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EXISTENCE AND SHARP REGULARITY RESULTS FOR LINEAR PARABOLIC NON-AUTONOMOUS INTEGRO-DIFFERENTIAL EQUATIONS

BY

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ABSTRACT

We study existence, uniqueness and maximal regularity of the strict solution $u \in C^1([0,T],E)$ of the integro-differential equation

$$u'(t) - A(t)u(t) - \int_0^t B(t,s)u(s)ds = f(t), \qquad t \in [0,T],$$

with the initial datum $u(0) = x$, in a Banach space E. $\{A(t)\}_{t\in[0,T]}$ is a family of generators of analytic semigroups whose domains $D_{A(t)}$ are not constant in t as well as (possibly) not dense in E, whereas $\{B(t)\}_{0\le s<t\le T}$ is a family of closed linear operators with $D_{B(t,s)} \supseteq D_{A(s)}$ $\forall t \in]s,T]$. We prove necessary and sufficient conditions for existence of the strict solution and for Hölder continuity of its derivative; well-posedness of the problem with respect to the Hölder norms is also shown.

0. Introduction

Let $\{A(t)\}_{t\in[0,T]}$ and $\{B(t,s)\}_{0\le s<t\le T}$ be two families of closed linear operators on a Banach space E. In this paper we study the linear problem

$$(0.1) \qquad \begin{cases} u'(t) - A(t)u(t) - \int_0^t B(t,s)u(s)ds = f(t), & t \in [0,T] \\ u(0) = x \end{cases}$$

where $x \in E$ and $f:[0,T]\to E$ is a continuous function. We suppose here that for each $t \in [0,T]$, $A(t)$ is the infinitesimal generator of a bounded analytic semigroup $\{e^{\xi A(T)}\}_{\xi\ge 0}$; the domains $D_{A(t)}$ may change with t, and are not assumed

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to be dense in E, i.e. the semigroups $e^{\xi A(t)}$ need not be strongly continuous at $\xi = 0$. About the operators $B(t,s)$ we require that for each $s \in [0,T[$ and $t \in]s,T]$ the domain $D_{B(t,s)}$ contains $D_{A(s)}$, and in addition we assume for $B(t,s)$ some smoothness in t.

Integro-differential equations as in (0.1) have physical interest: for example, they arise naturally in the study of heat flows in materials with memory. For more specific physical models in which the linear theory of parabolic integro-differential equations applies, see Coleman–Gurtin [9], Gurtin [21], Belleni Morante [6], Miller [29] (and the references therein) and Heard [22, Section 6] where a brief survey of the physical background is outlined.

There is a number of papers concerning problem (0.1) under different assumptions: we will just mention here the papers considering the parabolic case (i.e. when $A(t)$ generates an analytic semigroup). The first contribution is due to Friedman–Shinbrot [18] (see also Friedman [17], [16]): they suppose $D_{A(t)} \equiv D_{A(0)}$ (dense in E) and a Hölder condition on $t \to A(t)A(0)^{-1}$, which are the classical assumptions of the theory of linear parabolic evolution equations in the constant-domain case, whereas $B(t,s)$ has the convolution form $h'(t-s)A(s)$, h being a scalar function belonging to the Sobolev space $H^{2,p}(0,T)$ and such that $h(0) > 0$. They convert (0.1) into an equation of the form $u' = T(u) + F(u)$, where F is "small", and solve it by a fixed-point argument. Thus the fundamental solution of (0.1), or resolvent operator, $R(t,s)$ is found, and the solution of (0.1) is given by the variation of parameters formula; an explicit representation of $R(t,s)$ as an integral along a suitable path of the complex plane is given in the case $A(t) \equiv A$ by Laplace transform methods. Existence and uniqueness of solutions of classical type (see Definition 1 below) are proved for any $x \in D_{(-A)^\mu}$ (the domains of the μ-fractional power of $-A$) and any Hölder continuous f.

Several other papers concern the case $A(t) \equiv A$ and $B(t,s)$ of convolution type. Miller ([28], particularly Section 8) studies (0.1) in $[0, +\infty[$ by a method introduced by himself [27] in the finite-dimensional case: he transforms (0.1) into a problem $z'(t) = Cz(t)$, $z(0) = z_0$ in a larger Banach space Z, with a suitable operator C which is the infinitesimal generator of a strongly continuous semigroup. He solves this problem by the Hille–Yosida theory and proves existence and uniqueness of strict solutions (see Definition 1.4 below) for any $x \in D_A$ and f such that $f' + B(\cdot)x$ is uniformly continuous and bounded in $[0, +\infty[$. Well-posedness of (0.1) is also shown, i.e. the norm of the solution $\|u(\cdot)\|_E$ tends to 0 uniformly on compact subsets of $[0,\infty[$ as $\|x\|_{D_A} + \sup_{t \geqq 0} \|f(t)\|_E$ tends to 0.

Another method for the study of (0.1) in $[0,\infty[$ when $A(t) \equiv A$ and $B(t,s) =$

$B(t-s)$ rests on the formal application of Laplace transform to the equation in (0.1); the resolvent operator is then obtained by inverting the resulting equation, and again the variation of parameters formula yields the solution of (0.1). This approach is carried out by Grimmer–Pritchard [20]; in particular they assume that the Laplace transform $\hat{B}(\lambda)$ of $B(t)$ exists and is analytic in a sector $\{|\arg z| < \pi/2 + \delta\}$ with $\delta > 0$: this leads to a resolvent operator which is analytic in t. They find classical solutions whenever $x \in D_{(-A)^{\mu}}$ for some $\mu \in]0,1[$ and f is continuous with values in the same subspace.

More precise results can be found in Grimmer–Kappel [19]: they find the resolvent operator as the integral of the series $\Sigma_{j=0}^{\infty}[(\lambda - A)^{-1}\hat{B}(\lambda)]^{j}(\lambda - A)^{-1}$ along a suitable vertical line of the complex plane; classical (resp. strict) solutions are found provided $x \in D_{(-A)^{\mu}}$ (resp. $x \in D_A$) and f is either continuous with values in $D_{(-A)^{\mu}}$ or Hölder continuous with values in E. Their assumptions require that $\|\hat{B}(\lambda)\| \leqq \text{const} \cdot |\lambda|^{-\beta}$ for $\operatorname{Re}\lambda > 0$, where $\beta > 1$, or alternatively the same estimate in the larger sector $\{|\arg \lambda| < \pi/2 + \delta\}$ with any $\beta > 0$.

Sharp regularity results are proved in Da Prato–Iannelli [13]: their assumptions are slightly stronger than those of [19], since they require that $|\hat{B}(\lambda)| \leqq \text{const} \cdot |\lambda|^{-1}$ for $|\arg \lambda| < \pi/2 + \delta$; accordingly, their results are also finer. Indeed, by using the interpolation spaces $D_A(\mu,\infty)$ (see Definition 1.7 below) in place of $D_{(-A)^{\mu}}$, they prove existence results of the strict solution u which are analogous to those of [19], and in addition the maximal regularity property both in space and in time is proved: namely, u' and Au have exactly the same smoothness as f, where f is either continuous with values in E and bounded with values in $D_A(\mu,\infty)$ (space regularity), or μ-Hölder continuous with values in E (time regularity); in each case x has to be chosen in D_A with $Ax + f(0) \in D_A(\mu,\infty)$.

The general case of (0.1) (i.e. when $A(t)$ is variable with dense and possibly non-constant domains, and $B(t,s)$ is not necessarily of convolution type) is treated by Prüss [30] by a direct method. He takes for $A(t)$ the same assumptions as [18] in the constant-domain case, whereas in the case of variable domains a set of assumptions introduced by Yagi [39], which guarantees the solvability of the linear parabolic evolution equation, is taken: for $B(t,s)$, a Hölder condition with respect to (t,s) is assumed. He converts the equation for the resolvent operator of (0.1) into an integral equation in the space of bounded linear operators $E \to E$, and verifies (which is the main step) that the solution of the latter equation is in fact the resolvent operator of (0.1). He finds classical (resp. strict) solutions whenever f is Hölder continuous and $x \in E$ (resp. $x \in D_{A(0)}$), and shows in addition well-posedness with respect to the norm of E.

In Tanabe [36] the case of non-constant, dense domains is studied; the solvability of the linear parabolic evolution equation is assumed (this is guaranteed by different types of hypotheses: see Kato–Tanabe [23], Tanabe [34], Yagi [39], [40]), whereas $B(t,s)$ has the same form as in [18]. His method resembles that of [18], but enables him to find classical solutions for any $x \in E$ and Hölder continuous f.

All above papers consider the case of dense domains; on the other hand Lunardi–Sinestrari [26] in a recent paper treat the non-autonomous, constant-domain case, taking for $A(t)$ the same hypotheses of [18] except for the density of the domain. About $B(t,s)$ some regularity is assumed, which however in the convolution case $B(t,s) = B(t-s)$ reduces simply to require $\|B(\cdot)\| \in L^p(0,T)$ for a suitable $p>1$. The integral term is considered as a perturbation of the linear parabolic autonomous evolution equation with fixed $A = A(0)$; they use the existence, uniqueness and maximal regularity results proved by Sinestrari [31] for the latter equation, and find strict solutions of (0.1), without constructing the resolvent operator, for any $x \in D_{A(0)}$ and μ-Hölder continuous f satisfying in addition the compatibility condition $A(0)x + f(0) \in D_{A(0)}(\mu,\infty)$. Moroever u' and $A(\cdot)u(\cdot)$ are μ-Hölder continuous too, and the problem is well-posed with respect to the μ-Hölder norms.

In other papers, linear equations different from (0.1) are considered, in which however generators of analytic semigroups play a crucial role: see, among others, Carr–Hannsgen [7], [8], Da Prato–Iannelli [12], Da Prato–Iannelli–Sinestrari [14]. Nonlinear versions of (0.1) are treated in Webb [37], [38], Fitzgibbon [15], Tanabe [35], Heard [22], Sinestrari [32].

In this paper we study problem (0.1) by the same method used in [26], i.e. we treat the integral term as a perturbation of the linear non-autonomous parabolic evolution equation

$$(0.2) \qquad \begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [0,T] \\ u(0) = x \end{cases}$$

in the variable-domain case. In particular, we do not make use of the fundamental solution of (0.2), using instead the regularity results and the representation formula for the solutions of (0.2) proved in Acquistapace–Terreni [1]; by a fixed-point argument we deduce existence, uniqueness and maximal regularity in time for the strict solution of (0.1), in complete analogy with the results of [1] for the strict solution of (0.2). Our hypotheses relative to $A(t)$ are the same as Kato–Tanabe [23], except for the density of domains, which is not assumed here; about $B(t,s)$ we essentially require a Hölder condition for $t \to B(t,s)A(s)^{-1}$ (not uniform with respect to s) and L^p-integrability for $s \to B(t,s)A(s)^{-1}$ with a suitable $p > 1$. From this point of view, our assumptions are weaker than those of [30], whereas the hypotheses in [30] about $A(t)$ are independent of ours (see Remark 1.2 below). Moreover our results about strict solutions are stronger than those in [30]; on the other hand we do not consider classical solutions here (whereas this is done in [30]). This however could be done, again by a fixed point technique, with just a slight strengthening of the assumptions about $B(t,s)$, in order that the integral $\int_0^t B(t,s)u(s)ds$ makes sense for a classical solution u.

We describe now the subject of the next sections. In Section 1 we specify our assumptions, state our definitions and prove some preliminary results. Section 2 contains our main theorems about existence, uniqueness and maximal time regularity of strict solutions of (0.1), as well as their continuous dependence on the data. In Section 3 we give two significant examples which are analyzed in detail. Finally there is an appendix where some improvements are given of the results of [1] relative to the linear parabolic non-autonomous Cauchy problem (0.2), which had been used in the proofs of our main results in Section 2.

1. Notations, assumptions and preliminaries

Let Y be a Banach space and $[a,b]$ a finite interval of the real line. We will use the following Banach spaces of functions:

(a) for each $p \in [1,\infty[$

$$L^p(a,b;y) = \{f :]a,b[\to Y : f \text{ is Bochner measurable and } \textstyle\int_a^b \|f(s)\|_Y^p ds < \infty\}$$

and

$$C([a,b],Y) = \{f : [a,b] \to Y : f \text{ is continuous}\},$$

with norms

$$\|f\|_{L^p(a,b;Y)} = \left[\int_a^b \|f(s)\|_Y^p ds\right]^{1/p},$$

$$\|f\|_{C([a,b],Y)} = \sup_{s\in[a,b]} \|f(s)\|_Y;$$

(b) for each $\theta \in]0,1[$

$$C^\theta([a,b],Y)$$
$$= \left\{f \in C([a,b],Y) : [f]_{C^\theta([a,b],Y)} = \sup\left\{\frac{\|f(s)-f(t)\|_Y}{|s-t|^\theta} : t,s \in [a,b], t \neq s\right\} < \infty\right\}$$

with norm

$$\|f\|_{C^\theta([a,b],Y)} = \|f\|_{C([a,b],Y)} + [f]_{C^\theta([a,b],Y)};$$

(c) $C^1([a,b],Y) = \{f \in C([a,b],Y) : f$ is strongly differentiable in $[a,b]$ and $f' \in C([a,b],Y)\}$, with norm

$$\|f\|_{C^1([a,b],Y)} = \|f\|_{C([a,b],Y)} + \|f'\|_{C([a,b],Y)};$$

(d) for each $\theta \in]0,1[$

$$C^{1,\theta}([a,b],Y) = \{f \in C^1([a,b],Y) : f' \in C^\theta([a,b],Y)\}$$

with norm

$$\|f\|_{C^{1,\theta}([a,b],Y)} = \|f\|_{C([a,b],Y)} + \|f'\|_{C^\theta([a,b],Y)}.$$

We will also consider the function spaces

$$C]a,b],Y) = \bigcap_{\varepsilon \in]0,b-a[} C([a+\varepsilon,b],Y)$$

and

$$C^\theta(]a,b],Y), \quad C^1(]a,b],Y), \quad C^{1,\theta}(]a,b],Y)$$

which are defined similarly.

If X, Y are Banach spaces, $\mathscr{L}(X,Y)$ (or $\mathscr{L}(Y)$ if $X = Y$) will denote the Banach space of bounded linear operators $Q : X \to Y$, with norm

$$\|Q\|_{\mathscr{L}(X,Y)} = \sup\left\{\frac{\|Qx\|_Y}{\|x\|_X} : x \in X - \{0\}\right\}.$$

If A is a linear operator in a Banach space Y, we denote by D_A its domain, by $\sigma(A)$ and $\rho(A)$ its spectrum and resolvent set; for $\lambda \in \rho(A)$ we denote $(\lambda - A)^{-1}$ by $R(\lambda, A)$.

Let us list now our assumptions about the operators $\{A(t)\}_{t\in[0,T]}$ and $\{B(t,s)\}_{0\leq s<t\leq T}$. In what follows E is a fixed Banach space and T is a real positive number.

Hypothesis I. For each $t \in [0,T]$, $A(t) : D_{A(t)} \subseteq E \to E$ is a closed linear operator; in addition there exist $\theta_0 \in]\pi/2, \pi]$ and $M > 0$ such that

(i) $\rho(A(t)) \supseteq S_{\theta_0} := \{z \in \mathbf{C} : |\arg z| < \theta_0\} \cup \{0\} \quad \forall t \in [0,T]$;

(ii) $\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leq \dfrac{M}{|\lambda|+1} \quad \forall \lambda \in S_{\theta_0}, \quad \forall t \in [0,T]$.

Thus in particular there exists $\{M_n\}_{n\in\mathbf{N}} \subseteq]0,+\infty[$ such that

(iii) $\|A(t)^n e^{\xi A(t)}\|_{\mathscr{L}(E)} \leq M_n/\xi^n \quad \forall \xi > 0, \quad \forall n \in \mathbf{N}, \quad \forall t \in [0,T]$.

HYPOTHESIS II. For each $\lambda \in S_{\theta_0}$, $t \to R(\lambda, A(t))$ belongs to $C^1([0,T], \mathcal{L}(E))$ and there exist $L > 0$ and $\alpha \in]0,1[$ such that

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathcal{L}(E)} \leqq \frac{L}{|\lambda|^\alpha + 1} \quad \forall \lambda \in S_{\theta_0}, \quad \forall t \in [0,T].$$

HYPOTHESIS III. There exist $N > 0$, $\eta \in]0,1[$ such that

$$\left\| \frac{d}{dt} A(t)^{-1} - \frac{d}{ds} A(s)^{-1} \right\|_{\mathcal{L}(E)} \leqq N(t-s)^\eta \quad \forall t, s \in [0,T].$$

HYPOTHESIS IV. For $0 \leqq s < t \leqq T$, $B(t,s): D_{B(t,s)} \subseteq E \to E$ is a closed linear operator; in addition:

(i) $D_{B(t,s)} \supseteq D_{A(s)} \quad \forall t \in]s,T]$;

(ii) for each $t \in]0,T]$ and $x \in E$, $B(t,\cdot)A(\cdot)^{-1}x$ is Bochner measurable in E, and there exist $H > 0$, $\beta \in]0,1[$ such that

$$\| B(t,s)A(s)^{-1} \|_{\mathcal{L}(E)} \leqq \frac{H}{(t-s)^{1-\beta}} \qquad \text{for } 0 \leqq s < t \leqq T,$$

or, alternatively,

$$\| B(t,\cdot)A(\cdot)^{-1}x \|_{L^{1/(1-\beta)}(0,t)} \leqq H \| x \|_E \quad \forall t \in]0,T], \qquad \forall x \in E;$$

(iii) there exists a measurable function $\phi(\tau,s)$ such that $H_0 := \sup_{\tau \in]0,T]} \int_0^\tau \phi(\tau,s)ds < \infty$ and

$$\| B(t,s)A(s)^{-1} - B(\tau,s)A(s)^{-1} \|_{\mathcal{L}(E)} \leqq (t-\tau)^\beta \phi(\tau,s) \quad \text{for } 0 \leqq s < \tau \leqq t \leqq T,$$

where β is the number appearing in (ii).

