UNIVERSITA' DEGLI STUDI DI PISA
Facolta' di Scienze Matematiche Fisiche e Naturali
CORSO DI LAUREA MAGISTRALE IN MATEMATICA

Mechanism Design and its applications to real life

PISA, 31 MAGGIO 2013

Tesi di Laurea Magistrale

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Anno Accademico 2012/2013

To my family.

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Chapter 1

Introduction

“Philosophers and social scientists have long realized that it is not necessary that all citizens strive to enhance social welfare for the outcome of their joint actions to be nevertheless good for society at large. Adam Smith’s classical metaphor of the "invisible hand" suggests precisely this: how markets under ideal conditions lead to an efficient allocation of resources even when all agents are motivated by their self-interest.”

This is how Professor Weibull starts his speech to confer The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel, 2007, to Leonid Hurwicz, Eric S. Maskin, Roger B. Myerson *“for having laid the foundations of mechanism design theory”*.

“Mechanism design theory provides general methods for the analysis and development of mechanisms for resource allocation. This analysis is carried out in three steps. First, one makes a prediction of the behaviour that is expected under given rules. Here, game theory comes to use. Thereafter, one evaluates, according to the given goal, the resource allocations - such as consumption, production and environmental stress - that result. Finally, one looks for the mechanism, with due regard to its behavioural implications, that best meets the goal. The last step is the hardest. Here, the so-called revelation principle comes into use, a principle that was discovered by several researchers in the 1970s; according to this principle it suffices to look for the best possible direct mechanism that is compatible with individual incentives, a subclass of mechanisms that permits mathematical analysis. The user of a mechanism also desires that this not only can lead to the desired outcome

but in fact will lead to such an outcome. Expressed in more technical jargon, the mechanism should not permit suboptimal equilibria along with the optimal ones."

The above description is so complete and exhaustive, that we have nothing to add, but some examples. The theory of mechanism design applies to various fields of the real life, answering to question such as: how should shareholders' vote in a corporate assembly be conducted? How should an advertising campaign be organized? Which kind of auction is more useful for a given type of sale? And even, what are the optimal agreements for an engaged couple before the wedding? The theory of mechanism design studies the general structure underlying all these situations, as well as other ones.
The main mathematical tool in this theory is certainly game theory: in fact we can say that mechanism design consists in the study of formal rules for predicting how a game will be played. Of course, behind game theory there is a good amount of functional and convex analysis.
Let us describe now the content of this dissertation.
Chapter 1 deals with the preliminary definitions and results that are used throughout: in particular we explain some fundamental notions of convex analysis such as convexity, subgradient, Lipschtz regularity; some important theorems are proved, such as the extreme point theorem, Helly's compactness theorem and a regularity result linked to the Rademacher theorem.
Chapter 2 is devoted to a brief introduction to game theory, giving the notions of strategic game of complete information, strategic game of incomplete information and Nash equilibrium; moreover we prove the existence of a Nash equilibrium, by means of Kakutani's theorem.
In Chapter 3 we introduce the theory of mechanism design. We define the notions of mechanism and direct mechanism, and show how they are linked; we prove the so called revelation principle; we solve the problem of implementing efficient decision rules, by means of important theoretical results such as the revenue equivalence theorem, the payoff equivalence theorem and Rochet's theorem. In addition we introduce an important group of mechanisms, i.e. the VCG mechanisms.
Chapters 4 and 5 deal with some applications in economy: we present the problem of pricing goods, both linear and nonlinear, and we model it with suitable mechanisms.
In Chapter 6 we present another interesting application in the field of elections: we prove Arrow's theorem, and as a consequence we show the paradox

that under realistic hypotheses there are no really democratic political elections.

Chapter 2

Preliminaries

In this chapter we are going to give the preliminary results that will be used in this thesis.

2.1 Convex Analysis

This section gives us the tools of convex analysis which are going to be useful in the proofs of very important theorems regarding mechanism design.

2.1.1 Convexity, subgradient and directional derivatives

Definition 2.1.1. Let X be a vector space. A subset C of X is *convex* if for all $x, y \in C$, for all $t \in [0, 1]$, we have

$$tx + (1 - t)y \in C.$$

Definition 2.1.2. Let X be a vector space and C a convex subset of X. A function $F : C \to \mathbb{R}$ is *convex* if

$$F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y)$$

for all $t \in [0, 1]$, for all $x, y \in C$.

We will deal with the special case in which $X = \mathbb{R}^n$.

