}(I_5+I_7)](\tau)\|_E d\tau \le C\ T^\delta(\|x\|_E+\int_0^t\|f(s)\|_E ds\ \ \forall\varepsilon\in]0,1[,\forall t\in]0,T].$$

Finally the terms I_6 and I_8 need a more careful procedure, which rests on the following

<u>LEMMA 1.2</u> <u>Let</u> $Q = Q_0$ <u>be defined by</u> (1.7). <u>The</u> n-th <u>iterate</u> Q^n $(n\ge1)$ <u>is given by</u>

$$Q^n g(t) = \int_0^t Q_n(t,s)g(s)ds,\qquad t\in[0,T]$$

<u>where the kernel</u> $Q_n(t,s)$ <u>is defined inductively by</u>

$$Q_1(t,s) = Q(t,s),\qquad Q_n(t,s) = \int_s^t Q_{n-1}(t,\sigma)Q(\sigma,s)d\sigma,$$

<u>and satisfies for</u> $0\le s\le\sigma<t\le T$:

(i) $\|Q_n(t,s)\|_{\mathcal{L}(E)} \le \frac{K^n\Gamma(\delta)^n}{\Gamma(n\delta)}(t-s)^{n\delta-1}\quad \forall\ n\ge1,$

(ii) $\|Q(t,\sigma)-Q(t,s)\|_{\mathcal{L}(E)} \le B\,\frac{(\sigma-s)^\delta}{t-s}\ ,$

$$\text{(iii)} \quad \int_s^t \frac{\|Q_n(t,\sigma)-Q_n(t,s)\|_{\mathcal{L}(E)}}{\sigma-s}\,d\sigma \le C\,\frac{K^{n-1}\Gamma(\delta)^{n-1}}{\Gamma(n\delta-\delta)}\,(t-s)^{n\delta-1} \quad \forall\, n \ge 2.$$

Proof The first part is straightforward; the proof of (i) is easily obtained by induction, starting from (1.6). Part (ii) follows readily by [5, Lemma 2.3(iii)]. Concerning (iii), the result is obvious if $n=1$; otherwise we write

$$Q_n(t,\sigma)-Q_n(t,s) = \int_\sigma^t Q_{n-1}(t,r)[Q(r,\sigma)-Q(r,s)]dr - \int_s^\sigma Q_{n-1}(t,r)Q(r,s)dr,$$

and the result follows by (i) and (ii) after straightforward calculations.

Let us estimate the term I_6. We have by Lemma 1.2(iii)-(i)

$$\text{(1.13)} \quad \left\|\int_{\varepsilon t}^t [(1-Q)^{-1}I_6](\tau)d\tau\right\|_E \le \left\|\int_{\varepsilon t}^t I_6(\tau)d\tau\right\|_E + \sum_{n=1}^\infty \left\|\int_{\varepsilon t}^t [Q^n I_6](\tau)d\tau\right\|_E$$

$$\le \|(e^{tA(0)}-e^{\varepsilon tA(0)})x\|_E + M\sum_{n=1}^\infty \int_{\varepsilon t}^t \int_0^\tau \frac{\|Q_n(\tau,\sigma)-Q_n(\tau,0)\|_{\mathcal{L}(E)}}{\sigma}d\sigma d\tau\,\|x\|_E$$

$$+ \sum_{n=1}^\infty \left\|\int_{\varepsilon t}^t Q_n(\tau,0)(e^{\tau A(0)}-1)x d\tau\right\|_E$$

$$\le C(1+T^\delta)\sum_{n=1}^\infty \frac{[K\Gamma(\delta)T^\delta]^n}{n\Gamma(n\delta)}\|x\|_E \quad \forall\,\varepsilon\in]0,1[,\ \forall\, t\in]0,T].$$