REMARK 1.1. As usual, in Hypotheses I, II, III, IV the role of $A(t)$ may be played by $A(t) - \omega_0$, where ω_0 is any positive number. Indeed, the substitution $v(t) = e^{-\omega_0 t}u(t)$ leads to a problem like (0.1) with $A(t)$, $B(t,s)$, $f(t)$ replaced respectively by $A(t) - \omega_0$, $e^{-\omega_0(t-s)}B(t,s)$, $e^{\omega_0 t}f(t)$; hence the results of this paper can be applied to v, and consequently to the original u.

REMARK 1.2. Hypotheses I, II, III are classical (except for the lack of density of domains) in the theory of parabolic evolution equations with variable domains (see [23], [1]). Slight refinements are however possible in Hypothesis II: namely, one can just require that $t \to R(\lambda, A(t))x \in C^1([0,T], E)$ for each $x \in E$, with the same estimate on $\| \partial R(\lambda, A(t))/\partial t \|_{\mathcal{L}(E)}$. In Acquistapace–Terreni [4] a situation is considered in which Hypothesis II holds only in this weakened form (and not in the form stated above).

Hypothesis III can also be modified [39] and weakened [40], but for our purposes such modifications are not useful since under the assumptions of [39], [40] only the existence of a differentiable solution of (0.2) was proved, and not Hölder regularity of its derivative.

REMARK 1.3. Hypothesis IV is required in order to assure that $t \to \int_0^t B(t,s)v(s)ds$ is Hölder continuous whenever $A(\cdot)v(\cdot)$ is continuous (see Lemmata 1.10 and 1.11 below); consequently, it can be replaced by any (possibly) weaker statement for which that conclusion is still true.

In particular in the convolution case, e.g. $B(t,s) = Q(t-s)A(s)$ with $Q(\sigma) \in \mathscr{L}(E)$ $\forall \sigma \in [0,T]$, Hypothesis IV requires a sort of local Hölder continuity of $\sigma \to Q(\sigma)$. Actually this assumption can be dropped altogether but the proofs of our main theorems have to be changed and one obtains less general results: see Remark 2.3 below.

Let us define now our solutions. Let $x \in E$ and $f \in C([0,T],E)$.

DEFINITION 1.4. We say that $u:[0,T] \to E$ is a *strict solution* of (0.1) if $u \in C^1([0,T],E)$, $u(t) \in D_{A(t)}$ $\forall t \in [0,T]$ and $A(\cdot)u(\cdot) \in C([0,T],E)$, and

$$u'(t) - A(t)u(t) - \int_0^t B(t,s)u(s)ds = f(t) \qquad \forall t \in [0,T], \quad u(0) = x.$$

DEFINITION 1.5. We say that $u:[0,T] \to E$ is a *classical solution* of (0.1) if:

(i) $u \in C([0,T],E) \cap C^1(]0,T],E)$, $u(t) \in D_{A(t)}$ $\forall t \in]0,T]$ and $A(\cdot)u(\cdot) \in C]0,T],E)$;

(ii) there exists

$$\int_0^t B(t,s)u(s)ds := \lim_{\varepsilon \to 0^+} \int_\varepsilon^t B(t,s)u(s)ds \quad \forall t \in]0,T];$$

(iii) $u'(t) - A(t)u(t) - \int_0^t B(t,s)u(s)ds = f(t) \qquad \forall t \in]0,T], \quad u(0) = x.$

REMARK 1.6. We call "classical" the solution of Definition 1.5 because in the classical semigroup theory when $B(t,s) \equiv 0$ the construction of the evolution operator leads to this kind of solution. The name "strict solutions", relative to the functions of Definition 1.4, goes back to Da Prato–Grisvard [10] and has been often adopted (see Da Prato–Grisvard [11], Acquistapace–Terreni [1], [2], [3], Sinestrari [31], Lunardi [25]). However it should be noted that in [30] a different terminology is used: our classical (resp. strict) solutions are denoted strict (resp. strong) there.

Let us recall now the definition of the intermediate spaces $D_A(\theta,\infty)$ ($\theta \in]0,1[$) between D_A and E, where A is a closed linear operator in E, generating a bounded analytic semigroup (possibly not strongly continuous at 0).

DEFINITION 1.7. We set for each $\theta \in]0,1[$

$$D_A(\theta,\infty) = \left\{ x : \sup_{t>0} t^{-\theta} \| e^{tA}x - x \|_E < \infty \right\}.$$

$D_A(\theta,\infty)$ is a Banach space with norm

$$\| x \|_\theta = \| x \|_E + \sup_{t>0} t^{-\theta} \| e^{tA}x - x \|_E ;$$

in addition, we have $D_A \subseteq D_A(\theta,\infty) \subseteq D_A(\theta',\infty) \subseteq \bar{D}_A$ for $0 < \theta' \leqq \theta < 1$ with continuous inclusions. Equivalent definitions and further properties of these spaces can be found in [11], [31].

We list now some basic results which will be needed in the following section. First of all, set for any interval $[a,b] \subseteq [0,T]$

$$C([a,b],D_{A(\cdot)}) = \{u \in C([a,b],E) : u(t) \in D_{A(t)}$$

$$\forall t \in [a,b], A(\cdot)u(\cdot) \in C([a,b],E)\};$$

as $0 \in \rho(A(t))$ $\forall t \in [0,T]$, $C([a,b],D_{A(\cdot)})$ is a Banach space with norm

$$\| u \|_{C([a,b],D_{A(\cdot)})} = \| A(\cdot)u(\cdot) \|_{C([a,b],E)}.$$

Next, we give a survey of the main properties of the strict solution of the linear problem

$$\text{(1.1)} \qquad \begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [t_0,T] \\ u(t_0) = x \end{cases}$$

with initial time $t_0 \in [0,T[$. These properties in the case $t_0 = 0$ are essentially proved in [1], but for the general case only straightforward modifications are required.

PROPOSITION 1.8. *Under Hypotheses I, II, III let* $x \in D_{A(t_0)}$, $f \in C^\delta([t_0,T],E)$, $\delta \in]0,\eta \wedge \alpha]$; *then a strict solution* $u(t)$ *of* (1.1) *exists if and only if*

$$A(t_0)x + f(t_0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_0} A(t_0)x \in \overline{D_{A(t_0)}}.$$

In this case the strict solution is unique and is given by

$$u(t)=e^{(t-t_0)A(t)}x+\int_{t_0}^{t}e^{(t-s)A(t)}g(s)ds,\qquad t\in[t_0,T], \tag{1.2}$$

where g is the unique solution of the Volterra integral equation

$$g(t)+\int_{t_0}^{t}P(t,s)g(s)ds=f(t)-P(t,t_0)x,\qquad t\in[t_0,T], \tag{1.3}$$

whose kernel $P(t,s)$ is defined by

$$P(t,s)=\left[\frac{\partial}{\partial t}e^{\xi A(t)}\right]_{\xi=t-s},\qquad 0\leqq s<t\leqq T. \tag{1.4}$$

Moreover for each $t_1\in]t_0,T]$ and $\rho\in]0,\eta[$ we have

$$\begin{aligned}\|A(\cdot)u(\cdot)-A(t_0)x\|_{C([t_0,t_1],E)}\leqq C_1\Big\{&\frac{(t_1-t_0)^{\delta\wedge\rho}}{\eta-\rho}[\|A(t_0)x\|_E+\|f\|_{C^\delta([t_0,t_1],E)}]\\ &+\Big\|A(t_0)x+f(t_0)-\Big[\frac{d}{dt}A(t)^{-1}\Big]_{t=t_0}A(t_0)x\Big\|_E\Big\}\end{aligned} \tag{1.5}$$

where C_1 does not depend on t_0,t_1,ρ.

Finally we have $u',A(\cdot)u(\cdot)\in C^\delta(]t_0,T],E)$; in addition, $u',A(\cdot)u(\cdot)\in C^\delta([t_0,T],E)$ if and only if

$$A(t_0)x+f(t_0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_0}A(t_0)x\in D_{A(t_0)}(\delta,\infty), \tag{1.6}$$

and if this is the case, then

$$\begin{aligned}\|u'\|_{C^\delta([t_0,T],E)}+\|A(\cdot)u(\cdot)\|_{C^\delta([t_0,T],E)}\leqq C_2\Big\{&\|A(t_0)x\|_E+\|f\|_{C^\delta([t_0,T],E)}\\ &+\Big\|A(t_0)x+f(t_0)-\Big[\frac{d}{dt}A(t)^{-1}\Big]_{t=t_0}A(t_0)x\Big\|_{D_{A(t_0)}(\delta,\infty)}\Big\},\end{aligned} \tag{1.7}$$

PROOF. For the case $t_0=0$ see [1, Theorems 5.1 and 5.3] and Theorems A.5, A.6 and A.7 in the Appendix below.

LEMMA 1.9. *Under Hypotheses I, II suppose that u is a strict solution of* (1.1); *then we have*

$$A(t)u(t)+f(t)-\left[\frac{d}{dt}A(t)^{-1}\right]A(t)u(t)\in\overline{D_{A(t)}}\qquad\forall t\in[t_0,T].$$

PROOF. Fix $t_1\in[t_0,T]$; by definition $u(t_1)\in D_{A(t_1)}$ and moreover, as $t\to t_1$ we get

$$\frac{u(t)-u(t_1)}{t-t_1} \to A(t_1)u(t_1)+f(t_1) \qquad \text{in } E.$$

On the other hand

$$\frac{u(t)-u(t_1)}{t-t_1} = \frac{A(t)^{-1}-A(t_1)^{-1}}{t-t_1}A(t)u(t)+A(t_1)^{-1}\frac{A(t)u(t)-A(t_1)u(t_1)}{t-t_1} \qquad \forall t \in [t_0, T]$$

which implies

$$\frac{u(t)-u(t_1)}{t-t_1} - \frac{A(t)^{-1}-A(t_1)^{-1}}{t-t_1}A(t)u(t) \in D_{A(t_1)} \qquad \forall t \in [t_0, T];$$

as $t \to t_1$ we get

$$A(t_1)u(t_1)+f(t_1) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_1} A(t_1)u(t_1) \in \overline{D_{A(t_1)}}.$$

Let us consider now the operators $B(t,s)$ introduced in Hypothesis IV. First of all we have:

LEMMA 1.10. *Under Hypotheses I, IV we have*:

$$\int_\tau^t \|B(t,s)A(s)^{-1}\|_{\mathscr{L}(E)}ds \leqq C_3(t-\tau)^\beta \qquad \text{if } 0 \leqq \tau \leqq t \leqq T.$$

PROOF. Evident.

LEMMA 1.11. *Under Hypotheses I, IV fix $t_0 \in [0, T[$ and set*

$$Sv(t) = \int_{t_0}^t B(t,s)v(s)ds;$$

then for each $t_1 \in]t_0, T]$, S maps continuously $C([t_0,t_1], D_{A(\cdot)})$ into $\{u \in C^\beta([t_0,t_1], E) : u(t_0)=0\}$; in addition

$$\|Sv\|_{C^\beta([t_0,t_1],E)} \leqq C_4 \|v\|_{C([t_0,t_1],D_{A(\cdot)})}$$

where C_4 does not depend on t_0 and t_1.

PROOF. Let $v \in C([t_0,t_1], D_{A(\cdot)})$: then if $t_0 < \tau \leqq t \leqq t_1$

$$\begin{aligned} Sv(t)-Sv(\tau) &= \int_\tau^t [B(t,s)A(s)^{-1}]A(s)v(s)ds \\ &+ \int_{t_0}^\tau [B(t,s)A(s)^{-1} - B(\tau,s)A(s)^{-1}]A(s)v(s)ds; \end{aligned}$$

hence by Lemma 1.10 and Hypothesis IV(iii)

$$\|Sv(t)-Sv(\tau)\|_E \leqq C_3(t-\tau)^\beta \|v\|_{C([t_0,t_1],D_{A(\cdot)})}+(t-\tau)^\beta \int_{t_0}^{\tau} \phi(\tau,s)ds\,\|v\|_{C([t_0,t_1],D_{A(\cdot)})}$$

$$\leqq (C_3 \vee H_0)\|v\|_{C([t_0,t_1],D_{A(\cdot)})}(t-\tau)^\beta.$$

In addition

$$\|Sv(t)\|_E \leqq \int_{t_0}^{t} \|B(t,s)A(s)^{-1}\|_{\mathcal{L}(E)}ds\,\|v\|_{C([t_0,t_1],D_{A(\cdot)})}$$

$$\leqq C_3\|v\|_{C([t_0,t_1],D_{A(\cdot)})}(t-t_0)^\beta,$$

so that $[Sv](t_0)=0$ and the estimate follows.

2. Strict solutions

In this section we will show, by a fixed-point argument, existence and uniqueness of the strict solution of (0.1); next, we will prove its maximal regularity and well-posedness of the problem.

THEOREM 2.1. *Under Hypotheses I, II, III, IV, let* $x \in D_{A(0)}$, $f \in C^\delta([0,T],E)$, *where* $\delta \in]0,1[$; *then a strict solution* u *of* (0.1) *exists if and only if the vectors* x *and* $f(0)$ *satisfy*

$$A(0)x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x \in \overline{D_{A(0)}}; \tag{2.1}$$

if this is the case, the strict solution is unique, and there exists $c_5>0$ *such that*

$$\|u'\|_{C([0,T],E)}+\|A(\cdot)u(\cdot)\|_{C([0,T],E)} \leqq C_5\{\|A(0)x\|_E+\|f\|_{C^\delta([0,T],E)}\}. \tag{2.2}$$

PROOF. If a strict solution of (0.1) exists, then (2.1) follows by Lemma 1.9.

Suppose conversely that (2.1) holds: for each $t_0 \in]0,T]$ consider the affine submanifold of the Banach space $C([0,t_0],D_{A(\cdot)})$ defined by

$$C_x([0,t_0],D_{A(\cdot)}):=\{u \in C([0,t_0],D_{A(\cdot)}): u(0)=x\};$$

then $C_x([0,t_0],D_{A(\cdot)})$ is a complete metric space with distance

$$d(u,v)=\|u-v\|_{C([0,t_0],D_{A(\cdot)})}.$$

By Lemma 1.11 the operator

$$v \to [Sv](t)=\int_0^t B(t,s)v(s)ds$$

maps $C_x([0,t_0],D_{A(\cdot)})$ into $\{u \in C^\beta([0,t_0],E): u(0)=0\}$; hence for fixed $v \in C_x([0,t_0],D_{A(\cdot)})$ we can consider the problem

$$
\begin{cases} w'(t) - A(t)w(t) = Sv(t) + f(t), & t \in [0,t_0], \\ w(0) = x. \end{cases} \tag{2.3}
$$

Note that $Sv + f \in C^{\beta\wedge\delta}([0,t_0],E)$ and $[Sv](0)=0$. Hence, by Proposition 1.8, problem (2.3) has a unique strict solution $w(t)$ which is given by

$$
w(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}g(s)ds, \qquad t \in [0,t_0], \tag{2.4}
$$

where $g(t)$ solves the integral equation

$$
g(t) + \int_0^t P(t,s)g(s)ds = Sv(t) + f(t) - P(t,0)x, \qquad t \in [0,t_0], \tag{2.5}
$$

whose kernel $P(t,s)$ is defined by (1.4).

We have thus defined a map $\Gamma: v \to \Gamma(v)$, where, for each $v \in C_x([0,t_0],D_{A(\cdot)})$, $\Gamma(v) = w$ is the strict solution of (2.3); hence Γ maps $C_x([0,t_0],D_{A(\cdot)})$ into itself. Moreover, by (2.4) we deduce

$$
A(t)[\Gamma(v)](t) = A(t)e^{tA(t)}x + \int_0^t A(t)e^{(t-s)A(t)}[g(s)-g(t)]ds + (e^{tA(t)}-1)g(t), \quad t \in [0,t_0], \tag{2.6}
$$

with g defined in (2.5).

Let us prove that Γ is a contraction in the complete metric space $C_x([0,t_0],D_{A(\cdot)})$ provided t_0 is sufficiently small.