Definition 2.1.3. Let $F : C \to \mathbb{R}$ be a convex function, with $C \subset \mathbb{R}^n$. A vector $\bar{x} \in \mathbb{R}^n$ is a *subgradient* of F at $x \in C$ if for all $y \in C$ we have

$$F(y) \geq F(x) + \bar{x} \cdot (y - x).$$

We denote with $\partial F(x)$ the set of subgradients of F at x and we call *subdifferential* the set valued function

$$\partial F : x \mapsto \partial F(x).$$

A function $f : C \to \mathbb{R}^n$ is called a *selection* from ∂F if for all $x \in C$

$$f(x) \in \partial F(x).$$

Definition 2.1.4. Let $F : C \to \mathbb{R}$ be a convex function. We call *one-sided directional derivative* of F at x with respect to a vector $y \in \mathbb{R}^n$ the limit

$$\frac{\partial F^+}{\partial y}(x) = \lim_{\lambda \to 0^+} \frac{F(x + \lambda y) - F(x)}{\lambda}.$$

The next result links the concept of sub gradient with the one of directional derivative.

Theorem 2.1.1. *Let $F : C \to \mathbb{R}$ be a convex function. Let $x \in C$; if $\bar{x} \in \partial F(x)$, then for all $y \in C$ we have*

$$-\frac{\partial F^+}{\partial(-y)}(x) \leq \bar{x} \cdot y \leq \frac{\partial F^+}{\partial y}(x).$$

Proof. By definition of subgradient, we have

$$F(z) - F(x) \geq \bar{x} \cdot (z - x).$$

Consider now

$$z = x + \lambda y,$$

with $\lambda > 0$. Then we have

$$F(x + \lambda y) - F(x) \geq \bar{x} \cdot (x + \lambda y - x)$$

that is
$$\frac{F(x+\lambda y)-F(x)}{\lambda} \geq \bar{x}\cdot y.$$
Thus we have
$$\lim_{\lambda\downarrow 0}\frac{F(x+\lambda y)-F(x)}{\lambda} \geq \bar{x}\cdot y,$$
which means, by definition of directional derivative,
$$\frac{\partial F^+}{\partial y}(x) \geq \bar{x}\cdot y.$$
We can also conclude that
$$\frac{\partial F^+}{\partial(-y)}(x) \geq -\bar{x}\cdot y,$$
that is
$$-\frac{\partial F^+}{\partial(-y)}(x) \leq \bar{x}\cdot y.$$
The theorem is proved.

□

2.1.2 Lipschitzian and regular functions

We are going to deal with Lipschitzian functions, so we recall their definition.

Definition 2.1.5. Let $(X, \|\cdot\|)$ be a normed space. A function $F : X \to \mathbb{R}$ is *Lipschitzian* if there exists a constant L such that for all $x, y \in X$, we have
$$|F(x) - F(y)| \leq L\|x - y\|.$$

Definition 2.1.6. A function $F : \mathbb{R}^n \to \mathbb{R}$ is *locally Lipschitzian* if for all $x \in \mathbb{R}^n$ there exists a neighbourhood U_x of x and a constant λ such that
$$|F(y) - F(z)| \leq \lambda\|y - z\|$$
for all $y, z \in U_x$.

Definition 2.1.7. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitzian function. We call *generalized directional derivative* of F at x with respect to a vector $y \in \mathbb{R}^n$ the quantity

$$F^0(x, y) = \limsup_{z \to x, \lambda \to 0^+} \frac{F(z + \lambda y) - F(z)}{\lambda}.$$

Now we generalize the concept of subgradient to Lipschitzian functions.

Definition 2.1.8. Let F be a Lipschitzian function. A vector $\tilde{x} \in \mathbb{R}^n$ is a *generalized subgradient* of F at x if

$$\tilde{x} \cdot y \leq F^0(x, y)$$

for all $y \in \mathbb{R}^n$. We denote with $\partial F^0(x)$ the set of generalized subgradients of F at x. We call *generalized subdifferential* the set valued function

$$\partial F^0 : x \mapsto \partial F^0(x).$$

Definition 2.1.9. A Lipschitzian function $F : C \to \mathbb{R}$ is *regular* at x if for all $y \in C$, there exists the one-sided directional derivative of F at x with respect to y and it is equal to $F^0(x, y)$. F is *regular* if it is regular at x for all $x \in C$.

We shall use the following results.

Proposition 2.1.2. *Let C an open convex subset of $\mathbb{R}^n$ and $F : C \to \mathbb{R}$. If K be a compact subset of C, then*

$$F : K \to \mathbb{R}$$

is Lipschitzian.

To show it, we need a preliminary result.