Finally the term I_8 is treated in the following way:

$$\left\|\int_{\varepsilon t}^t [(1-Q)^{-1}I_8](\tau)d\tau\right\|_E \le \left\|\int_{\varepsilon t}^t \int_0^{\tau-\varepsilon\tau} A(s)e^{(\tau-s)A(s)}f(s)ds d\tau\right\|_E$$

$$+ \sum_{n=1}^\infty \left\|\int_{\varepsilon t}^t \int_0^\tau \int_0^{\sigma-\varepsilon\sigma} [Q_n(\tau,\sigma)-Q_n(\tau,s)]A(s)e^{(\sigma-s)A(s)}f(s)ds d\sigma d\tau\right\|_E$$

$$+ \sum_{n=1}^\infty \left\|\int_{\varepsilon t}^t \int_0^\tau \int_0^{\sigma-\varepsilon\sigma} Q_n(\tau,s)A(s)e^{(\sigma-s)A(s)}f(s)ds d\sigma d\tau\right\|_E$$

$$= \left\|\int_{\varepsilon t}^t [e^{(t-s)A(s)}-e^{[(\varepsilon t-s)\vee(s\varepsilon/(1-\varepsilon))]A(s)}]f(s)ds\right\|_E$$

$$+ M\sum_{n=1}^\infty \int_{\varepsilon t}^t \int_0^{\tau-\varepsilon\tau} \int_{s/(1-\varepsilon)}^\tau \frac{\|Q_n(\tau,\sigma)-Q_n(\tau,s)\|_{\mathcal{L}(E)}}{\sigma-s}\|f(s)\|_E d\sigma ds d\tau$$

$$+ \sum_{n=1}^\infty \left\|\int_{\varepsilon t}^t \int_0^{\tau-\varepsilon\tau} Q_n(\tau,s)[e^{(\tau-s)A(s)}-e^{(s\varepsilon/(1-\varepsilon))A(s)}]f(s)ds d\tau\right\|_E,$$

where we have used Fubini's Theorem. Consequently we get by Lemma 1.2 (iii)-(i), for each $\varepsilon\in]0,1[$ and $t\in]0,T]$:

$$\text{(1.14)} \quad \left\|\int_t^t [(1-Q)^{-1}I_8](\tau)d\tau\right\|_E \le C(1+T^\delta)\sum_{n=1}^\infty \frac{[K\Gamma(\delta)T^\delta]^n}{n\Gamma(n\delta)}\int_0^t \|f(s)\|_E ds .$$

Recalling (1.8) and collecting (1.10),(1.11),(1.12),(1.13) and (1.14) we can let $\varepsilon\to 0^+$ in (1.9), and (1.1) follows. Theorem 1.1 is completely proved.

REMARK 1.3 A generalized version of Theorem 1.1 can be proved for classical solutions belonging to the class $\bigcup_{0\le\mu<1+\delta} I_\mu(D_A)$; this class was defined in [5, formulas (1.1)-(1.2)]. Correspondingly, the data x,f have to be chosen in $\overline{D_{A(0)}}$ and in $L^1(0,T,E)\cap C(]0,T],E)$ respectively; the proof is essentially the same, but it requires much more technicalities.

2. THE FUNDAMENTAL SOLUTION

The argument used in the proof of Theorem 1.1 can be refined in order to get a deeper result. Namely, we have:

THEOREM 2.1 Let $f\in C([0,T],E)$, $x\in D_{A(0)}$ and suppose that $A(0)x+f(0) \in \overline{D_{A(0)}}$. If u is a strict solution of problem (0.1), then u is given by

$$\text{(2.1)} \quad u(t) = U(t,0)x + \int_0^t U(t,s)f(s)ds, \quad t\in[0,T],$$

where

$$\text{(2.2)} \quad U(t,s) = e^{(t-s)A(s)} + \int_s^t \Big\{\Big[(1-Q_s)^{-1}[A(\cdot)e^{(\cdot-s)A(\cdot)}-A(s)e^{(\cdot-s)A(s)}]\Big](\tau)$$

$$+ \int_s^\tau \sum_{n=1}^\infty [Q_n(\tau,\sigma)-Q_n(\tau,s)]A(s)e^{(\sigma-s)A(s)}d\sigma$$

$$+ \sum_{n=1}^\infty Q_n(\tau,s)[e^{(\tau-s)A(s)}-1]\Big\}d\tau , \quad 0\le s\le t\le T.$$

Proof As in the proof of Theorem 1.1 we arrive at (1.9); now we try to pass to the limit as $\varepsilon\to 0^+$ directly in this expression, in order to get a representation formula for $u(t)$. Recalling (1.8) and (1.10) we have:

$$\text{(2.3)}\qquad u(t) = x + \int_0^t f(s)ds + \lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}(I_3+I_5+I_6+I_7+I_8)](\tau)d\tau \;.$$