Fix $v_1, v_2 \in C_x([0,t_0],D_{A(\cdot)})$; then $\Gamma(v_1) - \Gamma(v_2)$ solves (2.3) with $x = 0$, $f \equiv 0$, $v \equiv v_1 - v_2$, so that by (2.6)

$$
A(t)[\Gamma(v_1)(t) - \Gamma(v_2)(t)] = \int_0^t A(t)e^{(t-s)A(t)}[\psi(s) - \psi(t)]ds + (e^{tA(t)}-1)\psi(t), \quad t \in [0,t_0], \tag{2.7}
$$

where

$$
\psi(t) + \int_0^t P(t,s)\psi(s)ds = [Sv_1](t) - [Sv_2](t), \qquad t \in [0,t_0].
$$

By Lemma 1.11 and Lemma A.2 in the Appendix below, we have in addition $\psi \in C^{\beta\wedge\alpha\wedge\eta}([0,t_0],E)$, $\psi(0)=0$ and

$$\|\psi\|_{C^{\beta\wedge\alpha\wedge\eta}([0,t_0],E)} \leqq C_6\|Sv_1 - Sv_2\|_{C^{\beta}([0,t_0],E)}$$

where C_6 does not depend on t_0; therefore by (2.7) we deduce that

$$\|A(t)[\Gamma(v_1)(t)-\Gamma(v_2)(t)]\|_E \leqq C_6\left[\frac{M_1}{\beta\wedge\alpha\wedge\eta}+M_0+1\right]t^{\beta\wedge\alpha\wedge\eta}\|Sv_1 - Sv_2\|_{C^{\beta}([0,t_0],E)}$$

$$\leqq C_7 t_0^{\beta\wedge\alpha\wedge\eta}\|Sv_1 - Sv_2\|_{C^{\beta}([0,t_0],E)}.$$

Hence Lemma 1.11 yields

$$\|\Gamma(v_1)-\Gamma(v_2)\|_{C([0,t_0],D_{A(\cdot)})} \leqq C_7C_4t_0^{\beta\wedge\alpha\wedge\eta}\|v_1 - v_2\|_{C([0,t_0],D_{A(\cdot)})}. \tag{2.8}$$

Choose now

$$t_0 \in \left]0,(2C_7C_4)^{-1/(\beta\wedge\alpha\wedge\eta)}\wedge\left(\frac{4C_1C_4}{\eta}\right)^{-1/(\delta\wedge\alpha\wedge\beta\wedge(\eta/2))}\wedge T\right];$$

then by (2.8) we get

$$\|\Gamma(v_1)-\Gamma(v_2)\|_{C([0,t_0],D_{A(\cdot)})} \leqq \tfrac{1}{2}\|v_1 - v_2\|_{C([0,t_0],D_{A(\cdot)})},$$

so that the operator $\Gamma: C_x([0,t_0],D_{A(\cdot)}) \to C_x([0,t_0],D_{A(\cdot)})$ has a unique fixed point u_0. This means that there exists a unique $u_0 \in C([0,t_0],D_{A(\cdot)}) \cap C^1([0,t_0],E)$ such that

$$\begin{cases} u_0'(t)-A(t)u_0(t)=\displaystyle\int_0^t B(t,s)u_0(s)ds+f(t), & t\in[0,t_0], \\ u_0(0)=x; \end{cases} \tag{2.9}$$

in addition by (1.5) and Lemma 1.11 we get, choosing $\rho = \eta/2$,

$$\|A(\cdot)u_0(\cdot)-A(0)x\|_{C([0,t_0],E)}$$

$$\leqq C_1\left\{\frac{2}{\eta}t_0^{\delta\wedge\alpha\wedge\beta\wedge(\eta/2)}[\|A(0)x\|_E+\|f\|_{C^{\delta}([0,t_0],E)}+C_4\|u_0\|_{C([0,t_0],D_{A(\cdot)})}]\right.$$

$$\left.+\left\|A(0)x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x\right\|_E\right\}$$

$$\leqq\frac{2C_1}{\eta}t_0^{\delta\wedge\alpha\wedge\beta\wedge(\eta/2)}[(1+C_4)\|A(0)x\|_E+\|f\|_{C^{\delta}([0,t_0],E)}$$

$$+C_4\|A(\cdot)u_0(\cdot)-A(0)x\|_{C([0,t_0],E)}]$$

$$+C_1\left\|A(0)x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x\right\|_E,$$

which implies, because of the choice of t_0,

$$\|A(\cdot)u_0(\cdot)\|_{C([0,t_0],E)} \leqq C_8\{\|A(0)x\|_E + \|f\|_{C^\delta([0,t_0],E)}\}. \tag{2.10}$$

Hence the desired strict solution of (0.1) is constructed up to $t = t_0$. In addition we have by Lemma 1.9

$$A(t_0)u_0(t_0) + \int_0^{t_0} B(t_0,s)u_0(s)ds + f(t_0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_0} A(t_0)u_0(t_0) \in \overline{D_{A(t_0)}}. \tag{2.11}$$

Now we can start again: set $x_1 = u_0(t_0)$, $t_1 = (2t_0) \wedge T$, and define

$$[S_1 v](t) = \int_{t_0}^{t} B(t,s)v(s)ds, \qquad v \in C([t_0,t_1],D_{A(\cdot)}).$$

By Lemma 1.11, $S_1 v \in C^\beta([t_0,t_1],E)$, $[S_1 v](t_0) = 0$ and

$$\|S_1 v\|_{C^\beta([t_0,t_1],E)} \leqq C_4\|v\|_{C([t_0,t_1],D_{A(\cdot)})} \quad \forall v \in C([t_0,t_1],D_{A(\cdot)}). \tag{2.12}$$

Consider the map $\Gamma_1 : C_{x_1}([t_0,t_1],D_{A(\cdot)}) \to C_{x_1}([t_0,t_1],D_{A(\cdot)})$ defined by $\Gamma_1(v) = w$, where w is the strict solution of

$$\begin{cases} w'(t) - A(t)w(t) = S_1 v(t) + \int_0^{t_0} B(t,s)u_0(s)ds + f(t), & t \in [t_0,t_1], \\ w(t_0) = x_1. \end{cases}$$

By Proposition 1.8 it is clear that w exists, since it is easily seen that $t \to [S_1 v](t) + \int_0^{t_0} B(t,s)u_0(s)ds + f(t)$ belongs to $C^{\beta\wedge\delta}([t_0,t_1],E)$ and since (2.11) holds; moreover, as before we easily get that Γ_1 is a contraction in $C_{x_1}([t_0,t_1],D_{A(\cdot)})$. Denote by u_1 its unique fixed point: then $u_1 \in C([t_0,t_1],D_{A(\cdot)}) \cap C^1([t_0,t_1],E)$ and

$$\begin{cases} u_1'(t) - A(t)u_1(t) = \int_0^{t_0} B(t,s)u_0(s)ds + \int_{t_0}^{t} B(t,s)u_1(s)ds + f(t), \\ \hfill t \in [t_0,t_1], \\ u_1(t_0) = x_1. \end{cases} \tag{2.13}$$

Hence by (1.5) and (2.12) we have for each $\rho \in]0,\eta[$

$$\begin{aligned} &\|A(\cdot)u_1(\cdot) - A(t_0)x_1\|_{C([t_0,t_1],E)} \\ &\leqq c_1 \frac{t_0^{\delta\wedge\alpha\wedge\beta\wedge\rho}}{\eta-\rho}\left\{\|A(t_0)x_1\|_E + \|f\|_{C^\delta([t_0,t_1],E)} + \left\|\int_0^{t_0} B(\cdot,s)u_0(s)ds\right\|_{C^\beta([t_0,t_1],E)}\right. \\ &\qquad \left. + C_4\|u_1\|_{C([t_0,t_1],D_{A(\cdot)})}\right\} \\ &+ C_1\left\|A(t_0)u_0(t_0) + \int_0^{t_0} B(t_0,s)u_0(s)ds + f(t_0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_0} A(t_0)u_0(t_0)\right\|_E. \end{aligned} \tag{2.14}$$

On the other hand it is easy to verify that

$$\left\| \int_0^{t_0} B(\cdot,s)u_0(s)ds \right\|_{C^\beta([t_0,t_1],E)} \leqq C_4 \| u_0 \|_{C([0,t_0],D_{A(\,)})}$$

so that, choosing $\rho = \eta/2$, (2.14) leads to

$$\begin{aligned}
&\| A(\cdot)u_1(\cdot) - A(t_0)x_1 \|_{C([t_0,t_1],E)} \\
&\quad \leqq \frac{2C_1}{\eta} t_0^{\delta\wedge\alpha\wedge\beta\wedge(\eta/2)} \{(1+C_4)\| A(t_0)x_1 \|_E + \| f \|_{C^\delta([t_0,t_1],E)} + C_4 \| A(\cdot)u_1(\cdot) \\
&\qquad\qquad - A(t_0)x_1 \|_{C([t_0,t_1],E)} + \| u_0 \|_{C([0,t_0],D_{A(\,)})}\} \\
&\quad + C_1 \left\| A(t_0)u_0(t_0) + \int_0^{t_0} B(t_0,s)u_0(s)ds + f(t_0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_0} A(t_0)u_0(t_0) \right\|_E,
\end{aligned}$$

and finally, by (2.10) and recalling that $(2/\eta)C_1C_4t_0^{\delta\wedge\alpha\wedge\beta\wedge(\eta/2)} < \frac{1}{2}$,

$$\| A(\cdot)u_1(\cdot) \|_{C([t_0,t_1],E)} \leqq C_9 \{\| A(0)x \|_E + \| f \|_{C^\delta([0,t_1],E)}\}. \tag{2.15}$$

Clearly, by (2.9) and (2.13) it follows that the function

$$u(t) = \begin{cases} u_0(t), & t \in [0,t_0] \\ u_1(t), & t \in [t_0,t_1] \end{cases}$$

belongs to $C([0,t_1],D_{A(\cdot)}) \cap C^1([0,t_1],E)$ and solves (0.1) in $[0,t_1]$; in particular by (2.10) and (2.15)

$$\| A(\cdot)u(\cdot) \|_{C([0,t_1],E)} \leqq C_{10} \{\| A(0)x \|_E + \| f \|_{C^\delta([0,t_1],E)}\}$$

where $C_{10} = C_8 \vee C_9$, and by Lemma 1.9

$$A(t_1)u(t_1) + \int_0^{t_1} B(t_1,s)u(s)ds + f(t_1) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=t_1} A(t_1)u(t_1) \in \overline{D_{A(t_1)}},$$

so that we can start again. In a finite number of steps we obtain the solution in the whole interval $[0,T]$ and (2.2) also follows.

Let us study now the regularity properties of the strict solution of (0.1).

THEOREM 2.2. *Under Hypotheses I, II, III, IV, let* $x \in D_{A(0)}$, $f \in C^\delta([0,T],E)$, $\delta \in]0, \alpha \wedge \eta \wedge \beta]$, *and suppose that*

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in \overline{D_{A(0)}}.$$

Then the strict solution u of (0.1) *is such that $u', A(\cdot)u(\cdot) \in C^{\delta}(]0,T],E)$. In addition $u', A(\cdot)u(\cdot) \in C^{\delta}([0,T],E)$ if and only if*

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in D_{A(0)}(\delta,\infty), \tag{2.16}$$

and if this is the case, then

$$\begin{aligned} \|u'\|_{C^{\delta}([0,T],E)} + \|A(\cdot)u(\cdot)\|_{C^{\delta}([0,T],E)} \leqq C_{11} \Big\{ &\|A(0)x\|_E + \|f\|_{C^{\delta}([0,T],E)} \\ &+ \left\| A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \right\|_{D_{A(0)}(\delta,\infty)} \Big\}. \end{aligned} \tag{2.17}$$

PROOF. We know that $u \in C([0,T], D_{A(\cdot)})$; hence in particular Lemma 1.11 yields $t \to Su(t) = \int_0^t B(t,s)u(s)ds \in C^{\beta}([0,T],E)$ and $[Su](0) = 0$. Thus u satisfies

$$\begin{cases} u'(t) - A(t)u(t) = g(t), & t \in [0,T] \\ u(0) = x \end{cases}$$

where we have set $g = Su + f$. Note that $g \in C^{\delta}([0,T],E)$ and $g(0) = f(0)$, so that

$$A(0)x + g(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in \overline{D_{A(0)}}.$$

All conclusions then follow by Proposition 1.8.

REMARK 2.3. In the convolution case described in Remark 1.3, we can drop Hypothesis IV(iii), but the results are less precise. Namely, under Hypotheses I, II, III and IV(i)–(ii), it can be shown that if $x \in D_{A(0)}$ and $f \in C^{\delta}([0,T],E)$, where $\delta \in]0, \eta \wedge \alpha]$, then a unique strict solution u of (0.1) exists if and only if (2.16) holds, and in this case $u', A(\cdot)u(\cdot) \in C^{\delta}([0,T],E)$ and (2.17) is true. Hence to get a strict solution a little more regularity on the data x, f is needed. The proof also has to be changed: one has essentially to apply the contraction principle in the space $C^{\delta}([0,T], D_{A(\cdot)})$ (whose definition is clear) instead of $C([0,T], D_{A(\cdot)})$; for the details see [26], where this method is employed in the constant-domain case.

3. Examples

Consider the problem

$$
(3.1)\quad \begin{cases} u_t(t,x) - a(t,x)u_{xx}(t,x) - b(t,x)u_x(t,x) - c(t,x)u(t,x) \\ \qquad - \int_0^t [p(t,s,x)u_{xx}(s,x) + q(t,s,x)u_x(s,x) + r(t,s,x)u(s,x)]ds \\ \qquad = f(t,x), \qquad (t,x) \in [0,T]\times[0,1]; \\ \alpha_0(t)u(t,0) - \beta_0(t)u_x(t,0) = \alpha_1(t)u(t,1) + \beta_1(t)u_x(t,1) = 0, \quad t \in [0,T]; \\ u(0,x) = \phi(x), \qquad x \in [0,1], \end{cases}
$$

with prescribed data f, ψ, under the following assumptions:

$$
(3.2)\quad \begin{cases} a,b,c \in C([0,T]\times[0,1],\mathbf{R}), \\ a(\cdot,x), b(\cdot,x), c(\cdot,x) \in C^{1,\eta}([0,T],\mathbf{R}) \\ \text{with norms independent of } x \in [0,1], \text{ for some } \eta \in]0,1[, \\ a > 0,\ c \leqq 0 \text{ in } [0,T]\times[0,1]; \end{cases}
$$

$$
(3.3)\quad \begin{cases} \alpha_0, \beta_0, \alpha_1, \beta_1 \in C^{1,\eta}([0,T],[0,+\infty[)\ (\eta \text{ defined in (3.2)}), \\ \alpha_0 + \beta_0 > 0, \qquad \alpha_1 + \beta_1 > 0 \qquad \text{in } [0,T], \\ \alpha_0 + \alpha_1 + \int_0^1 |c(\cdot,y)|dy > 0 \qquad \text{in } [0,T]; \end{cases}
$$

$$
(3.4)\quad p,q,r : \{(t,s) : 0 \leqq s < t \leqq T\} \to \mathbf{C} \text{ are measurable functions;}
$$

$$
(3.5)\quad \begin{cases} p(t,s,\cdot), q(t,s,\cdot), r(t,s,\cdot) \in C^2([0,1],\mathbf{C}) \\ \text{and there exist } H > 0,\ \beta \in]0,1[\text{ such that} \\ |p(t,s,x)| + |q(t,s,x)| + |r(t,s,x)| \leqq H/(t-s)^{1-\beta} \\ \forall t \in]0,T], \quad \forall s \in [0,t[, \quad \forall x \in [0,1], \\ \text{or, alternatively} \\ \int_0^t \Big[\sup_{x\in[0,1]} |p(t,s,x)| + \sup_{x\in[0,1]} |q(t,s,x)| \\ \qquad + \sup_{x\in[0,1]} |r(t,s,x)| \Big]^{1/(1-\beta)} ds \leqq H \quad \forall t \in [0,T]; \end{cases}
$$

$$(3.6)\quad \begin{cases} p(\cdot,s,x), q(\cdot s,x), r(\cdot s,x) \in C^{\beta}(]s,T]) \ (\beta \text{ as in } (3.5)), \\ \text{and there exists a measurable non-negative function } \phi(\tau,s) \\ \qquad \text{such that } H_0 = \sup_{\tau\in]0,T]} \int_0^{\tau} \phi(\tau,s)ds < \infty \quad \text{and} \\ \sup_{x\in[0,1]} |p(t,s,x)-p(\tau,s,x)| + \sup_{x\in[0,1]} |q(t,s,x)-q(\tau,s,x)| \\ \quad + \sup_{x\in[0,1]} |r(t,s,x)-r(\tau,s,x)| \leqq |t-\tau|^{\beta}\phi(\tau,s) \\ \qquad \forall s\in[0,T[, \quad \forall t,\tau\in]s,T]. \end{cases}$$

In order to apply the results of Section 2, we set

$$E = C([0,1]), \qquad \|u\|_E = \sup_{x\in[0,1]} |u(x)| = : \|u\|_\infty,$$

and define for each $t \in]0,T]$ and $s \in [0,t[$:

$$(3.7)\quad \begin{cases} D_{A(t)} = \{u \in C^2([0,1]) : \alpha_0(t)u(0) - \beta_0(t)u'(0) = \alpha_1(t)u(1) + \beta_1(t)u'(1) = 0\}, \\ \qquad [A(t)u](x) = a(t,x)u''(x) + b(t,x)u'(x) + c(t,x)u(x), \end{cases}$$

$$(3.8)\quad \begin{cases} D_{B(t,s)} = \{u \in C([0,1]) : p(t,s,\cdot)u'' + q(t,s,\cdot)u' + r(t,s,\cdot)u \in C([0,1])\}, \\ \qquad [B(t,s)u](x) = p(t,s,x)u''(x) + q(t,s,x)u'(x) + r(t,s,x)u(x). \end{cases}$$

We will verify now that the operators $\{A(t)\}$ and $\{B(t,s)\}$ defined in (3.7) and (3.8) satisfy Hypotheses I, II, III and IV of Section 1.