Proposition 2.1.3. *Let $F : C \to \mathbb{R}$ be a convex function, bounded over $B(x_0, r) \subseteq C$. Then F is locally Lipschitzian over $B(x_0, r)$.*

Proof (of proposition 2.1.3). Let $x \in B(x_0, r)$ and let $\delta > 0$ such that $B(x, 2\delta) \subset B(x_0, r)$. We are going to prove that F is Lipschitzian over $B(x, \delta)$. Since f is bounded over $B(x_0, r)$, we have

$$N_x = \sup_{B(x, 2\delta)} |f| < \infty.$$

Thus, if $y, z \in B(x, \delta)$, putting

$$d = \|y - z\|$$

and

$$u = z + \frac{\delta}{d}(z - y),$$

we have that $u \in B(x, 2\delta)$ and

$$z = \frac{\delta}{d+\delta} y + \frac{d}{d+\delta} u.$$

Since F is convex, we obtain

$$f(z) \leq \frac{\delta}{d+\delta} f(y) + \frac{d}{d+\delta} f(u),$$

from which

$$f(z) - f(y) \leq \frac{d}{d+\delta}[f(u) - f(y)] \leq \frac{d}{\delta}|f(u) - f(y)| \leq \frac{2N_x}{\delta}\|y - z\|.$$

Exchanging y and z, we obtain

$$|f(z) - f(x)| \leq \frac{2N_x}{\delta}\|y - z\|,$$

for all $y, z \in B(x, \delta)$.

□

Proof (of proposition 2.1.2) Thanks to proposition 2.1.3, we can conclude that F is locally Lipschitzian. Thus, for all $x \in K$, there exists an open neighbourhood U_x of x and a constant λ_x such that for all $y, z \in U_x$, we have

$$|F(z) - F(y)| \leq \lambda_x \|z - y\|.$$

Since K is compact, there exist $x_1, \cdots x_n$ such that

$$K \subseteq U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n};$$

if we put

$$\lambda = \max_{i=1,\cdots,n} \lambda_{x_i},$$

we have

$$|F(z) - F(y)| \leq \lambda \|z - y\|,$$

for all $z, y \in K$.

□

Proposition 2.1.4. *Let C be a compact and convex subset of $\mathbb{R}^n$ and $F : C \to \mathbb{R}$ a convex function. Then F is regular.*

Proof. Fix $\epsilon > 0$; by definition of $F^0(x, y)$, there exists a number $\delta > 0$ such that for all $x \in B(x, \delta) - \{x\}$, for all $\lambda \in (0, \delta)$, we have

$$\frac{F(z + \lambda y) - F(z)}{\lambda} \leq F^0(x, y) + \epsilon.$$

The function F is continuous at x; thus considering the limit for $z \to x$, we have

$$\frac{F(x + \lambda y) - F(x)}{\lambda} \leq F^0(x, y) + \epsilon;$$

as $\lambda \to 0^+$, we obtain

$$\frac{\partial F^+}{\partial y}(x) \leq F^0(x, y) + \epsilon,$$

which means

$$\frac{\partial F^+}{\partial y}(x) \leq F^0(x, y).$$

By definition of F^0, we can also state that for all $m \in \mathbb{N}^+$, there exists $z_m \in B(x, \frac{1}{m})$ and $\lambda_m \in (0, \frac{1}{m})$, such that

$$F^0(x, y) - \epsilon < \frac{F(z_m + \lambda_m y) - F(z_m)}{\lambda_m}.$$

Thus, since λ_m is definitively less than λ, there exists m_ϵ such that

$$F^0(x, y) - \epsilon < \frac{F(z_m + \lambda y) - F(z_m)}{\lambda}$$

for all $m \geq m_\epsilon$. Since F is continuous, as $m \to \infty$ we obtain

$$F^0(x, y) - \epsilon \leq \frac{F(x + \lambda y) - F(x)}{\lambda}$$

for all $\lambda > 0$; thus as $\lambda \to 0^+$ we have

$$F^0(x, y) - \epsilon \leq \frac{\partial F^+}{\partial y}(x),$$

which means

$$F^0(x,y) \leq \frac{\partial F^+}{\partial y}(x).$$

Thus F is regular.

□

Proposition 2.1.5. *Let C be a compact subset of $\mathbb{R}^n$ and $F : C \to \mathbb{R}$ a continuously differentiable function. Then F is Lipschitzian and regular.*

Proof. F is Lipschitzian since, being continuously differentiable, we have

$$F(x) - F(y) = \int_0^1 \frac{d}{dt} F((1-t)x + ty)dt = \int_o^1 (\nabla F((1-t)x + ty), y - x)_{\mathbb{R}^n} dt,$$

where $(\, , \,)_{\mathbb{R}^n}$ denotes the scalar product in $\mathbb{R}^n$; thus we have

$$|F(x) - F(y)| \leq \|\nabla F\|_\infty |y - x|,$$

that is F is Lipschitzian.