Now it is easily seen that

$$\text{(2.4)}\qquad \lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}I_5](\tau)d\tau = \int_0^t \Big[(1-Q)^{-1}[A(\cdot)e^{\cdot A(\cdot)} - A(0)e^{\cdot A(0)}]x\Big](\tau)d\tau,$$

whereas

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}I_7](\tau)d\tau$$
$$= \int_0^t \Big[(1-Q)^{-1}\Big[\int_0^{\cdot} [A(\cdot)e^{(\cdot-s)A(\cdot)} - A(s)e^{(\cdot-s)A(s)}]f(s)ds\Big]\Big](\tau)d\tau \;;$$

but if we split the operator $(1-Q)^{-1}$ into its Neumann series, then a simple calculation shows, via Fubini's Theorem, that the last equality can be rewritten as:

$$\text{(2.5)}\qquad \lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}I_7](\tau)d\tau$$
$$= \int_0^t \int_s^t \Big[(1-Q_s)^{-1}[A(\cdot)e^{(\cdot-s)A(\cdot)} - A(s)e^{(\cdot-s)A(s)}]\Big](\tau)d\tau f(s)ds \;.$$

Next, concerning I_6 we have:

$$\text{(2.6)}\qquad \lim_{\varepsilon\to 0^+} \int_t^t [(1-Q)^{-1}I_6](\tau)d\tau = \lim_{\varepsilon\to 0^+} [e^{tA(0)} - e^{\varepsilon t A(0)}]x$$
$$+ \sum_{n=1}^{\infty}\int_{\varepsilon t}^t \int_0^{\tau} [Q_n(\tau,\sigma)-Q_n(\tau,0)]A(0)e^{\sigma A(0)}x\,d\sigma d\tau$$
$$+ \sum_{n=1}^{\infty}\int_0^t Q_n(\tau,0)[e^{\tau A(0)}-1]x\,d\tau$$
$$= [e^{tA(0)}-1]x + \sum_{n=1}^{\infty}\int_0^t \int_0^{\tau} [Q_n(\tau,\sigma)-Q_n(\tau,0)]A(0)e^{\sigma A(0)}x\,d\sigma d\tau$$
$$+ \sum_{n=1}^{\infty}\int_0^t Q_n(\tau,0)[e^{\tau A(0)}-1]x\,d\tau \;.$$

Finally we consider together I_3 and I_8:

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}(I_3+I_8)](\tau)d\tau = \lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t \Big[(1-Q)^{-1}[e^{\varepsilon\cdot A(\cdot)}-1]f(\cdot)\Big](\tau)d\tau$$

$$+ \int_{\varepsilon t}^t \int_0^{\tau-\varepsilon\tau} A(s)e^{(\tau-s)A(s)}f(s)dsd\tau$$
$$+ \sum_{n=1}^{\infty}\int_{\varepsilon t}^t \int_0^{\tau}\int_0^{\sigma-\varepsilon\sigma} [Q_n(\tau,\sigma)-Q_n(\tau,s)]A(s)e^{(\sigma-s)A(s)}f(s)dsd\sigma d\tau$$
$$+ \sum_{n=1}^{\infty}\int_{\varepsilon t}^t \int_0^{\tau}\int_0^{\sigma-\varepsilon\sigma} Q_n(\tau,s)A(s)e^{(\sigma-s)A(s)}f(s)dsd\sigma d\tau \;.$$

Using once more Fubini's Theorem, we easily get:

$$\lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}(I_3+I_8)](\tau)d\tau = \lim_{\varepsilon\to 0^+}\Big\{\int_0^t \Big[(1-Q)^{-1}[e^{\varepsilon\cdot A(\cdot)}-1]f(\cdot)\Big](\tau)d\tau$$
$$+ \int_0^{t-\varepsilon t} [e^{(t-s)A(s)} - e^{[(\varepsilon t-s)V(s\varepsilon/(1-\varepsilon))]A(s)}]f(s)ds$$
$$+ \sum_{n=1}^{\infty}\int_0^t\int_0^{\tau}\int_0^{\sigma} [Q_n(\tau,\sigma)-Q_n(\tau,s)]A(s)e^{(\sigma-s)A(s)}f(s)dsd\sigma d\tau$$
$$+ \sum_{n=1}^{\infty}\int_0^t\int_0^{\tau-\varepsilon\tau} Q_n(\tau,s)[e^{(\tau-s)A(s)} - e^{[(\varepsilon\tau-s)V(s\varepsilon/(1-\varepsilon))]A(s)}]f(s)dsd\tau\Big\}$$
$$= -\int_0^t [(1-Q)^{-1}f](\tau)d\tau + \int_0^t e^{(t-s)A(s)}f(s)ds$$
$$+ \sum_{n=1}^{\infty}\int_0^t\int_s^t\int_s^{\tau} [Q_n(\tau,\sigma)-Q_n(\tau,s)]A(s)e^{(\sigma-s)A(s)}f(s)d\sigma d\tau ds$$
$$+ \sum_{n=1}^{\infty}\int_0^t\int_s^t Q_n(\tau,s)e^{(\tau-s)A(s)}f(s)d\tau ds$$
$$+ \lim_{\varepsilon\to 0^+}\int_0^t \Big[(1-Q)^{-1}\Big[[e^{\varepsilon\cdot A(\cdot)} - e^{(\varepsilon\cdot/(1-\varepsilon))A(\cdot)}]f(\cdot)\Big]\Big](\tau)d\tau \;.$$

The last limit is 0 since

$$\|e^{\varepsilon\tau A(\tau)} - e^{(\varepsilon\tau/(1-\varepsilon))A(\tau)}\|_{\mathcal{L}(E)} \le M\int_{\varepsilon\tau}^{\varepsilon\tau/(1-\varepsilon)} \frac{d\sigma}{\sigma} = M\log\frac{1}{1-\varepsilon} \;;$$

hence

$$\text{(2.7)}\qquad \lim_{\varepsilon\to 0^+} \int_{\varepsilon t}^t [(1-Q)^{-1}(I_3+I_8)](\tau)d\tau = \int_0^t e^{(t-s)A(s)}f(s)ds - \int_0^t f(s)ds$$
$$+ \int_0^t \sum_{n=1}^{\infty}\int_s^t Q_n(\tau,s)[e^{(\tau-s)A(s)}-1]d\tau f(s)ds$$
$$+ \int_0^t \sum_{n=1}^{\infty}\int_s^t\int_s^{\tau} [Q_n(\tau,\sigma)-Q_n(\tau,s)]A(s)e^{(\sigma-s)A(s)}d\sigma d\tau f(s)ds \;.$$

By (2.3), (2.4), (2.5), (2.6) and (2.7) we readily get (2.1) with $U(t,s)$ given by (2.2). The result is proved.

REMARK 2.2 The operator $U(t,s)$ defined by (2.2) enjoys all the usual properties of fundamental solutions. Indeed, it is clear by definition that $U(t,s)\in \mathcal{L}(E)$ for $0\le s\le t\le T$ and $U(t,t)=1$; however we have

$$\lim_{t\to s^+} U(t,s)x = x \iff \lim_{t\to s^+} e^{(t-s)A(s)}x = x \iff x\in \overline{D_{A(s)}}$$

(see [7, Proposition 1.2]). Moreover by (2.2) and by the representation formula for $A(\cdot)u(\cdot)$ proved in [5] it follows easily that

$$(2.8)\qquad \frac{\partial}{\partial t}U(t,s) = A(t)U(t,s) \qquad \forall\, t\in\,]s,T].$$

We also have

$$\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s) \qquad \forall\, s\in [0,t[,$$

in the sense that

$$(2.9)\qquad \lim_{h\to 0} h^{-1}[U(t,s+h)-U(t,s)]A(s)^{-1} = -U(t,s) \qquad \forall\, s\in [0,t[.$$

The proof of (2.9) is not evident; it is necessary to split patiently the ratio $h^{-1}[U(t,s+h)-U(t,s)]$ into 16 terms and to consider each of them separately (but sometimes two terms or more have to be assembled in order to get convergence as $h\to 0$). The result is just (2.9).

REMARK 2.3 Again, a generalized version of Theorem 2.1 holds (with much more tedious proof) for classical solutions belonging to the class $\bigcup_{0<\mu<1+\delta} I_\mu(D_A)$ with data x,f taken from $\overline{D_{A(0)}}$ and $L^1(0,T;E)\cap C(]0,T],E)$ respectively (compare with Remark 1.3).

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