To begin with, in order to verify the assumptions for $\{A(t)\}$, let us prove the following *a priori* estimate:

PROPOSITION 3.1. *Let $a, b, c, \alpha_0, \alpha_1, \beta_0, \beta_1$ be as in* (3.1), (3.2), *and suppose that $u \in C^2([0,1])$ is a solution of*

$$(3.9)\quad \begin{cases} \lambda u - a(t,\cdot)u'' - b(t,\cdot)u' - c(t,\cdot)u = f \in C([0,1]), \\ \alpha_0(t)u(0) - \beta_0(t)u'(0) = z_0 \in \mathbf{C}, \\ \alpha_1(t)u(1) + \beta_1(t)u'(1) = z_1 \in \mathbf{C}, \end{cases}$$

where $t \in [0,T]$ is fixed and λ is a complex number lying in the sector

$$(3.10)\quad \Sigma_K = \{z \in \mathbf{C} : \operatorname{Re} z \geqq 0\} \cup \{z \in \mathbf{C} : |\operatorname{Im} z| > K|\operatorname{Re} z|\} \qquad (K > 0).$$

Then there exists $M > 0$, depending on $K, a, b, c, \alpha_0, \beta_0, \alpha_1, \beta_1$, but independent of t, such that

$$[1+|\lambda|]\|u\|_E + [1+|\lambda|^{1/2}]\|u'\|_E + \|u''\|_E \leqq M\{\|f\|_E + [1+|\lambda|^{\nu}][|z_0|+|z_1|]\} \tag{3.11}$$

where

$$\nu = \begin{cases} \frac{1}{2} & \text{if } \mu := \min_{t\in[0,T]} [\beta_0(t) \wedge \beta_1(t)] > 0, \\ 1 & \text{if } \mu = 0. \end{cases} \tag{3.12}$$

PROOF. The function u solves the equation

$$-(\psi(t,\cdot)u')' + [\lambda - c(t,\cdot)]\gamma(t,\cdot)u = f\gamma(t,\cdot) \qquad \text{in } [0,1], \tag{3.13}$$

where

$$\psi(t,x) = \exp\left(\int_0^x \frac{b(t,y)}{a(t,y)}dy\right), \qquad \gamma(t,x) = \frac{\psi(t,x)}{a(t,x)}. \tag{3.14}$$

Set

$$\begin{cases} p = \min_{t,x} \psi(t,x); \quad m = \min_{t,x} a(t,x); \\ \delta = \min\left\{\min_t\left(\alpha_0(t) + \alpha_1(t) + \int_0^1 |c(t,y)|dy\right),\right. \\ \qquad \left. \min_t(\alpha_0(t)+\beta_0(t)), \min_t(\alpha_1(t)+\beta_1(t))\right\}. \end{cases} \tag{3.15}$$

In order to prove (3.11) suppose first that

$$\lambda \in \Sigma_K \cap \{z \in \mathbf{C} : |z| \leqq \varepsilon_0\} \tag{3.16}$$

with ε_0 to be fixed later. Multiplying both members of (3.13) by $\bar{u}$ and integrating over $[0,1]$, we get

$$\begin{aligned} &\sum_{i=0}^{1}(-1)^i\psi(t,i)u'(i)\overline{u(i)} + \int_0^1 \psi|u'|^2dx + \int_0^1 |c|\gamma|u|^2dx \\ &= \int_0^1 f\gamma\bar{u}dx - \lambda\int_0^1 \gamma|u|^2dx; \end{aligned} \tag{3.17}$$

on the other hand, due to the endpoint conditions in (3.9), we can write for $i = 0,1$

$$(-1)^i\psi(t,i)u'(i)\overline{u(i)}=\begin{cases}\psi(t,i)\dfrac{\beta_i(t)}{\alpha_i(t)}|u'(i)|^2+(-1)^i\psi(t,i)u'(i)\dfrac{\bar{z}_i}{\alpha_i(t)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \beta_i(t)\leqq\alpha_i(t), \\ \psi(t,i)\dfrac{\alpha_i(t)}{\beta_i(t)}|u(i)|^2-\psi(t,i)\overline{u(i)}\dfrac{z_i}{\beta_i(t)} \quad \text{if } \beta_i(t)>\alpha_i(t).\end{cases}$$

Hence if we set

$$\rho_i(t,u):=\begin{cases}\dfrac{\beta_i(t)}{\alpha_i(t)}|u'(i)|^2 & \text{if } \beta_i(t)\leqq\alpha_i(t), \\ \dfrac{\alpha_i(t)}{\beta_i(t)}|u(i)|^2 & \text{if } \beta_i(t)>\alpha_i(t),\end{cases} \tag{3.18}$$

$$Q_i(t,u):=\begin{cases}|u'(i)|\dfrac{|z_i|}{\alpha_i(t)} & \text{if } \beta_i(t)\leqq\alpha_i(t), \\ |u(i)|\dfrac{|z_i|}{\beta_i(t)} & \text{if } \beta_i(t)>\alpha_i(t),\end{cases} \tag{3.19}$$

by (3.17) and (3.15) we easily obtain

$$\begin{aligned} &p\sum_{i=0}^{1}\rho_i(t,u)+\int_0^1\psi|u'|^2dx+\int_0^1|c|\gamma|u|^2dx \\ &\leqq\|\psi\|_\infty\sum_{i=0}^{1}Q_i(t,u)+\frac{\|\psi\|_\infty}{m}(\|f\|_\infty\|u\|_\infty+\varepsilon_0\|u\|_\infty^2).\end{aligned} \tag{3.20}$$

Now observe that for $i=0,1$

$$Q_i(t,u)\leqq\left|\begin{cases}\dfrac{|z_i|}{\alpha_i(t)}\|u'\|_\infty & \text{if } \beta_i(t)\leqq\alpha_i(t), \\ \dfrac{|z_i|}{\beta_i(t)}\|u\|_\infty & \text{if } \beta_i(t)>\alpha_i(t),\end{cases}\right.$$

so that, in any case,

$$\sum_{i=0}^{1}Q_i(t,u)\leqq\frac{2}{\delta}(|z_0|+|z_1|)(\|u\|_\infty+\|u'\|_\infty). \tag{3.21}$$

Next, by Landau's inequality:

$$\|u'\|_\infty\leqq 2\|u''\|_\infty^{1/2}\|u\|_\infty^{1/2},$$

and by the equation in (3.9) we derive

$$\| u'' \|_\infty \leqq \frac{1}{m}[(\varepsilon_0 + \| c \|_\infty)\| u \|_\infty + 2\| b \|_\infty \| u'' \|^{1/2} \| u \|_\infty^{1/2} + \| f \|_\infty].$$

Hence we easily check

$$\| u'' \|_\infty \leqq C_{12} \| u \|_\infty + \frac{2}{m} \| f \|_\infty, \tag{3.22}$$

$$\| u' \|_\infty \leqq (C_{12} + 1) \| u \|_\infty + \frac{2}{m} \| f \|_\infty, \tag{3.23}$$

where

$$c_{12} = \frac{2}{m}\left[\varepsilon_0 + \| c \|_\infty + \frac{2}{m} \| b \|_\infty^2 \right];$$

finally by (3.20) and (3.21) we have

$$\begin{aligned} &p \sum_{i=0}^{1} \rho_i(t,u) + \int_0^1 \psi |u'|^2 dx + \int_0^1 |c| \gamma |u|^2 dx \\ &\leqq \varepsilon_0 \frac{\| \psi \|_\infty}{m} \| u \|_\infty^2 + C_{13}[\| f \|_\infty + |z_0| + |z_1|] \| u \|_\infty + \frac{4 \| \psi \|_\infty}{\delta m} \| f \|_\infty [|z_0| + |z_1|], \end{aligned} \tag{3.24}$$

where

$$C_{13} = \max\left\{ \frac{\| \psi \|_\infty}{m}, \frac{2}{\delta} \| \psi \|_\infty (C_{12} + 2) \right\}.$$

On the other hand for each $x, y \in [0,1]$ we have

$$|u(x)|^2 \leqq 2|u(y)|^2 + 2\int_0^1 |u'(s)|^2 ds. \tag{3.25}$$

Now three cases can occur:

(a) $\int_0^1 |c(t,y)| dy \geqq \delta/3$,

(b) there exists $i \in \{0,1\}$ such that $\alpha_i(t) \geqq \delta/3$ and $\beta_i(t) > \alpha_i(t)$,

(c) there exists $i \in \{0,1\}$ such that $\alpha_i(t) \geqq \delta/3$ and $\beta_i(t) \leqq \alpha_i(t)$.

In case (a), multiplying (3.25) by $|c(t,y)|$ and integrating over $[0,1]$ with respect to y, (3.25) yields

$$\| u \|_\infty^2 \leqq \frac{6 \| a \|_\infty}{p\delta} \int_0^1 |c| \gamma |u|^2 dx + \frac{2}{p} \int_0^1 \psi |u'|^2 dx;$$

in case (b), choosing $y = i$ in (3.25) and recalling (3.18) we get

$$\|u\|_\infty^2 \leqq \frac{6}{\delta}\|\beta_i\|_\infty \rho_i(t,u) + \frac{2}{p}\int_0^1 \psi\,|u'|^2 dx;$$

in case (c) we note that

$$|u(i)|^2 \leqq 2|u'(i)|^2\left[\frac{\beta_i(t)}{\alpha_i(t)}\right]^2 + 2\frac{|z_i|^2}{[\alpha_i(t)]^2} \leqq 2\rho_i(t,u) + \frac{18}{\delta^2}|z_i|^2,$$

and hence, choosing again $y = i$ in (3.25), we obtain

$$\|u\|_\infty^2 \leqq 4\rho_i(t,u) + \frac{36}{\delta^2}|z_i|^2 + \frac{2}{p}\int_0^1 \psi\,|u'|^2 dx.$$

Thus in any case we have

$$\|u\|_\infty^2 \leqq C_{15}\left\{p\sum_{i=0}^{1}\rho_i(t,u) + \int_0^1 \psi|u'|^2 dx + \int_0^1 |c|\gamma|u|^2 dx + [|z_0| + |z_1|]^2\right\} \tag{3.26}$$

where

$$C_{15} = \max\left\{\frac{6\|a\|_\infty}{\delta p}, \frac{6}{\delta p}\|\beta_0\|_\infty, \frac{6}{\delta p}\|\beta_1\|_\infty, \frac{4}{p}, \frac{36}{\delta^2}\right\}.$$

By (3.26) and (3.24) we get

$$\begin{aligned}\|u\|_\infty^2 \leqq\ & C_{15}\varepsilon_0\frac{\|\psi\|_\infty}{m}\|u\|_\infty^2 + C_{15}C_{13}[\|f\|_\infty + |z_0| + |z_1|]\|u\|_\infty \\ & + \left(C_{15} \vee \frac{4\|\psi\|_\infty}{\delta m}\right)\cdot[\|f\|_\infty + |z_0| + |z_1|]^2.\end{aligned} \tag{3.27}$$

Choose now

$$\varepsilon_0 = \frac{m}{2C_{15}\|\psi\|_\infty};$$

then by (3.27), (3.22) and (3.23) we easily conclude that

$$\|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty \leqq C_{16}\{\|f\|_\infty + |z_0| + |z_1|\} \tag{3.28}$$

with

$$C_{16} = 2(C_{12}+1)\left(C_{15}C_{13} + \left[C_{15}^2C_{13}^2 + \left(C_{15} \vee \frac{4\|\psi\|_\infty}{\delta m}\right)\right]^{1/2}\right) + \frac{4}{m}.$$

This proves the result provided (3.16) holds, i.e. when $|\lambda| \leqq \varepsilon_0$. Suppose now $\lambda \in \Sigma_K$ and $|\lambda| \geqq \varepsilon_0$. The following argument is a slight modification of that in Acquistapace–Terreni [2, Section 8]. Let x_0 be a point of maximum for $|u|$ in

$[0,1]$; take $\rho = |\lambda|^{-1/2} \wedge 1$ and let $\sigma > 1$ be fixed later. Consider a function $\theta \in C^\infty(\mathbf{R},\mathbf{R})$ such that

$$0 \leqq \theta \leqq 1, \quad \theta \equiv 1 \text{ in } I_\rho := [x_0 - \rho, x_0 + \rho], \quad \theta \equiv 0 \text{ outside } I_{\sigma\rho}, \quad |\theta'| \leqq \frac{2}{(\sigma-1)\rho}.$$

Multiply (3.13) by $\bar{u}\theta^2$ and integrate over $[0,1]$. By estimating separately the real and imaginary parts, one easily gets (since $ab \leqq \varepsilon a^2/2 + b^2/(2\varepsilon)$ $\forall \varepsilon > 0$)

$$\begin{aligned}(3.29)\qquad &\operatorname{Re}\lambda \int_0^1 \gamma |u|^2\theta^2 dx + \int_0^1 |c|\gamma |u|^2 \theta^2 dx + \frac{1}{2}\int_0^1 \psi |u'|^2\theta^2 dx + p\sum_{i=0}^1 \rho_i(t,u)\theta(i)^2 \\ &\leqq \frac{8\cdot\sigma}{(\sigma-1)^2\rho}\|\psi\|_\infty \|u\|_\infty^2 + 2\sigma\rho\frac{\|\psi\|_\infty}{m}\|f\|_\infty\|u\|_\infty + \|\psi\|_\infty \sum_{i=0}^1 Q_i(t,u),\end{aligned}$$

$$\begin{aligned}(3.30)\qquad &|\operatorname{Im}\lambda| \int_0^1 \gamma |u|^2\theta^2 dx \leqq \varepsilon \int_0^1 \psi |u'|^2 \theta^2 dx \\ &+ \frac{4\sigma}{\varepsilon(\sigma-1)^2\rho}\|\psi\|_\infty\|u\|_\infty^2 + 2\sigma\rho\frac{\|\psi\|_\infty}{m}\|f\|_\infty\|u\|_\infty + \|\psi\|_\infty\sum_{i=0}^1 Q_i(t,u) \quad \forall\varepsilon > 0,\end{aligned}$$

with $\rho_i(t,u)$, $Q_i(t,u)$ given by (3.18), (3.19).