Let's show regularity. If $\epsilon > 0$, by definition of $F^0(x,y)$, there exists a number $\delta > 0$ such that for all $x \in B(x,\delta) - \{x\}$, for all $\lambda \in (0,\delta)$, we have

$$\frac{F(z + \lambda y) - F(z)}{\lambda} \leq F^0(x,y) + \epsilon.$$

Since F is continuous in x, being continuously differentiable, if we consider the limit for $z \to x$, we obtain

$$\frac{F(x + \lambda y) - F(x)}{\lambda} \leq F^0(x,y) + \epsilon;$$

as $\lambda \to 0^+$, we have

$$\frac{\partial F^+}{\partial y}(x) \leq F^0(x,y) + \epsilon,$$

that is

$$\frac{\partial F^+}{\partial y}(x) \leq F^0(x,y).$$

Let's show the other inequality. By definition of F^0, for all $m \in \mathbb{N}^+$, there exists $z_m \in B(x, \frac{1}{m})$ and $\lambda_m \in (0, \frac{1}{m})$, such that

$$F^0(x,y) - \epsilon < \frac{F(z_m + \lambda_m y) - F(z_m)}{\lambda_m}.$$

Notice that

$$\begin{aligned}&\left|\frac{F(z_m + \lambda_m y) - F(z_m)}{\lambda_m} - (\nabla F(x), y)_{\mathbb{R}^n}\right| \\ &= \left|\int_0^1 [(\nabla F(z_m + t\lambda_m y), y) - (\nabla F(x), y)]\, dt\right| \to 0\end{aligned}$$

as $m \to \infty$ thanks to the dominated convergence theorem. Thus we have

$$F^0(x,y) - \epsilon \leq \frac{\partial F^+}{\partial y}(x),$$

that is

$$F^0(x,y) \leq \frac{\partial F^+}{\partial y}(x).$$

In other words, F is regular.

□

2.1.3 An application of Rademacher Theorem

Definition 2.1.10. Let $C \subset \mathbb{R}^n$, let $a, b \in C$. A *smooth path* joining a to b is a function

$$\alpha : [0,1] \to C$$

which is continuously differentiable in $(0,1)$ and such that $\alpha(0) = a$, $\alpha(1) = b$. Let $f : C \to \mathbb{R}^n$; we call *line integral* of f along α the quantity

$$\int_\alpha f \cdot r\, ds = \int_0^1 (f(\alpha(s)), \alpha'(s))_{\mathbb{R}^n} ds.$$

We now give this fundamental result.

Theorem 2.1.6. *Let $F : C \to \mathbb{R}$ be a convex function, with C an open and convex subset of $\mathbb{R}^n$. Let f be a measurable selection from ∂F. Then for every smooth path α from a to b we have*

$$\int_\alpha f \cdot r \, ds = F(b) - F(a).$$

In order to demonstrate this result, we need the following important theorem.

Theorem 2.1.7. *(Rademacher Theorem) If $A \subseteq \mathbb{R}^n$ is an open set and $f : A \to \mathbb{R}$ is locally Lipschitzian, thus f is differentiable at almost every $x \in \mathbb{R}^n$.*

Now we have all the instruments to prove theorem 2.1.6.

Proof (of theorem 2.1.6). Let α be a smooth path between a and b. Let H be the convex hull of the compact set $\alpha([0,1])$.
Since $H \subset C$ is compact and $F : H \to \mathbb{R}$ is convex by hypothesis, using proposition 2.1.2 we can say that F is Lipschitzian. Let $\Phi : [0,1] \to \mathbb{R}$ defined as

$$\Phi(r) = F(\alpha(r)).$$

We now use this lemma.

Lemma 2.1.8. *Let $\alpha \in C^1[0,1]$, let F be a convex function defined over an open set containing $H = \alpha([0,1])$. Thus the function*

$$\phi = F \circ \alpha : [0,1] \to \mathbb{R}$$

is Lipschitzian and regular.

Proof. It is obvious that ϕ is Lipschitzian. Now we prove regularity. Both F and α are Lipschitzian and regular thanks to what we have already shown. We have to prove that there exists

$$\phi'_+(r) = \lim_{\lambda \to 0^+} \frac{\phi(r+\lambda) - \phi(r)}{\lambda}$$

and

$$\phi'_+(r) = \limsup_{s \to r, \lambda \to 0^+} \frac{\phi(s+\lambda) - \phi(s)}{\lambda}.$$

We have

$$\begin{aligned}
\phi(r+h)-\phi(r) &= F(\alpha(r+h))-F(\alpha(r)) \\
&= F\left(\alpha(r)+h\frac{\alpha(r+h)-\alpha(r)}{h}\right)-F(\alpha(r)) \\
&= F\left(\alpha(r)+h\left[\frac{\alpha(r+h)-\alpha(r)}{h}-\alpha'(r)\right]+h\alpha'(r)\right)-F(\alpha(r)) \\
&= F\left(\alpha(r)+h\left[o(1)+\alpha'(r)\right]\right)-F(\alpha(r)) \\
&= F\left(\alpha(r)+h\alpha'(r)\right)-F(\alpha(r))+F\left(\alpha(r)+h\left[o(1)+\alpha'(r)\right]\right) \\
&- F\left(\alpha(r)+h\alpha'(r)\right) \\
&= h\left[\frac{\partial F^+}{\partial\alpha'(r)}(\alpha(r))+o(1)\right]+ho(1)= \\
&= h\left[\frac{\partial F^+}{\partial\alpha'(r)}(\alpha(r))+o(1)\right];
\end{aligned}$$

thus we have proved that there exists $\phi'_+(r)$ and

$$\phi'_+(r)=\frac{\partial F^+}{\partial\alpha'(r)}(\alpha(r)).$$

In a similar way, we can prove that there exists

$$\phi'_-(r)=\lim_{\lambda\to 0^-}\frac{\phi(r+\lambda)-\phi(r)}{\lambda}$$

and

$$\phi'_-(r)=-\frac{\partial F^+}{\partial(-\alpha'(r))}(\alpha(r)).$$

Moreover in each point in which $\partial F(x)=\{f(x)\}$, that is almost everywhere, there exists $\phi'(r)$ and

$$\phi'(r)=\frac{\partial F}{\partial\alpha'(r)}(\alpha(r))=(f(\alpha(r)),\alpha'(r)).$$

Now we prove regularity. We have

$$\begin{aligned}\limsup_{s\to r,\lambda\to 0^+}\frac{\phi(s+\lambda)-\phi(s)}{\lambda} &= \limsup_{s\to r,\lambda\to 0^+}\frac{F(\alpha(s+\lambda))-F(\alpha(s))}{\lambda}\\ &= \limsup_{s\to r,\lambda\to 0^+}\frac{F(\alpha(s)+\lambda\alpha'(s)+\lambda\omega_s(\lambda))-F(\alpha(s))}{\lambda}\\ &= \limsup_{s\to r,\lambda\to 0^+}\frac{F(\alpha(s)+\lambda\alpha'(r)+\lambda\eta_{r,s}(\lambda))-F(\alpha(s))}{\lambda}\\ &= \limsup_{s\to r,\lambda\to 0^+}\left[\frac{F(\alpha(s)+\lambda\alpha'(r))-F(\alpha(s))}{\lambda}\right.\\ &\qquad\left.+\frac{F(\alpha(s)+\lambda\alpha'(r)+\lambda\eta_{r,s}(\lambda))-F(\alpha(s)+\lambda\alpha'(r))}{\lambda}\right],\end{aligned}$$

where

$$\begin{aligned}\omega_s(\lambda) &= \alpha(s+\lambda)-\alpha(s)-\lambda\alpha'(s),\\ \eta_{r,s}(\lambda) &= \omega_s(\lambda)+\alpha'(s)-\alpha'(r).\end{aligned}$$

Since F is Lipschitzian, we have

$$\frac{F(\alpha(s)+\lambda\alpha'(r)+\lambda\eta_{r,s}(\lambda))-F(\alpha(s)+\lambda\alpha'(r))}{\lambda}\le K|\eta_{r,s}(\lambda)|\to 0,$$

since

$$\begin{aligned}|\eta_{r,s}(\lambda)| &\le |\alpha'(r)-\alpha'(s)|+|\omega_s(\lambda)|\\ &\le |\alpha'(r)-\alpha'(s)|+|\alpha(s+\lambda)-\alpha(s)-\lambda\alpha'(s)|\\ &\le |\alpha'(r)-\alpha'(s)|+|\lambda(\alpha'(\xi)-\alpha'(s))|\\ &\le |\alpha'(r)-\alpha'(s)|+|\lambda|\left[|\alpha'(\xi)-\alpha'(r)|-|\alpha'(s)-\alpha'(r)|\right]\to 0,\end{aligned}$$

as $\lambda\to 0$. Thus

$$\begin{aligned}\limsup_{s\to r,\lambda\to 0^+}\frac{\phi(s+\lambda)-\phi(s)}{\lambda} &= \limsup_{s\to r,\lambda\to 0^+}\frac{F(\alpha(s)+\lambda\alpha'(r))-F(\alpha(s))}{\lambda}\\ &= F^0(\alpha(r),\alpha'(r))\\ &= \frac{\partial F^+}{\partial\alpha'(r)}(\alpha(r))=\phi'_+(r).\end{aligned}$$

In other words, ϕ is regular.