Assume $\operatorname{Re}\lambda \leqq 0$; then $|\operatorname{Re}\lambda| < K^{-1}|\operatorname{Im}\lambda|$, so that, choosing in (3.30) $\varepsilon = K/4$, by (3.29) and (3.30) we easily get

$$\begin{aligned}(3.31)\qquad \frac{1}{4}\int_0^1 \psi|u'|^2\theta^2 dx &\leqq C_{17}\frac{\sigma}{(\sigma-1)^2\rho}\|u\|_\infty^2 + C_{18}\sigma\rho\|f\|_\infty\|u\|_\infty \\ &\quad + C_{19}\sum_{i=0}^1 Q_i(t,u) \qquad \text{if } \operatorname{Re}\lambda \leqq 0\end{aligned}$$

where

$$C_{17} = 8\|\psi\|_\infty\left(1 + \frac{2}{K^2}\right), \quad C_{18} = \frac{2\|\psi\|_\infty}{m}\left(1 + \frac{1}{K}\right), \quad C_{19} = \|\psi\|_\infty\left(1 + \frac{1}{K}\right).$$

On the other hand, if $\operatorname{Re}\lambda > 0$ we can use (3.29), thus obtaining in any case

$$(3.32)\qquad \int_0^1 \psi|u'|^2\theta^2 dx \leqq 4C_{17}\frac{\sigma}{(\sigma-1)^2\rho}\|u\|_\infty^2 + 4C_{18}\sigma\rho\|f\|_\infty\|u\|_\infty + 4C_{19}\sum_{i=0}^1 Q_i(t,u).$$

By (3.32) and (3.30) we have easily

$$\begin{aligned}(3.33)\qquad &|\operatorname{Im}\lambda| \int_0^1 \gamma|u|^2\theta^2 dx \\ &\leqq C_{20}\frac{\sigma}{(\sigma-1)^2\rho}\|u\|_\infty^2 + C_{21}\sigma\rho\|f\|_\infty\|u\|_\infty + C_{22}\sum_{i=0}^1 Q_i(t,u),\end{aligned}$$

where

$$C_{20} = KC_{17} + \frac{16}{K}\|\psi\|_\infty, \quad C_{21} = KC_{18} + \frac{2\|\psi\|_\infty}{m}, \quad C_{22} = KC_{19} + \|\psi\|_\infty.$$

Finally, by summation of (3.29) and (3.33) (if $\operatorname{Re}\lambda > 0$) or of (3.32) and (3.33) (if $\operatorname{Re}\lambda \leqq 0$) we check, setting $B_\rho := I_\rho \cap [0,1]$ and recalling that $\rho^{-2} \geqq |\lambda| \geqq \varepsilon_0\rho^{-2}$:

$$\begin{aligned} &\frac{1}{\rho^2}\int_{B_\rho} \gamma|u^2|\,dx + \int_{B_\rho} \psi|u'|^2 dx \\ &\leqq C_{23}\frac{\sigma}{(\sigma-1)^2\rho}\|u\|_\infty^2 + C_{24}\sigma\rho\,\|f\|_\infty\|u\|_\infty + C_{25}\sum_{i=0}^{1} Q_i(t,u), \end{aligned} \tag{3.34}$$

where

$$C_{23} = \frac{1}{\varepsilon_0}\max\left\{16\|\psi\|_\infty + C_{20}, 4C_{17} + \left(1+\frac{1}{K}\right)C_{20}\right\}.$$

$$C_{24} = \frac{1}{\varepsilon_0}\max\left\{4\frac{\|\psi\|_\infty}{m} + C_{21}, 4C_{18} + \left(1+\frac{1}{K}\right)C_{21}\right\},$$

$$C_{25} = \frac{1}{\varepsilon_0}\max\left\{2\|\psi\|_\infty + C_{22}, 4C_{19} + \left(1+\frac{1}{K}\right)C_{22}\right\}.$$

On the other hand, as in (3.25) we have for each $y \in B_\rho$

$$\|u\|_\infty^2 = |u(x_0)|^2 \leqq 2|u(y)|^2 + 2\int_{B_\rho} |u'(t)|^2 dt \cdot |x_0 - y|,$$

so that integration over B_ρ with respect to y yields

$$\|u\|_\infty^2 \leqq \frac{2}{p}[\|a\|_\infty \vee 1]\rho\left[\frac{1}{\rho^2}\int_{B_\rho} \gamma|u|^2\theta^2 dx + \int_{B_\rho} \psi|u'|^2\theta^2 dx\right]. \tag{3.35}$$

By (3.35) and (3.34) we get

$$\|u\|_\infty^2 \leqq \frac{2}{p}[\|a\|_\infty \vee 1]\cdot\left[C_{23}\frac{\sigma}{(\sigma-1)^2}\|u\|_\infty^2 + C_{24}\sigma\rho^2\|f\|_\infty\|u\|_\infty + C_{25}\rho\sum_{i=0}^{1} Q_i(t,u)\right]. \tag{3.36}$$

Choose now

$$\sigma = 2\left[\frac{2}{p}(\|a\|_\infty \vee 1)C_{23} + 1\right];$$

then

$$\frac{2}{p}(\|a\|_\infty \vee 1)\,C_{23}\frac{\sigma}{(\sigma-1)^2} < \frac{1}{2},$$

so that (3.36) implies

$$\|u\|_\infty^2 \leqq \frac{4}{p}[\|a\|_\infty \vee 1]\left[C_{24}\sigma\rho^2\|f\|_\infty\|u\|_\infty + C_{25}\rho\sum_{i=0}^{1} Q_i(t,u)\right]. \tag{3.37}$$

Now by (3.19) se deduce instead of (3.21), due to the endpoint conditions satisfied by u,

$$\sum_{i=0}^{1} Q_i(t,u) \leqq \begin{cases} \dfrac{1}{\mu}(|z_0|+|z_1|)\|u\|_\infty \\ \\ \dfrac{2}{\delta}(|z_0|+|z_1|)(\|u\|_\infty + \|u'\|_\infty) \end{cases} \tag{3.38}$$

where $\mu = \min_t[\beta_0(t) \wedge \beta_1(t)]$.

On the other hand, by Landau's inequality and by the equation in (3.9) we easily find

$$\|u''\|_\infty \leqq C_{26}|\lambda|\|u\|_\infty + C_{28}\|f\|_\infty, \tag{3.39}$$

$$\|u'\|_\infty \leqq C_{27}|\lambda|^{1/2}\|u\|_\infty + C_{28}|\lambda|^{-1/2}\|f\|_\infty, \tag{3.40}$$

where

$$C_{26} = \frac{2}{m}\left[1 + \frac{\|c\|_\infty}{\varepsilon_0} + 2\frac{\|b\|_\infty^2}{\varepsilon_0 m}\right], \quad C_{27} = 2C_{26}^{1/2} + 1, \quad C_{28} = \frac{2}{m}.$$

Let us go back to (3.37). If $\mu > 0$ by (3.37) and (3.38) it follows that

$$\|u\|_\infty^2 \leqq \frac{C_{29}}{|\lambda|}[\|\rho\|_\infty + |\lambda|^{1/2}(|z_0|+|z_1|)]\|u\|_\infty,$$

where

$$C_{29} = \frac{4}{p}[\|a\|_\infty \vee 1]\left[(C_{24}\sigma) \vee \frac{C_{25}}{\mu}\right],$$

which implies

$$|\lambda|\|u\|_\infty \leqq C_{29}[\|f\|_\infty + |\lambda|^{1/2}(|z_0|+|z_1|)] \qquad \text{if } \mu > 0; \tag{3.41}$$

otherwise if $\mu = 0$ by (3.37), (3.38) and (3.40) we derive that

$$\|u\|_\infty^2 \leqq \frac{C_{30}}{|\lambda|}[\|f\|_\infty + |\lambda|(|z_0|+|z_1|)]\|u\|_\infty + \frac{C_{31}}{|\lambda|}\|f\|_\infty[|z_0|+|z_1|]$$

where

$$C_{30} = \frac{4}{p}[\|a\|_\infty \vee 1]\left[(C_{24}\sigma) \vee \left(C_{25}\frac{2}{\delta}\left(\frac{1}{\varepsilon_0^{1/2}} + C_{27}\right)\right)\right], \qquad C_{31} = [\|a\|_\infty \vee 1]\frac{8C_{25}C_{28}}{p\delta\varepsilon_0},$$

and hence

$$(3.42)\qquad |\lambda|\|u\|_\infty \leqq (C_{30}+C_{31}^{1/2})[\|f\|_\infty+|\lambda|(|z_0|+|z_1|)] \qquad \text{if } \mu=0.$$

By (3.41), (3.42), (3.40) and (3.39) we get in any case when $|\lambda|\geqq\varepsilon_0$

$$(3.43)\qquad |\lambda|\|u\|_\infty+|\lambda|^{1/2}\|u'\|_\infty+\|u''\|_\infty \leqq C_{32}[\|f\|_\infty+|\lambda|^\nu(|z_0|+|z_1|)]$$

where ν is defined by (3.12) and

$$C_{32}=\begin{cases} C_{29}(1+C_{27}+C_{26})+2C_{28} & \text{if } \mu>0,\\ (C_{30}+C_{31}^{1/2})(1+C_{27}+C_{26})+2C_{28} & \text{if } \mu=0.\end{cases}$$

By (3.43) and (3.28) the result follows with

$$(3.44)\qquad M=C_{16}+C_{32}.$$

As a consequence of the above proposition, we have:

PROPOSITION 3.2. *Let $a,b,c,\alpha_0,\alpha_1,\beta_0,\beta_1$ be as in* (3.1), (3.2); *let $\{A(t)\}_{t\in[0,T]}$ be defined by* (3.7). *Then we have*:

(i) $\sigma(A(t))\subseteq\,]-\infty,0[\ \forall t\in[0,T]$;

(ii) *for each $K>0$ there exists $M(K)>0$ (depending also on $a,b,c,\alpha_0,\alpha_1,\beta_0,\beta_1$) such that*

$$(3.45)\qquad \|R(\lambda,A(t))\|_{\mathscr{L}(E)}\leqq\frac{M(K)}{1+|\lambda|}\qquad \forall\lambda\in\Sigma_K,\quad \forall t\in[0,T];$$

(iii) $R(\lambda,A(\cdot))\in C^{1,\eta}([0,T],\mathscr{L}(E))$ *if $\lambda\notin\,]-\infty,0[$, and for each $K>0$ there exists $L(K)>0$ (depending also on $a,b,c,\alpha_0,\alpha_1,\beta_0,\beta_1$) such that*

$$\left\|\frac{\partial}{\partial t}R(\lambda,A(t))\right\|_{\mathscr{L}(E)}\leqq\frac{L(K)}{1+|\lambda|^{3/2-\nu}}\qquad \forall\lambda\in\Sigma_K,\quad \forall t\in[0,T],$$

where Σ_K and ν are defined in (3.10), (3.12).

PROOF. (i)–(ii) Fix $t\in[0,T]$; let us first prove that $0\in\rho(A(t))$. By Proposition 3.1 we get that 0 is not an eigenvalue of $A(t)$. Let u_1, u_2 be the (unique) solutions of

$$\begin{cases} a(t,\cdot)u_0''+b(t,\cdot)u_0'+c(t,\cdot)u_0=0,\\ u_0(0)=\beta_0(t),\quad u_0'(0)=\alpha_0(t),\end{cases}\qquad \begin{cases} a(t,\cdot)u_1''+b(t,\cdot)u_1'+c(t,\cdot)u_1=0,\\ u_1(1)=\beta_1(t),\quad u_1'(1)=-\alpha_1(t).\end{cases}$$

As both u_0 and u_1 solve (3.13) with $\lambda=0$, $f\equiv0$, it is readily seen by differentiation that

$$\psi(t,\cdot)(u_0'u_1 - u_1'u_0) = \text{const} =: Q(t)$$

(ψ is defined in (3.14)), and the constant $Q(t)$ cannot be zero, for otherwise we would get $u_0, u_1 \in D_{A(t)}$ and $A(t)u_0 = A(t)u_1 = 0$, which is impossible since 0 is not an eigenvalue of $A(t)$. Hence it is easy to verify that for each $f \in E$ the function

$$u(x) = -\frac{1}{Q(t)}\left[u_1(x)\int_0^x u_0(y)f(y)\frac{\psi(t,y)}{a(t,y)}dy + u_0(x)\int_x^1 u_1(y)f(y)\frac{\psi(t,y)}{a(t,y)}dy\right]$$

is a solution of (3.9) with $\lambda = 0$, $z_0 = z_1 = 0$. Thus by Proposition 3.1 we have $0 \in \rho(A(t))$. Recalling now that $\lambda \in \rho(A(t))$ provided $\lambda_0 \in \rho(A(t))$ and $|\lambda - \lambda_0| < \|R(\lambda_0, A(t))\|^{-1}_{\mathscr{L}(E)}$, by a standard argument we deduce that $\Sigma_K \subseteq \rho(A(t))$ and (3.45) holds for each $\lambda \in \Sigma_K$ with a suitable constant $M(K)$. This proves (i) and (ii).

(iii) Set for each $t \in [0,T]$ and $g \in C^2([0,1])$ (here D denotes the derivative with respect to x):

$$A(t,\cdot D)g = a(t,\cdot)g'' + b(t,\cdot)g' + c(t,\cdot)g,$$

$$\Gamma(t,D)g = (\alpha_0(t)g(0) - \beta_0(t)g'(0), \alpha_1(t)g(1) + \beta_1(t)g'(1)),$$

and, for fixed $\lambda \in \Sigma_K$, $f \in E$,

$$u(t,\cdot) = R(\lambda, A(t))f, \quad v(t,s,\cdot) = \frac{u(t,\cdot) - u(s,\cdot)}{t-s}, \qquad t,s \in [0,T], \quad t \neq s.$$

The functions $u(t,\cdot) - u(\tau,\cdot)$ and $v(t,s,\cdot) - v(t,\sigma,\cdot)$ solve respectively

$$\text{(3.46)} \qquad \begin{cases} [\lambda - A(t,\cdot,D)][u(t,\cdot) - u(\tau,\cdot)] = [A(t,\cdot,D) - A(\tau,\cdot D)]u(\tau,\cdot), \\ \Gamma(t,D)[u(t,\cdot) - u(\tau,\cdot)] = -[\Gamma(t,D) - \Gamma(\tau,D)]u(\tau,\cdot); \end{cases}$$

$$\text{(3.47)} \qquad \begin{cases} [\lambda - A(t,\cdot,D)][v(t,s,\cdot) - v(t,\sigma,\cdot)] \\ \quad = \dfrac{A(t,\cdot,D) - A(s,\cdot,D)}{t-s}[u(s,\cdot) - u(\sigma,\cdot)] \\ \quad + \left[\dfrac{A(t,\cdot,D) - A(s,\cdot,D)}{t-s} - \dfrac{A(t,\cdot,D) - A(\sigma,\cdot,D)}{t-\sigma}\right]u(\sigma,\cdot), \\ \Gamma(t,D)[v(t,s,\cdot) - v(t,\sigma,\cdot)] = -\dfrac{\Gamma(t,D) - \Gamma(s,D)}{t-s}[u(s,\cdot) - u(\sigma,\cdot)] \\ \quad - \left[\dfrac{\Gamma(t,D) - \Gamma(s,D)}{t-s} - \dfrac{\Gamma(t,D) - \Gamma(\sigma,D)}{t-\sigma}\right]u(\sigma,\cdot). \end{cases}$$

As the coefficients of $A(t,\cdot,D)$ and $\Gamma(t,D)$ are differentiable in t, by (3.46) and Proposition 3.1 we check (denoting by D the derivative with respect to x):

$$\begin{aligned}
&[1+|\lambda|]\|u(t,\cdot)-u(\tau,\cdot)\|_\infty \\
&\quad +[1+|\lambda|^{1/2}]\|Du(t,\cdot)-Du(\tau,\cdot)\|_\infty+\|D^2u(t,\cdot)-D^2u(\tau,\cdot)\|_\infty \\
&\leqq M\cdot|t-\tau|\{C_{33}\|u(\tau,\cdot)\|_{C^2([0,1])}+C_{34}(1+|\lambda|^\nu)\|u(\tau,\cdot)\|_{C^1([0,1])}\} \\
&\leqq M^2(C_{33}\vee C_{34})|t-\tau|[1+|\lambda|^{\nu-1/2}]\|f\|_E,
\end{aligned}$$

where ν is defined in (3.12), M is given by (3.44) and

$$C_{33}=\max\left\{\sup_{x\in[0,1]}\|a(\cdot,x)\|_{C^1([0,T])},\ \sup_{x\in[0,1]}\|b(\cdot,x)\|_{C^1([0,T])},\ \sup_{x\in[0,1]}\|c(\cdot,x)\|_{C^1([0,T])}\right\},$$

$$C_{34}=\max\{\|\alpha_0\|_{C^1([0,T])},\|\alpha_1\|_{C^1([0,T])},\|\beta_0\|_{C^1([0,T])},\|\beta_1\|_{C^1([0,T])}\}.$$

Similarly by (3.47), using (3.48) we get

$$\begin{aligned}
&[1+|\lambda|]\|v(t,s,\cdot)-v(t,\sigma,\cdot)\|_\infty+[1+|\lambda|^{1/2}]\|Dv(t,s,\cdot)-Dv(t,\sigma,\cdot)\|_\infty \\
&\quad +\|D^2v(t,s,\cdot)-D^2v(t,\sigma,\cdot)\|_\infty \\
&\leqq M\cdot\{[C_{33}\|u(s,\cdot)-u(\sigma,\cdot)\|_{C^2([0,1])}+C_{35}|s-\sigma|^\eta\|u(\sigma,\cdot)\|_{C^2([0,1])}] \\
&\quad +[1+|\lambda|^\nu][C_{34}\|u(s,\cdot)-u(\sigma,\cdot)\|_{C^1([0,1])}+C_{36}|s-\sigma|^\eta\|u(\sigma,\cdot)\|_{C^1([0,1])}\} \\
&\leqq C_{37}|s-\sigma|^\eta(1+|\lambda|^{2\nu-1})\|f\|_E
\end{aligned}\tag{3.49}$$

where

$$C_{35}=\max\left\{\sup_{x\in[0,1]}\|a(\cdot,x)\|_{C^{1,\eta}([0,T])},\ \sup_{x\in[0,1]}\|b(\cdot,x)\|_{C^{1,\eta}([0,T])},\ \sup_{x\in[0,1]}\|c(\cdot,x)\|_{C^{1,\eta}([0,T])}\right\},$$

$$C_{36}=\max\{\|\alpha_0\|_{C^{1,\eta}([0,T])},\|\alpha_1\|_{C^{1,\eta}([0,T])},\|\beta_0\|_{C^{1,\eta}([0,T])},\|\beta_1\|_{C^{1,\eta}([0,T])}\},$$

$$C_{37}=2M^2\{M(C_{33}\vee C_{34})(C_{33}+C_{34})+C_{35}+C_{36}\}.$$

By (3.49) we deduce that there exists

$$w(t,\cdot)=\lim_{s\to t}v(t,s,\cdot)=\frac{d}{dt}u(t,\cdot)=\frac{\partial}{\partial t}R(\lambda,A(t))f$$

in the C^2-norm and by (3.48) we check as $\tau\to t$

$$\begin{aligned}
&[1+|\lambda|]\|w(t,\cdot)\|_\infty+[1+|\lambda|^{1/2}]\|Dw(t,\cdot)\|_\infty+\|D^2w(t,\cdot)\|_\infty \\
&\leqq M^2(C_{33}\vee C_{34})[1+|\lambda|^{\nu-1/2}]\|f\|_E
\end{aligned}\tag{3.50}$$

which implies in particular

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) \right\|_{\mathscr{L}(E)} \leqq \frac{L(K)}{1+|\lambda|^{3/2-\nu}} \tag{3.51}$$

with $L(K) = 2M^2(C_{33} \vee C_{34})$.

To conclude the proof of (iii) we have to show that

$$t \to \frac{\partial}{\partial t} R(\lambda, A(t)) \in C^{\eta}([0,T], \mathscr{L}(E)).$$

Now observe that by (3.46) it follows that $w(t,\cdot)$ is the solution of

$$\begin{cases} [\lambda - A(t,\cdot,D)]w(t,\cdot) = \dot{A}(t,\cdot,D)u(t,\cdot), \\ \Gamma(t,D)w(t,\cdot) = -\dot{\Gamma}(t,D)u(t,\cdot), \end{cases} \tag{3.52}$$

whereas $w(t,\cdot) - w(\tau,\cdot)$ solves

$$\begin{cases} [\lambda - A(t,\cdot,D)][w(t,\cdot) - w(\tau,\cdot)] = \dot{A}(t,\cdot,D)[u(t,\cdot) - u(\tau,\cdot)] \\ + [\dot{A}(t,\cdot,D) - \dot{A}(\tau,\cdot,D)]u(\tau,\cdot) + [A(t,\cdot,D) - A(\tau,\cdot,D)]w(\tau,\cdot), \\ \Gamma(t,D)[w(t,\cdot) - w(\tau,\cdot)] = -\dot{\Gamma}(t,\cdot,D)[u(t,\cdot) - u(\tau,\cdot)] \\ -[\dot{\Gamma}(t,\cdot,D) - \dot{\Gamma}(\tau,\cdot,D)]u(\tau,\cdot) - [\Gamma(t,\cdot,D) - \Gamma(\tau,\cdot,D)]w(\tau,\cdot); \end{cases} \tag{3.53}$$

in (3.52) and (3.53) we have set for $g \in C^2([0,1])$

$$\dot{A}(t,\cdot,D)g = \frac{\partial a}{\partial t}(t,\cdot)g'' + \frac{\partial b}{\partial t}(t,\cdot)g' + \frac{\partial c}{\partial t}(t,\cdot)g,$$

$$\dot{\Gamma}(t,D)g = \left(\frac{d\alpha_0}{dt}(t)g(0) - \frac{d\beta_0}{dt}(t)g'(0), \frac{d\alpha_1}{dt}(t)g(1) + \frac{d\beta_1}{dt}g'(1) \right).$$

By (3.53), Proposition 3.1, (3.48) and (3.50) we readily obtain

$$\begin{aligned} &[1+|\lambda|]\| w(t,\cdot) - w(\tau,\cdot)\|_\infty \\ &+ [1+|\lambda|^{1/2}]\| Dw(t,\cdot) - Dw(\tau,\cdot)\|_\infty + \| D^2 w(t,\cdot) - D^2 w(\tau,\cdot)\|_\infty \\ &\leqq C_{38} |t-\tau|^{\eta}(1+|\lambda|^{2\nu-1})\| f\|_\infty \end{aligned} \tag{3.54}$$

where

$$C_{38} = M^2(4M(C_{33} \vee C_{34})(C_{33} + C_{34}) + C_{35} + 2C_{36});$$

this in particular yields the result.

By Proposition 3.2 we see that the operators $\{A(t)\}_{t\in[0,T]}$ defined in (3.7) actually satisfy Hypotheses I, II (with $\alpha = 1/2$ if $\mu = 0$, and any $\alpha \in]0,1[$ if $\mu > 0$) and III. On the other hand it is easy to see, by (3.5), (3.6) and (3.11), that the operators $\{A(t)\}_{t\in[0,T]}$ and $\{B(t,s)\}_{0\le s<t\le T}$ defined in (3.7) and (3.8) also fulfil Hypothesis IV.

In order to apply the results of Section 2 to problem (3.1), we need to characterize the spaces $D_{A(0)}$ and $D_{A(0)}(\theta,\infty)$, $\theta \in]0,1[$.

It is easy to see that for each $t \in [0,T]$

$$(3.55)\quad \overline{D_{A(t)}} = \begin{cases} C([0,1]) & \text{if } \beta_0(t) \wedge \beta_1(t) > 0, \\ \{u \in C([0,1]) : u(i) = 0\} & \text{if } \beta_i(t) = 0 < \beta_j(t), \\ & \quad \{i,j\} = \{0,1\}, \\ \{u \in C([0,1]) : u(0) = u(1) = 0\} & \text{if } \beta_0(t) = \beta_1(t) = 0. \end{cases}$$

On the other hand it is known that if $\theta \in]0,1/2[$

$$(3.56)\quad D_{A(t)}(\theta,\infty) = C^{2\theta}([0,1]) \cap \overline{D_{A(t)}},$$

whereas if $\theta \in]\frac{1}{2},1[$

$$(3.57)\quad \begin{aligned} &D_{A(t)}(\theta,\infty) \\ &= \{u \in C^{1,2\theta-1}([0,1]) : \alpha_0(t)u(0) - \beta_0(t)u'(0) = \alpha_1(t)u(1) + \beta_1(t)u'(1) = 0\}; \end{aligned}$$

finally in the special case $\theta = \frac{1}{2}$ one obtains

$$(3.58)\quad D_{A(t)}(\tfrac{1}{2},\infty) = \begin{cases} \left\{u \in C^{*,1}([0,1]) : \sup\limits_{x\in[0,1]-\{i\}} \dfrac{|u(x)-u(i)|}{|x-i|} < \infty, i = 0,1\right\} \\ \quad \text{if } \beta_0(t) \wedge \beta_1(t) > 0, \\ \left\{u \in C^{*,1}([0,1]) : u(i) = 0, \sup\limits_{x\in[0,1]-\{j\}} \dfrac{|u(x)-u(j)|}{|x-j|} < \infty\right\} \\ \quad \text{if } \beta_i(t) = 0 < \beta_j(t), \{i,j\} = \{0,1\}, \\ \{u \in C^{*,1}([0,1]) : u(0) = u(1) = 0\} \\ \quad \text{if } \beta_0(t) = \beta_1(t) = 0, \end{cases}$$

where $C^{*,1}([0,1])$ is the "Zygmund class" of functions, defined by

$$C^{*,1}([0,1]) = \left\{u \in C([0,1]) : \sup \frac{|u(x)+u(y)-2u((x+y)/2)|}{|x-y|} : x,y \in [0,1], x \neq y < \infty\right\}.$$

In the case $\beta_0(t)=\beta_1(t)=0$, the characterizations (3.56), (3.57) are proved in Da Prato–Grisvard [11] and Lunardi [24], where also a proof of (3.58) can be found; when $\beta_0(t)\wedge\beta_1(t)>0$ the proof of (3.56) and (3.57) is in Acquistapace–Terreni [3, Section 6], whereas (3.58) is proved in Acquistapace–Terreni [5]. The remaining cases ($\beta_i(t)=0<\beta_j(t)$, $\{i,j\}=\{0,1\}$) have not been proved explicitly: however the proof follows in a standard way by employing the procedure of [5] relative to the case $\theta=\frac{1}{2}$ in the one-dimensional setting described in [3, Section 6].

Let us go back now to (3.1). Assume that

$$\text{(3.59)}\qquad \begin{cases} f\in C([0,T]\times[0,1]) \quad\text{and}\quad \sup_{x\in[0,1]}|f(t,x)-f(s,x)|\leqq [f]_\sigma |t-s|^\sigma \\ \forall t,s\in[0,T] \quad (\sigma\in\,]0,\eta\wedge\beta]); \end{cases}$$

$$\text{(3.60)}\qquad \begin{cases} \psi\in C^2([0,1]) \quad\text{and} \\ \alpha_0(0)\psi(0)-\beta_0(0)\psi'(0)=\alpha_1(0)\psi(1)+\beta_1(0)\psi'(1)=0. \end{cases}$$

By Theorem 2.1, a unique strict solution of (3.1) exists if and only if condition (2.1) holds; in addition, by Theorem 2.2, such a solution belongs to $C^{1,\delta}([0,T],E)\cap C^\delta([0,T],D_{A(\cdot)})$, $\delta\in\,]0,\sigma]$, if and only if condition (2.16) holds. Thus we have only to write down the concrete meaning of conditions (2.1) and (2.16) in the present situation.

Set

$$w=\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\psi.$$

Then w is in $C^2([0,1])$ and solves

$$\text{(3.61)}\qquad \begin{cases} a(0,x)w''(x)+b(0,x)w'(x)+c(0,x)w(x) \\ \quad =\dfrac{\partial a}{\partial t}(0,x)\psi''(x)+\dfrac{\partial b}{\partial t}(0,x)\psi'(x)+\dfrac{\partial c}{\partial t}(0,x)\psi(x), \quad x\in[0,1], \\ \alpha_0(0)w(0)-\beta_0(0)w'(0)=-\alpha_0'(0)\psi(0)+\beta_0'(0)\psi'(0), \\ \alpha_1(0)w(1)+\beta_1(0)w'(1)=-\alpha_1'(0)\psi(1)-\beta_1'(0)\psi'(1). \end{cases}$$

Consequently it is easy to see that condition (2.1) is automatically true if $\beta_0(0)\wedge\beta_1(0)>0$, otherwise it becomes respectively:

$$\text{(3.62)}\qquad \begin{gathered} f(0,0)+a(0,0)\psi''(0)+\left[b(0,0)-\frac{\beta_0'(0)}{\alpha_0(0)}\right]\psi'(0)=0 \\ \text{if } \beta_0(0)=0<\beta_1(0), \end{gathered}$$

$$f(0,1)+a(0,1)\psi''(1)+\left[b(0,1)+\frac{\beta_1'(0)}{\alpha_1(0)}\right]\psi'(1)=0 \tag{3.63}$$
$$\text{if } \beta_0(0)>0=\beta_1(0);$$

finally if $\beta_0(0)=\beta_1(0)=0$ then (2.1) is equivalent to both (3.62) and (3.63).

The meaning of condition (2.16) is a little more involved: set

$$g=f(0,\cdot)+A(0)\psi=f(0,\cdot)+a(0,\cdot)\psi''+b(0,\cdot)\psi'+c(0,\cdot)\psi; \tag{3.64}$$

then we have

$$A(0)\psi+f(0,\cdot)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\psi=g-w,$$

where w is the solution of (3.61). As $w\in C^2([0,1])$, $g-w$ belongs to $C^\delta([0,1])$, $C^{*,1}([0,1])$, $C^{1,\delta}([0,1])$ as soon as g does; hence it is easy to see that condition (2.16) is equivalent to:

$$\begin{cases} g\in C^{2\delta}([0,1]) & \text{if } \beta_0(0)\wedge\beta_1(0)>0,\\ g\in C^{2\delta}([0,1]) \text{ and (3.62) holds} & \text{if } \beta_0(0)=0<\beta_1(0),\\ g\in C^{2\delta}([0,1]) \text{ and (3.63) holds} & \text{if } \beta_0(0)>0=\beta_1(0),\\ g\in C^{2\delta}([0,1]) \text{ and both (3.62) and (3.63) hold} & \text{if } \beta_0(0)=\beta_1(0)=0, \end{cases} \tag{3.65}$$

if $\delta\in\,]0,1/2[$;

$$g\in C^{*,1}([0,1]) \quad \text{and} \quad \begin{cases} \sup_{x\in]0,1]}\dfrac{|g(x)-g(0)|}{x}<\infty,\ \sup_{x\in[0,1[}\dfrac{|g(x)-g(1)|}{1-x}<\infty \\ \qquad \text{if } \beta_0(0)\wedge\beta_1(0)>0,\\ \sup_{x\in[0,1[}\dfrac{|g(x)-g(1)|}{1-x}<\infty \text{ and (3.62) holds}\\ \qquad \text{if } \beta_0(0)=0<\beta_1(0),\\ \sup_{x\in]0,1]}\dfrac{|g(x)-g(0)|}{x}<\infty \text{ and (3.63) holds}\\ \qquad \text{if } \beta_0(0)>0=\beta_1(0),\\ \text{(3.62) and (3.63) hold if } \beta_0(0)=\beta_1(0)=0, \end{cases} \tag{3.66}$$

if $\delta=\frac{1}{2}$;

$$g\in C^{1,2\delta-1}([0,1]) \quad \text{and} \begin{cases} \alpha_0(0)g(0)-\beta_0(0)g'(0)+\alpha_0'(0)\psi(0)-\beta_0'(0)\psi'(0)=0,\\ \alpha_1(0)g(1)+\beta_1(0)g'(1)+\alpha_1'(0)\psi(1)+\beta_1'(0)\psi'(1)=0, \end{cases} \tag{3.67}$$

if $\delta\in\,]\frac{1}{2},1[$.

Now we are ready to state the final result.

THEOREM 3.3. *Assume that* (3.2), (3.3), (3.4), (3.5), (3.6) *hold and let* $f(t,x)$, $\psi(x)$ *be functions satisfying* (3.59), (3.60). *Then*:

(i) *if* $\beta_0(0)\wedge\beta_1(0)>0$, *problem* (3.1) *has a unique strict solution* $u\in C^1([0,T]\times[0,1])$ *such that* $u_{xx}\in C([0,T]\times[0,1])$;

(ii) *if* $\beta_0(0)=0<\beta_1(0)$ (*resp.* $\beta_0(0)>0=\beta_1(0)$) *the conclusion of* (i) *is true if and only if* f *and* ψ *satisfy* (3.62) (*resp.* (3.63));

(iii) *if* $\beta_0(0)=\beta_1(0)=0$ *the conclusion of* (i) *is true if and only if* f *and* ψ *satisfy both* (3.62) *and* (3.63);

(iv) *if the strict solution* u *exists, then*

$$u_t(\cdot,x),u_{xx}(\cdot,x)\in\begin{cases}C^{\sigma}(]0,T])\ \textit{uniformly in } x & \textit{if } \mu>0\\[2ex] C^{\sigma\wedge(1/2)}(]0,T])\ \textit{uniformly in } x & \textit{if } \mu=0\end{cases}$$

where $\mu=\min_{t\in[0,T]}(\beta_0(t)\wedge\beta_1(t))$;

(v) *if the strict solution* u *exists, then*

$$u_t(\cdot,x),u_{xx}(\cdot,x)\in\begin{cases}C^{\sigma}([0,T])\ \textit{uniformly in } x & \textit{if } \mu>0\\[2ex] C^{\sigma\wedge(1/2)}([0,T])\ \textit{uniformly in } x & \textit{if } \mu=0\end{cases}$$

if and only if f *and* ψ *are such that*

$$\begin{cases}(3.65)\ \textit{holds with } \delta=\sigma & \textit{if } \sigma\in]0,\tfrac12[\\[1ex] (3.66)\ \textit{holds} & \textit{if } \sigma=\tfrac12\\[1ex] (3.67)\ \textit{holds with } \delta=\sigma\ (\textit{resp. } (3.66)\ \textit{holds}) & \textit{if } \sigma\in]\tfrac12,1[\ \textit{and}\\[1ex] & \mu>0\ (\textit{resp. } \mu=0),\end{cases}$$

where the function g, *appearing in* (3.65), (3.66) *and* (3.67), *is defined by* (3.64).