□

Now we conclude the proof of the theorem. We have shown that there exists $\phi'(r)$ and it is a measurable and bounded function. Moreover, since

$$\partial F(x) = \{f(x)\}$$

at almost each x, we have

$$\begin{aligned} F(b) - F(a) = \phi(1) - \phi(0) &= \int_0^1 \phi'(r)dr \\ &= \int_0^1 \frac{\partial F}{\partial \alpha'(r)}(\alpha(r))dr = \int_0^1 (f(\alpha(r)), \alpha'(r))dr \\ &= \int_\alpha f \cdot rds. \end{aligned}$$

The theorem is thus proved.

□

2.1.4 Extreme points and the Extreme Point Theorem

We conclude this paragraph introducing the notion of extreme point.

Definition 2.1.11. Let C be a convex subset of a vector space X. An element $c \in C$ is an *extreme point* of C if for every $x \in X - \{0\}$, either $c + x \notin C$ or $c - x \notin C$ or both.

Now we give a theorem regarding the existence of extreme points over a compact subset.

Theorem 2.1.9. *For any nonempty compact subset C of a normed linear space X, there exists an extreme point in C.*

We need this definition.

Definition 2.1.12. A nonempty and closed subset S of C is an *extremal subset* of C if for all $x, y \in S$ such that

$$\lambda x + (1 - \lambda)y \in C$$

for some $\lambda \in (0, 1)$, thus $x, y \in C$.

Proof (of 2.1.9). Let $\mathcal{A}$ be the class of extremal subset of C. Since $S \in \mathcal{A}$, we have $\mathcal{A} \neq \emptyset$. Moreover $(\mathcal{A}, \subseteq)$ is obviously a poset. Let $(\mathcal{B}, \subseteq)$ be a loset such that $\mathcal{B} \subseteq \mathcal{A}$. It is easy to see that

$$\cap\mathcal{B} \in \mathcal{A}.$$

Then, by Zorn's Lemma, $(\mathcal{A}, \subseteq)$ must possess a maximal element: there exists a set $A \in \mathcal{A}$ such that $B \subset A$ does not hold for any $B \in \mathcal{A}$.
Now we want to show that A is singleton. Suppose that there exists two distinct points $x, y \in A$. By Hanh Banach Theorem, there exists a nonzero $L \in X^*$ such that $L(x) \neq L(y)$. Now we define the set

$$B = \{z \in A : L(z) = \max L\}.$$

Since A is compact (being a closed set in a compact one) and L is continuous, we have $B \neq \emptyset$; moreover, since L is linear, B is also convex; B is also closed, by continuity of L. Thus $B \in \mathcal{A}$ and $B \subseteq A$; thus, by definition of A, we have $A = B$. This means thet L is constant on A, which is impossible since $L(x) \neq L(y)$.
Thus A is singleton and the unique element of A must be an extreme point of S.

□

It will be useful the following theorem.

Theorem 2.1.10. *(Extreme Point Theorem) Let C be a convex and compact subset of a normed vector space and let $F : C \to \mathbb{R}$ be a continuous linear function. Then there exists an extreme point c such that*

$$F(c) \geq F(x)$$

for all $x \in C$.

Proof.

Consider the set

$$A = \{y \in C : F(y) = \max F\};$$

by Weierstrass' theorem, the set A is not empty; moreover, by continuity of F, A is a closed set. Since C is compact, A is compact too. By theorem 2.1.9, there exists an extreme point x of A. We want to show that x is an extreme point os C too. Indeed, if this was not the case, we could find $\lambda \in (0,1)$ and two distinct points y and z such that

$$x = \lambda y + (1-\lambda)z.$$

By linearity of L, we have

$$\lambda L(y) + (1-\lambda)L(z) = \max L,$$

so that

$$L(y) = L(z) = \max L.$$

Thus $y, z \in A$, contradicting that x is an extreme point of A.

□

2.2 Helly's Compactness Theorem

Studying the applications of mechanism design, we will need to know the topological properties of the set:

$$\{F : \mathbb{R} \to \mathbb{R} : F \text{ is increasing and } |F(x)| \leq c \text{ for all } x \in \mathbb{R}\}.$$

Thus, we give the following theorem.