Second Example

Let $\Omega\subseteq\mathbf{R}^n$ be a bounded connected open set, with boundary $\partial\Omega$ of class C^3. Consider the differential operator

$$A(t,x,D)=\sum_{i,j=1}^n a_{ij}(t,x)D_{x_i}D_{x_j}+\sum_{i=1}^n b_i(t,x)D_{x_i}+c(t,x)I,\tag{3.68}$$

$$(t,x)\in[0,T]\times\bar{\Omega},$$

under the following assumptions:

$$\left\{\begin{array}{l} a_{ij}, b_i, c \in C([0,T]\times\bar{\Omega},\mathbf{C}); \qquad \dfrac{\partial a_{ij}}{\partial t}, \dfrac{\partial b_i}{\partial t}, \dfrac{\partial c}{\partial t} \in C([0,T]\times\bar{\Omega},\mathbf{C}), \\ a_{ij}(\cdot,x), b_i(\cdot,x), c(\cdot,x) \in C^{1,\eta}([0,T],\mathbf{C}) \\ \quad \text{with bounds independent of } x \quad (\eta \in]0,1[), \end{array}\right. \tag{3.69}$$

$$\operatorname{Re}\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geqq N|\xi|^2 \qquad \forall \xi \in \mathbf{R}^n, \quad \forall t \in [0,T]\times\bar{\Omega} \quad (N>0). \tag{3.70}$$

Consider also the boundary differential operator

$$\Gamma(t,x,D) = \sum_{i=1}^{n} \beta_i(t,x)D_{x_i} + \alpha(t,x)I, \qquad (t,x)\in[0,T]\times\partial\Omega \tag{3.71}$$

where it is assumed that

$$\left\{\begin{array}{l} \beta = (\beta_1,\dots,\beta_n)\in C^1([0,T]\times\partial\Omega,\mathbf{R}^n), \quad \alpha \in C^1([0,T]\times\partial\Omega,\mathbf{R}), \\ \beta(t,\cdot), \dfrac{\partial\beta}{\partial t}(t,\cdot) \in C^2(\partial\Omega,\mathbf{R}^n), \alpha(t,\cdot), \dfrac{\partial\alpha}{\partial t}(t,\cdot)\in C^2(\partial\Omega,\mathbf{R}) \\ \qquad\qquad \text{with bounds independent of } t, \\ \beta(\cdot,x)\in C^{1,\eta}([0,T],\mathbf{R}^n), \alpha(\cdot,x)\in C^{1,\eta}([0,T],\mathbf{R}) \\ \qquad\qquad \text{with bounds independent of } x, \end{array}\right. \tag{3.72}$$

$$\alpha(t,x)\geqq 0, \quad \sum_{i=1}^{n}\beta_i(t,x)\nu_i(x)\geqq\delta_0 \quad \forall (t,x)\in[0,T]\times\partial\Omega \quad (\delta_0>0) \tag{3.73}$$

where $\nu(x) = (\nu_1(x),\dots,\nu_n(x))$ is the exterior normal unit vector at $x \in \partial\Omega$. Next, define for $0 \leqq s < t \leqq T$

$$B(t,s,x,D) = \sum_{i=1}^{n} p_i(t,s,x)D_{x_i} + q(t,s,x)I, \qquad x \in \bar{\Omega}, \tag{3.74}$$

where we suppose that:

$$\left\{\begin{array}{l} p_i, q \text{ are complex-valued functions, defined in} \\ \{(t,s,x): 0\leqq s<t\leqq T,\, x\in\bar{\Omega}\}, \text{ measurable with respect} \\ \text{to } (t,s), \text{ continuously differentiable in } x, \text{ and such that:} \\ \qquad \displaystyle\sum_{i=1}^{n}|p_i(t,s,x)| + |q(t,s,x)| \leqq \frac{H}{(t-s)^{1-\beta}} \\ \forall t\in]0,T],\ \forall s\in[0,t[,\ \forall x\in\bar{\Omega}, \text{ or, alternatively,} \\ \qquad \displaystyle\int_0^t \left[\sum_{i=1}^{n}\sup_{x\in\bar{\Omega}}|p_i(t,s,x)| + \sup_{x\in\bar{\Omega}}|q(t,s,x)|\right]^{1/(1-\beta)} ds \leqq H \\ \forall t\in[0,T] \ (H>0, \beta\in]0,1[) \end{array}\right. \tag{3.75}$$

and

$$\sum_{i=1}^{n}\sup_{x\in\bar\Omega}|p_i(t,s,x)-p_i(r,s,x)|+\sup_{x\in\bar\Omega}|q(t,s,x)-q(r,s,x)|$$

$$\text{(3.76)}\qquad \leqq (t-r)^{\beta}\phi(r,s)\qquad \forall t,r\in\,]0,T],\quad \forall s\in[0,r[$$

where β is the same as in (3.75) and $\phi(r,s)$ is a function such that H_0: $\sup_{r\in]0,T]}\int_0^r\phi(r,s)ds<\infty$. We want to apply the results of Section 2 to the problem

$$\text{(3.77)}\qquad \begin{cases} u_t(t,\cdot)-A(t,\cdot,D)u-\displaystyle\int_0^t B(t,s,\cdot,D)uds=f(t,\cdot) & \text{in } \bar\Omega;\quad t\in[0,T],\\ \Gamma(t,\cdot,D)u=0 \quad \text{in } \partial\Omega;\quad t\in[0,T], \\ u(0,\cdot)=\psi \quad \text{in } \bar\Omega, \end{cases}$$

with prescribed data f,ψ.

Set $E=C(\bar\Omega)$, $\|u\|_E=\sup_{x\in\bar\Omega}|u(x)|=:\|u\|_\infty$; define for each $t\in[0,T]$

$$\text{(3.78)}\qquad \begin{cases} D_{A(t)}=\left\{u\in\displaystyle\bigcap_{1\leqq q<\infty}H^{2,q}(\Omega): A(t,\cdot,D)u\in C(\bar\Omega),\Gamma(t,\cdot,D)u\equiv 0 \text{ in } \partial\Omega\right\},\\ A(t)u=A(t,\cdot,D)u; \end{cases}$$

$$\text{(3.79)}\qquad \begin{cases} D_{B(t,s)}=\{u\in C(\bar\Omega): B(t,s,\cdot,D)u\in C(\bar\Omega)\},\\ B(t,s)u=B(t,s,\cdot,D)u. \end{cases}$$

In order to verify Hypotheses I, II, III and IV in the present case, we first consider the operators $\{A(t)\}_{t\in[0,T]}$. We have

PROPOSITION 3.4. *Suppose that* (3.68), (3.69), (3.70), (3.71), (3.72) *and* (3.73) *hold, and let* $\{A(t)\}_{t\in[0,T]}$ *be defined by* (3.78). *Then there exist* $\omega,K,M,L>0$ *such that*

(i) $\sigma(A(t))\supseteq\Sigma_{K,\omega}:=\{\lambda\in\mathbf{C}:\mathrm{Re}\,\lambda\geqq\omega\}\cup\{\lambda\in\mathbf{C}:|\mathrm{Im}\,\lambda|>K|\mathrm{Re}\,\lambda-\omega|\}$ $\forall t\in[0,T]$;

(ii) $$[1+|\lambda-\omega|]\|R(\lambda,A(t))f\|_E + [1+|\lambda-\omega|^{1/2}]\sum_{i=1}^{n}\|D_{x_i}R(\lambda,A(t))f\|_E\leqq M\|f\|_E$$
$\forall f\in E,\forall\lambda\in\Sigma_{K,\omega},\forall t\in[0,T]$;

(iii) *for each* $\lambda\in\Sigma_{K,\omega}$ *we have* $R(\lambda,A(\cdot))\in C^{1,\eta}([0,T],\mathscr{L}(E))$ *and*

$$\left\|\frac{\partial}{\partial t}R(\lambda,A(t))\right\|_{\mathscr{L}(E)}\leqq\frac{L}{1+|\lambda-\omega|}\qquad\forall\lambda\in\Sigma_{K,\omega},\quad\forall t\in[0,T].$$

PROOF. This result is due to Stewart [33] in a very general situation; a simplified proof for the present case can be found in Acquistapace–Terreni [1, Section 6]. Thus Hypotheses, I, II (with any $\alpha \in]0,1[$) and III are fulfilled by the operators $\{A(t)-\omega I\}_{t\in[0,T]}$ (and not by $\{A(t)\}_{t\in[0,T]}$). On the other hand, by (3.75), (3.76) and Proposition 3.4(ii) it is easily seen that the operators $\{B(t,s)\}_{0\leq s<t\leq T}$ and $\{A(t)-\omega I\}_{t\in[0,T]}$ also satisfy Hypothesis IV.

Let us characterize the spaces $\overline{D_{A(0)}}$, $D_{A(0)}(\theta,\infty)$ ($\theta \in]0,1[$), obviously coinciding with $\overline{D_{A(0)-\omega I}}$, $D_{A(0)-\omega I}(\theta,\infty)$. In Acquistapace–Terreni [5] it is proved that under the above assumptions

$$(3.80)\quad D_{A(t)}(\theta,\infty)=\begin{cases} C^{2\theta}(\bar\Omega) & \text{if } \theta\in]0,\tfrac12[,\\ \Big\{u\in C^{*,1}(\bar\Omega):\sup\Big\{\dfrac{|u(x)-u(x-\sigma\beta(x))|}{\sigma}: \\ \qquad x\in\partial\Omega,\sigma>0,x-\sigma\beta(x)\in\bar\Omega\Big\}<\infty\Big\} & \text{if } \theta=\tfrac12,\\ \Big\{u\in C^{1,2\theta-1}(\bar\Omega): \\ \qquad \alpha(t,\cdot)u-\sum_{i=1}^{n}\beta_i(t,\cdot)D_{x_i}u\equiv 0 \text{ in } \partial\Omega\Big\} & \text{if } \delta\in]0,\tfrac12[, \end{cases}$$

where the Zygmund class $C^{*,1}(\bar\Omega)$ is defined by

$$C^{*,1}(\bar\Omega)=\Big\{u\in C(\bar\Omega):\sup\Big\{\frac{|u(x)+u(y)-2u((x+y)/2)|}{|x-y|}:x,y,\frac{x+y}{2}\in\bar\Omega\Big\}<\infty\Big\}.$$

This in particular shows that

$$(3.81)\qquad \overline{D_{A(t)}}=C(\bar\Omega).$$

Now take the data f,ψ such that

$$(3.82)\qquad f\in C([0,T]\times\bar\Omega)\ \text{ and }\ \sup_{x\in\bar\Omega}|f(t,x)-f(s,x)|\leq[f]_\sigma|t-s|^\sigma$$
$$\forall t,s\in[0,T]\ (\sigma\in]0,\eta\wedge\beta]),$$

$$(3.83)\quad \psi\in\bigcap_{1\leq q<\infty}H^{2,q}(\Omega), A(0,\cdot,D)\psi\in C(\bar\Omega),\qquad \Gamma(0,\cdot,D)\psi\equiv 0\quad\text{in }\partial\Omega.$$

Clearly, by (3.81), condition (2.1) is always true in the present situation. About condition (2.16), set

$$(3.84)\qquad w=\Big[\frac{d}{dt}A(t)^{-1}\Big]_{t=0}A(0)\psi,\qquad g=f(0,\cdot)+A(0,\cdot,D)\psi,$$

so that

$$A(0)\psi + f(0,\cdot) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\psi = g - w.$$

The function w solves

$$A(0,\cdot,D)w = F := \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial t}(0,\cdot)D_{x_i}D_{x_j}\psi + \sum_{i=1}^{n} \frac{\partial b_i}{\partial t}(0,\cdot)D_{x_i}\psi + \frac{\partial c}{\partial t}(0,\cdot)\psi \quad \text{in } \bar\Omega,$$

$$\Gamma(0,\cdot D)w = G := -\frac{\partial \alpha}{\partial t}(0,\cdot)\psi - \sum_{i=1}^{n} \frac{\partial \beta_i}{\partial t}(0,\cdot)D_{x_i}\psi \quad \text{in } \partial\Omega,$$

and obviously $F \in \bigcap_{1\le q<\infty} L^q(\Omega)$, $G \in \bigcap_{1\le q<\infty} H^{1-1/q,q}(\partial\Omega)$. This clearly implies $w \in \bigcap_{1<q<\infty} H^{2,q}(\Omega)$, and hence $w \in \bigcap_{0<\alpha<1} C^{1,\alpha}(\bar\Omega)$. Consequently, $g - w$ belongs to $C^{\delta}(\bar\Omega)$, $C^{*,1}(\bar\Omega)$, $C^{1,\delta}(\bar\Omega)$ as soon as g does. Therefore we easily derive that condition (2.16) is equivalent to:

$$(3.85) \quad \begin{cases} g \in C^{2\delta}(\bar\Omega) \qquad \text{if } \delta \in]0,\tfrac12[, \\ g \in C^{*,1}(\bar\Omega) \text{ and } \sup\left\{\dfrac{|g(x)-g(x-\beta(x))|}{\sigma}:\right. \\ \qquad\qquad \left. x \in \partial\Omega,\ \sigma > 0,\ x - \sigma\beta(x) \in \bar\Omega\right\} < \infty \quad \text{if } \delta = \tfrac12, \\ g \in C^{1,2\delta-1}(\bar\Omega) \text{ and } \alpha(0,\cdot)g + \displaystyle\sum_{i=1}^{n} \beta_i(0,\cdot)D_{x_i}g \\ \quad + \dfrac{\partial\alpha}{\partial t}(0,\cdot)\psi + \displaystyle\sum_{i=1}^{n} \frac{\partial\beta_i}{\partial t}(0,\cdot)D_{x_i}\psi \equiv 0 \quad \text{in } \partial\Omega, \text{ if } \delta \in]\tfrac12,1[, \end{cases}$$

Hence we can conclude with the following

THEOREM 3.5. *Let $\Omega \subseteq \mathbf{R}^n$ be a bounded connected open set with boundary $\partial\Omega$ of class C^3. Assume that* (3.68), (3.69), (3.70), (3.71), (3.72), (3.73), (3.74), (3.75) *and* (3.76) *hold, and let $f(t,x)$, $\psi(x)$ be functions satisfying* (3.82), (3.83). *Then*:

(i) *problem* (3.1) *has a unique strict solution $u \in C^1([0,T]\times\bar\Omega)$ such that $u \in C([0,T],H^{2,q}(\Omega))$ $\forall q \in [1,\infty[$, and $A(\cdot,\cdot D)u \in C([0,T]\times\bar\Omega)$*;

(ii) *the strict solution u satisfies in addition $u \in C^{\sigma}(]0,T],H^{2,q}(\Omega))$ $\forall q \in [1,\infty[$ and $u_t(\cdot,x), A(\cdot,x,D)u \in C^{\sigma}(]0,T])$ uniformly in x*;

(iii) *the strict solution u satisfies $u \in C^{\sigma}([0,T],H^{2,q}(\Omega))$ $\forall q \in [1,\infty[$ and $u_t(\cdot,x), A(\cdot,x,D)u \in C^{\sigma}([0,T])$ uniformly in x, if and only if f and ψ are such that* (3.85) *holds with $\delta = \sigma$, where the function g, appearing in* (3.85), *is defined in* (3.84).

Appendix

We will use here the same notations as in Acquistapace–Terreni [1]. Under Hypotheses I, II, III consider the problem

$$\text{(A.1)} \qquad \begin{cases} u'(t) - A(t)u(t) = f(t), & t \in]0,T] \text{ (resp. } t \in [0,T]), \\ u(0) = x, \end{cases}$$

where $x \in \overline{D_{A(0)}}$, $f \in C([0,T],E) \cap C^{\sigma}(]0,T],E)$ (resp. $x \in D_{A(0)}$, $f \in C^{\sigma}([0,T],E)$ and $A(0)x + f(0) - [(d/dt)A(t)^{-1}]_{t=0}A(0)x \in \overline{D_{A(0)}}$), and $\sigma \in]0,1[$. By [1, Theorems 4.1 and 5.1] a unique classical (resp. strict) solution u of (A.1) exists and is given by

$$\text{(A.2)} \qquad u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}g(s)ds, \qquad t \in [0,T],$$

where g is the unique solution of the integral equation

$$\text{(A.3)} \qquad g(t) + \int_0^t P(t,s)g(s)ds = f(t) - P(t,0)x, \qquad t \in [0,T]$$

with $P(t,s)$ defined by (1.4).

In [1, Theorems 4.1, 5.1 and 5.3] it is shown that if $\sigma \in]0,\eta[\cap]0,\alpha]$, then $u', A(\cdot)u(\cdot) \in C^{\sigma}(]0,T],E)$ (resp. $u', A(\cdot)u(\cdot) \in C^{\sigma}([0,T],E)$ provided $A(0)x + f(0) - [(d/dt)A(t)^{-1}]_{t=0}A(0)x \in D_{A(0)}(\sigma,\infty)$). We want to prove here a slight refinement of these results, namely that the same conclusions hold if $\sigma \in]0, \eta \wedge \alpha]$.

In addition we will prove estimates (1.5) and (1.7) for $A(\cdot)u(\cdot)$ and u', which were not explicitly stated in [1]. Let us begin with some refinements of the results of [1, Section 3].