Theorem 2.2.1. *(Helly's Compactness Theorem) If $\{f_n\}$ is a uniformly bounded sequence of increasing functions $f_n : \mathbb{R} \to \mathbb{R}$, there exists a subsequence which converges pointwise over $\mathbb{R}$.*

Proof. If we fix a real number $x \in \mathbb{R}$, thanks to Bolzano Weierstrass Theorem, we can conclude that there exists a subsequence of $\{f_n\}$ converging in x. Repeating this remark and using a diagonal procedure, we can prove that there exists a subsequence (f_{n_k}) of (f_n) such that there exists the limit

$$\lim_{k\to\infty} f_{n_k}(q) = F(q)$$

for all $q \in \mathbb{Q}$. The function F defined over $\mathbb{Q}$ is obviously increasing. For all $s \in \mathbb{R}$, we define the functions

$$F^+(s) = \lim_{q \to s^+, q \in \mathbb{Q}} F(q)$$

and

$$F^-(s) = \lim_{q \to s^-, q \in \mathbb{Q}} F(q).$$

Notice that, thanks to monotonicity, we have

$$F^+(s) \geq F^-(s);$$

moreover F^+ is right continuous, F^- is left continuous: let's show these facts. Firstly, we want to prove that

$$F^+(s) = \lim_{r \to s^+} F^+(r)$$

for all $s \in \mathbb{R}$. By definition we have

$$F^+(s) \leq F(q)$$

for all $q \geq r > s, q \in \mathbb{Q}$; thus

$$F^+(s) \leq F^+(r) = \lim_{q \to r^+, q \in \mathbb{Q}} F(q),$$

which means

$$F^+(s) \leq \lim_{r \to s^+} F^+(r)$$

By definition of F^+, for all $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$s < q < s + \delta \Longrightarrow F^+(s) \leq F(q) < F^+(s) + \epsilon.$$

If $s < r < q < s + \delta$, we have

$$F(q) < F^+(s) + \epsilon,$$

thus

$$F^+(r) < F^+(s) + \epsilon.$$

We can conclude that

$$\lim_{r \to s^+} F^+(r) \leq F^+(s) + \epsilon,$$

which means

$$\lim_{r\to s^+} F^+(r) \leq F^+(s).$$

We have proved the right continuity of F^+.

Left continuity of F^- can be shown in the same way. It is sufficient to show that

$$F^-(s) = \lim_{r\to s^-} F^+(r).$$

We have

$$F^-(s) \geq F(q)$$

for all $q \leq r < s, q \in \mathbb{Q}$; thus

$$F^-(s) \geq F^-(r),$$

for all $r < s$, that is

$$F^-(s) \geq \lim_{r\to s^-} F^-(r).$$

As before, for all $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$s - \delta < q < s \Longrightarrow F^-(s) \geq F(q) > F^-(s) - \epsilon.$$

If $s - \delta < q < r < s$, we have

$$F(q) > F^-(s) - \epsilon,$$

that is

$$F^-(r) > F^-(s) - \epsilon.$$

We can conclude that

$$\lim_{r\to s^-} F^-(r) \geq F^-(s) - \epsilon,$$

that is

$$\lim_{r\to s^-} F^-(r) \geq F^-(s).$$

Left continuity is also proved.

Notice that for all $s \in \mathbb{R}$, for all $q \in \mathbb{Q}$, we have

$$\limsup_{k\to\infty} f_{n_k}(s) \leq \lim_{k\to\infty} f_{n_k}(q) = F(q) \text{ if } s < q,$$

and

$$\liminf_{k\to\infty} f_{n_k}(s) \geq \lim_{k\to\infty} f_{n_k}(q) = F(q) \text{ if } s > q.$$

Thus, as $q \to s^-$ and $q \to s^+$

$$F^-(s) \leq \liminf_{k\to\infty} f_{n_k}(s) \leq \limsup_{k\to\infty} f_{n_k}(s) \leq F^+(s).$$

The subsequence $\{f_{n_k}\}$ is thus convergent everywhere except at the points in which $F^+(s) \neq F^-(s)$, which are at most countably many. It is now sufficient to find a suitable further subsequence to obtain the convergence everywhere on $\mathbb{R}$.

$\square$

Notice that the theorem is still true if we consider monotone functions on arbitrary intervals, either closed or open: to show it, we can repeat the same proof with trivial modifications.

Chapter 3

Introduction to game theory

Game theory is the mathematical instrument on which the mechanism design is based. Thus, we are going to get a glimpse of this fascinating theory.

3.1 Strategic Games of Complete Information

Let's start speaking about strategic games, modelling interactions between different players who make their choice once for all and simultaneously.

Definition 3.1.1. A *strategic game of complete information* is defined by:

- a finite set $I = \{1, \dots, N\}$;
- for all $i \in I$, a non empty set A_i;
- for all $i \in I$, a function $u_i : A_1 \times \cdots \times A_N \to \mathbb{R}$.