LEMMA A.1. *Set $P\phi(t) = \int_0^t P(t,s)\phi(s)ds$, $t \in [0,T]$. Under Hypotheses I, II, III we have:*

(i) *if $\phi \in L^1(0,T;E) \cap C^{\delta}(]0,T],E)$, $\delta \in]0,1[$, then $P\phi \in C^{\eta\wedge\alpha}(]0,T],E)$;*
(ii) *if $\phi \in C^{\delta}([0,T],E)$, $\delta \in]0,1[$, then $P\phi \in C^{\eta\wedge\alpha}([0,T],E)$.*

PROOF. (i) Let $t > \tau \geqq \varepsilon$. We have

$$\begin{aligned} P\phi(t) - P\phi(\tau) &= \int_\tau^t P(t,s)\phi(s)ds \\ &+ \left[\int_0^{\varepsilon/2} + \int_{\varepsilon/2}^{\tau}\right] \frac{1}{2\pi i}\int_\gamma e^{(t-s)\lambda}\left[\frac{\partial}{\partial t}R(\lambda,A(t)) - \frac{\partial}{\partial t}R(\lambda,A(\tau))\right][\phi(s)-\phi(\tau)]d\lambda ds \end{aligned}$$

$$\text{(A.4)} \quad + \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}[e^{(t-\varepsilon/2)\lambda} - e^{(t-\tau)\lambda}]\left[\frac{\partial}{\partial t}R(\lambda,A(t)) - \frac{\partial}{\partial t}R(\lambda,A(\tau))\right]\phi(\tau)d\lambda$$

$$+ \left[\int_0^{\varepsilon/2} + \int_{\varepsilon/2}^{\tau}\right]\int_{\tau-s}^{t-s}\frac{1}{2\pi i}\int_\gamma \lambda e^{\xi\lambda}\frac{\partial}{\partial\tau}R(\lambda,A(\tau))\phi(s)d\lambda d\xi ds;$$

consequently (by [1, formula (1.3) and Lemma 3.1])

$$
\begin{aligned}
\|P\phi(t)-P\phi(\tau)\|_E \leqq{} & C\|\phi\|_{C([\varepsilon,T],E)}(t-\tau)^\alpha \\
& + c(\varepsilon)[\|\phi\|_{L^1(0,\varepsilon/2;E)}+\|\phi\|_{C([\varepsilon,T],E)}][(t-\tau)^\eta+(t-\tau)] \\
& + c\|\phi\|_{C^\delta([\varepsilon/2,T],E)}\int_{\varepsilon/2}^{\tau}\left[\frac{(t-\tau)^\eta}{t-s}+\frac{t-\tau}{(t-s)^{2-\alpha}}\right](\tau-s)^\delta ds \\
& + c\|\phi\|_{C([\varepsilon,T],E)}\left[(t-\tau)^\eta+(t-\tau)\left(\frac{1}{(t-\varepsilon/2)^{1-\delta}}-\frac{1}{(t-\tau)^{1-\delta}}\right)\right] \\
& + c(\varepsilon)\|\phi\|_{L^1(0,\varepsilon/2;E)}(t-\tau)+c\|\phi\|_{C([\varepsilon/2,T],E)}\int_{\varepsilon/2}^{\tau}\int_{\tau-s}^{t-s}\frac{d\xi}{\xi^{2-\alpha}}ds \\
\leqq{} & c(\varepsilon)[(t-\tau)^\eta+(t-\tau)^\alpha].
\end{aligned}
$$

(ii) Let $t>\tau\geqq 0$. By (A.4) we get

$$
\begin{aligned}
\|P\phi(t)-P\phi(\tau)\|_E \leqq{} & c\|\phi\|_{C^\delta([0,T],E)}\left\{(t-\tau)^\alpha+\int_0^\tau\left[\frac{(t-\tau)^\eta}{t-s}+\frac{t-\tau}{(t-s)^{2-\alpha}}\right](\tau-s)^\alpha ds\right. \\
& \left.+(t-\tau)^\eta+(t-\tau)\left[\frac{1}{t^{1-\alpha}}+\frac{1}{(t-\tau)^{1-\alpha}}\right]+\int_0^\tau\int_{\tau-s}^{t-s}\frac{d\xi}{\xi^{2-\alpha}}ds\right\} \\
\leqq{} & c\|\phi\|_{C^\delta([0,T],E)}[(t-\tau)^\eta+(t-\tau)^\alpha].
\end{aligned}
\tag{A.5}
$$

Lemma. A.2. *Set $\psi=(1+P)^{-1}\phi$, i.e. $\psi+P\psi=\phi$. Under Hypotheses I, II, III we have*:

(i) *if $\phi\in L^1(0,T;E)\cap C^\delta(]0,T],E)$, $\delta\in]0,1[$, then $\psi\in C^{\delta\wedge\eta\wedge\alpha}(]0,T],E)$;*
(ii) *if $\phi\in C^\delta([0,T],E)$, $\delta\in]0,1[$, then $\psi\in C^{\delta\wedge\eta\wedge\alpha}([0,T],E)$.*

Proof. (i) It follows by [1, Proposition 3.6(ii) and 3.5(iv)], Lemma A.1(i) and the integral equation $\psi+P\psi=\phi$.

It follows by [1, Proposition 3.6(i) and 3.5(v)], Lemma A.1(ii) and the integral equation $\psi+P\psi=\phi$.

Lemma A.3. *Set $T\phi(t)=\int_0^t e^{(t-s)A(t)}\phi(s)ds$. Under Hypotheses I, II, III we have*:

(i) *if $\phi\in L^1(0,T;E)\cap C^\delta(]0,T],E)$, $\delta\in]0,1[$, then $T\phi\in C^{1,\delta\wedge\eta\wedge\alpha}(]0,T],E)$;*
(ii) *if $\phi\in C^\delta([0,T],E)$, $\delta\in]0,1[$, and moreover $\phi(0)=0$, then $T\phi\in C^{1,\delta\wedge\eta\wedge\alpha}([0,T],E)$.*

Proof. (i) By [1, Proposition 3.7(iv)] we have $T\phi\in C^1(]0,T],E)$ and for each $t\in]0,T]$

$$(A.6)\quad (T\phi)'(t)=\int_0^t A(t)e^{(t-s)A(t)}[\phi(s)-\phi(t)]ds+e^{tA(t)}\phi(t)+\int_0^t P(t,s)\phi(s)ds.$$

Let $t>\tau\geqq\varepsilon$. As in the proof of [1, Proposition 3.7(vi)] we have

$$(A.7)\quad \begin{aligned}&\left\|\int_0^t A(t)e^{(t-s)A(t)}[\phi(s)-\phi(t)]ds-\int_0^\tau A(\tau)e^{(\tau-s)A(\tau)}[\phi(s)-\phi(\tau)]ds\right\|_E\\ &\leqq c(\varepsilon)(t-\tau)^\delta,\end{aligned}$$

$$(A.8)\qquad \|e^{tA(t)}\phi(t)-e^{\tau A(\tau)}\phi(\tau)\|_E\leqq c(\varepsilon)(t-\tau)^\delta,$$

whereas by Lemma A.1(i)

$$(A.9)\qquad \left\|\int_0^t P(t,s)\phi(s)ds-\int_0^\tau P(\tau,s)\phi(s)ds\right\|_E\leqq c(\varepsilon)(t-\tau)^{\eta\wedge\alpha}.$$

By (A.6), (A.7), (A.8) and (A.9) the result follows.

(ii) By [1, Proposition 3.7(vii)], $T\phi\in C^1([0,T],E)$ and (A.5) holds for each $t\in[0,T]$; in particular $(T\phi)'(0)=0$. Now let $t>\tau\geqq 0$. As in the proof of [1, Proposition 3.7(vii)], we get

$$(A.10)\quad \begin{aligned}&\left\|\int_0^t A(t)e^{(t-s)A(t)}[\phi(s)-\phi(t)]ds-\int_0^\tau A(\tau)e^{(\tau-s)A(\tau)}[\phi(s)-\phi(\tau)]ds\right\|_E\\ &\leqq c\|\phi\|_{C^\delta([0,T],E)}(t-\tau)^\delta,\end{aligned}$$

$$(A.11)\qquad \|e^{tA(t)}\phi(t)-e^{\tau A(\tau)}\phi(\tau)\|_E\leqq c\|\phi\|_{C^\delta([0,T],E)}(t-\tau)^\delta;$$

on the other hand Lemma A.1(ii) and (A.5) yield

$$(A.12)\quad \left\|\int_0^t P(t,s)\phi(s)ds-\int_0^\tau P(\tau,s)\phi(s)ds\right\|_E\leqq c\|\phi\|_{C^\delta([0,T],E)}(t-\tau)^{\eta\wedge\alpha},$$

and the proof is complete.

We are now ready to prove our regularity results. We will always assume Hypotheses I, II and III.

THEOREM A.4. *If $x\in\overline{D_{A(0)}}$ and $f\in C([0,T],E)\cap C^\sigma(]0,T],E)$, $\sigma\in]0,1[$, then the classical solution u of* (A.1) *belongs to $C^{1,\sigma\wedge\eta\wedge\alpha}(]0,T],E)$.*

PROOF. By [1, Theorem 4.1], problem (A.1) has a unique classical solution $u(t)$, which is represented by (A.2). Now, by [1, Proposition 3.4 (i)–(vi)], $t\to e^{tA(t)}x\in C^{1,\eta}(]0,T],E)$. On the other hand, by [1, Proposition 3.6(iii)], $P(\cdot,0)x\in L^1(0,T;E)\cap C^\eta(]0,T],E)$, so that $f-P(\cdot,0)x\in L^1(0,T;E)\cap C^{\sigma\wedge\eta}(]0,T],E)$; consequently Lemma A.2(i) yields $g=(1+P)^{-1}(f-P(\cdot,0)x)\in L^1(0,T;E)\cap C^{\sigma\wedge\eta\wedge\alpha}(]0,T],E)$, and finally by Lemma A.3(i) we get $Tg\in C^{1,\sigma\wedge\eta\wedge\alpha}(]0,T],E)$.

THEOREM A.5. *If* $x \in D_{A(0)}$, $f \in C^{\sigma}([0,T],E)$, $\sigma \in]0,1[$, *and moreover*

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in \overline{D_{A(0)}},$$

then the strict solution u *of* (A.1) *belongs to* $C^{1,\sigma\wedge\eta\wedge\alpha}(]0,T],E)$.

PROOF. By [1, Theorem 5.1], problem (A.1) has a unique strict solution which is given by (A.2). The result follows by Theorem A.4 since u is, in particular, a classical solution.

THEOREM A.6. *If* $x \in D_{A(0)}$, $f \in C^{\sigma}([0,T],E)$, $\sigma \in]0,1[$, *and moreover*

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in \overline{D_{A(0)}},$$

then for each $t_1 \in]0,T]$ *the strict solution* u *of* (A.1) *satisfies for each* $\rho \in]0,\eta[$

$$\|A(\cdot)u(\cdot) - A(0)x\|_E \leqq C_1 \left\{ \frac{t_1^{\sigma\wedge\alpha\wedge\rho}}{\eta-\rho}[\|A(0)x\|_E + \|f\|_{C^{\sigma}([0,T],E)}] + \left\| A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \right\|_E \right\}$$

with C_1 *independent of* t_1.

PROOF. The function u is given by (A.2), so that

$$A(t)u(t) = \begin{cases} A(t)e^{tA(t)}x + \int_0^t A(t)e^{(t-s)A(t)}[g(s)-g(t)]ds \\ \quad + [e^{tA(t)} - 1]g(t), \quad \text{if } t \in]0,T], \\ A(0)x, \quad \text{if } t = 0. \end{cases}$$

Now by a recognition of the proof of [1, Theorem 5.1] it is not difficult to see that

$$A(t)e^{tA(t)}x = O(t^{\eta\wedge\alpha})\|A(0)x\|_E + e^{tA(0)}A(0)x$$
$$- tA(0)e^{tA(0)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \qquad \text{as } t \to 0^+,$$

$$\int_0^t A(t)e^{(t-s)A(t)}[g(s)-g(t)]ds = O(t^{\sigma\wedge\alpha\wedge\rho})\frac{1}{\eta-\rho}\{\|A(0)x\|_E + \|f\|_{C^{\sigma}([0,t_1],E)}\}$$
$$+ [tA(0)e^{tA(0)} - e^{tA(0)}(e^{tA(0)}-1)]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \qquad \text{as } t \to 0^+,$$

$$(e^{tA(t)} - 1)g(t) = O(t^{\sigma\wedge\alpha\wedge\eta})[\|A(0)x\|_E + \|f\|_{C^{\sigma}([0,t_1],E)}]$$
$$+ (e^{tA(0)}-1)f(0) + (e^{tA(0)}-1)(e^{tA(0)}-1)\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \qquad \text{as } t \to 0^+,$$

so that as $t \to 0^+$

$$A(t)u(t) - A(0)x = O(t^{\sigma\wedge\alpha\wedge\rho})\frac{1}{\eta-\rho}[\|A(0)x\|_E + \|f\|_{C^\sigma([0,t_1],E)}]$$

$$+ (e^{tA(0)} - 1)\left[A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x\right],$$

and the result follows.

Concerning maximal regularity of the strict solution, we have finally:

THEOREM A.7. *Let $x \in D_{A(0)}$ and $f \in C^\sigma([0,T],E)$, $\sigma \in]0, \eta \wedge \alpha]$; suppose in addition that*

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in \overline{D_{A(0)}},$$

and let u be the strict solution of (A.1). *Then $u \in C^{1,\sigma}([0,T],E)$ if and only if*

$$A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x \in D_{A(0)}(\sigma,\infty).$$

Moreover if this is the case, then

$$\|u'\|_{C^\sigma([0,T],E)} + \|A(\cdot)u(\cdot)\|_{C^\sigma([0,T],E)}$$

$$\leqq C_2\left\{\|A(0)x\|_E + \|f\|_{C^\sigma([0,T],E)} + \left\|A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x\right\|_{D_{A(0)}(\sigma,\infty)}\right\}.$$

PROOF. Consider the following problem:

$$\begin{cases} z'(t) - A(t)z(t) = A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x, & t \in [0,T], \\ z(0) = 0. \end{cases}$$

By [1, Theorem 5.1] it has a unique strict solution z, given by

$$\text{(A.13)} \qquad \begin{cases} z(t) = \int_0^t e^{(t-s)A(t)}h(s)ds, \\ h = (1+P)^{-1}\left(A(0)x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x\right). \end{cases}$$

Define now

$$\text{(A.14)} \qquad w(t) = u(t) - z(t) - A(t)^{-1}A(0)x,$$

we claim that $w \in C^{1,\sigma}([0,T],E)$. Indeed, w is the strict solution of

$$w'(t) - A(t)w(t) = \phi(t) := [f(t) - f(0)]$$

$$- \left[\frac{d}{dt} A(t)^{-1} - \left[\frac{d}{dt} A(t)^{-1} \right]_{t=0} \right] A(0)x, \qquad t \in [0,T],$$

$$w(0) = 0,$$

and hence $w(t) = T(1+P)^{-1}\phi$, where ϕ satisfies $\phi \in C^{\sigma}([0,T],E)$ and $\phi(0) = 0$: hence our claim is proved by Lemmata A.2(ii) and A.3(ii). Moreover by (A.10), (A.11) and (A.12) we easily get

$$\text{(A.15)} \quad \| w' \|_{C^{\sigma}([0,T],E)} + \| A(\cdot)w(\cdot) \|_{C^{\sigma}([0,T],E)} \leqq C\{[f]_{C^{\sigma}([0,T],E)} + \| A(0)x \|_E\}.$$

Thus, by (A.14) we deduce that $u \in C^{1,\sigma}([0,T],E)$ if and only if $z \in C^{1,\sigma}([0,T],E)$. Now, by (A.13) and [1, Proposition 3.7(v)], we have

$$z'(t) = \int_0^t A(t)e^{(t-s)A(t)}[(h(s) - h(0)) - (h(t) - h(0))]ds$$

$$+ e^{tA(t)}(h(t) - h(0)) + e^{tA(t)}h(0) + \int_0^t P(t,s)(h(s) - h(0))ds$$

$$+ \int_0^t P(t,s)h(0)ds = \frac{d}{dt}[T(h(t) - h(0))] + e^{tA(t)}h(0) + P(h(0)).$$

As $h \in C^{\eta \wedge \alpha}([0,T],E)$ by Lemma A.2(ii), we deduce by Lemmata A.3(ii) and A.1(ii) that $z \in C^{1,\sigma}([0,T],E)$ if and only if $t \to e^{tA(t)}h(0) \in C^{\sigma}([0,T],E)$, i.e. (by [1, Proposition 3.4(iii)]) if and only if $h(0) \in D_{A(0)}(\sigma,\infty)$; in addition, if this is the case, then

$$\text{(A.16)} \qquad \| z' \|_{C^{\sigma}([0,T],E)} + \| A(\cdot)z(\cdot) \|_{C^{\sigma}([0,T],E)} \leqq C \| h(0) \|_{D_{A(0)}(\sigma,\infty)},$$

as is easily seen by (A.10), (A.11), (A.12) and by a revision of the proof of [1, Proposition 3.4(iii)]. Hence we conclude that $u' \in C^{1,\sigma}([0,T],E)$ if and only if $h(0) \in D_{A(0)}(\sigma,\infty)$, and in this case we also get, by (A.14), (A.15) and (A.16),

$$\| u' \|_{C^{\sigma}([0,T],E)} + \| A(\cdot)u(\cdot) \|_{C^{\sigma}([0,T],E)} \leqq C_2\{\| A(0)x \|_E + \| f \|_{C^{\sigma}([0,T],E)} + \| h(0) \|_{D_{A(0)}(\sigma,\infty)}\}.$$

The proof is complete since

$$h(0) = A(0)x + f(0) - \left[\frac{d}{dt} A(t)^{-1} \right]_{t=0} A(0)x.$$

REMARK A.8. Of course, Theorem A.7 has a generalized version which holds for the strict solution of the problem with initial time $t_0 \in]0, T[$:

$$\text{(A.17)} \qquad \begin{cases} u'(t) - A(t)u(t) = f(t), & t \in [t_0, T], \\ u(t_0) = y. \end{cases}$$

As a consequence we have:

THEOREM A.9. *Let* $x \in \overline{D_{A(0)}}$, $f \in C([0, T], E) \cap C^{\sigma}(]0, T], E)$, $\sigma \in]0, \eta \wedge \alpha]$, *and let* u *be the classical solution of* (A.1). *Then for each* $t_0 \in]0, T[$ *we have*

$$A(t_0)u(t_0) + f(t_0) - \left[\frac{d}{dt} A(t)^{-1}\right]_{t=t_0} A(t_0)u(t_0) \in D_{A(0)}(\sigma, \infty).$$

PROOF. For each $t_0 \in]0, T[$, u is a strict solution of (A.17) with $y = u(t_0)$. Moreover $u \in C^{1,\sigma}([t_0, T], E)$ by Theorem A.4. Hence the result follows by the generalized version of Theorem A.7.

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