Let's explain this definition: I is the set of players; A_i is the set of possible actions for the player i and it is called the set of *pure strategies*; u_i is the *payoff function* and represents the amount the player i receives when each player's strategy is established. Each player's objective is to maximize his own payoff function, and in doing that he could help or hurt the other players. The idea behind strategic games is modelling situations in which the players are asked to make their choice at the same time, without the possibility of changing it during the game; moreover they know the structure of

the strategic form and the other players' payoffs.
We denote with A the product $A_1 \times \cdots \times A_N$, with A_{-i} the product without the i-th set A_i; if $s = (s_1, \cdots, s_N) \in A$, we write s_{-i} for $(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots s_N)$. If $a_i \in A_i$ and $s_{-i} \in A_{-i}$, we set $(a_i, s_{-i}) = (s_1, \cdots, s_{i-1}, a_i, s_{i+1}, \cdots, s_N)$.

We are going to focus on the case in which the sets of pure strategies are finite: we call such a game *finite*.

The concept of utility function induces over the set $A = A_1 \times \cdots \times A_N$ a preference relation, that is for all i a binary relation $\geq_i$ on A which is complete (for all $a, b \in A$, $a \geq_i b$ or $b \geq_i a$), reflexive (for all $a \in A$, $a \geq_i a$) and transitive (if $a \geq_i b$ and $b \geq_i c$, then $a \geq_i c$). Indeed it is sufficient to write, for all $a, b \in A$,

$$a \geq_i b \text{ if and only if } u_i(a) \geq u_i(b).$$

The key concept in game theory is that of Nash equilibrium, representing a steady state of the game in which no player has anything to gain by changing only his own strategy unilaterally. More formally, we give the following:

Definition 3.1.2. A *Nash equilibrium* of a strategic game is an element $\hat{a} \in A$ such that for all $i \in I$

$$(\hat{a}_{-i}, \hat{a}_i) \geq_i (\hat{a}_{-i}, a_i)$$

for all $a_i \in A_i$.

In other words, no player can profitably deviate if other players' actions are established.

Let's give an example of a strategic game and its Nash equilibrium.

Example 3.1.1. (*The Prisoner's Dilemma*) Two suspects in a crime are put into separate cells. If neither confesses, they both will spend one year in prison. If they both confess, they both will spend three years in prison. If only one of them confesses, he will be free while the other suspect will spend four years in prison. Notice that whatever a suspect does, the other prefers confessing to not confessing. So the game has a unique Nash equilibrium, that's to say (Confess, Confess).

Not every strategic game has a Nash equilibrium and the conditions of existence have been investigated extensively. We now recall the most important result. To do it, we need a preliminary result:

Lemma 3.1.2. (Kakutani's fixed point theorem) *Let K be a compact, convex and non empty subset of $\mathbb{R}^n$ and let $f : K \to \mathcal{P}(K)$ such that*

- *for all $x \in K$, $f(x)$ is non empty and convex;*
- *for all sequences $\{x_n\}, \{y_n\} \subset K$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$ and $y_n \to y$, we have $y \in f(x)$ (we say that the graph of f is closed).*

Then there exists a fixed point x^* for f, that is a point $x^* \in K$ such that $x^* \in f(x^*)$.

We also need two definitions:

Definition 3.1.3. A binary relation $\geq$ on a set $A \subset \mathbb{R}^n$ is continuous if for all sequences (a_k) and (b_k) converging respectively to a and b and satisfying $a_k \geq b_k$ for all k, we have $a \geq b$.

Definition 3.1.4. A binary relation $\geq_i$ is quasi-concave on A_i if for all $a^* \in A$, the set

$$\{a_i \in A_i : (a^*_{-i}, a_i) \geq_i a^*\}$$

is convex.

We now have the tools to prove the existence of a Nash equilibrium.

Theorem 3.1.3. *Suppose that a strategic game $(I, (A_i), (u_i))$ satisfies the following conditions:*

- *for all $i \in I$, the set A_i is a compact, convex and nonempty subset of $\mathbb{R}^n$;*
- *for all $i \in I$, the preference relation $\geq_i$ induced by u_i is continuous;*
- *for all $i \in I$, $\geq_i$ is quasi-concave on A_i.*

Then there exists a Nash equilibrium for the game.

Proof. Define

$$f : A \to \mathcal{P}(A)$$

by

$$f(a) = B_1(a_{-1}) \times \cdots \times B_N(a_{-N})$$

where

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \geq_i (a_{-i}, a_i') \text{ for all } a_i' \in A_i\}$$

Notice that a Nash equilibrium is a profile a^* such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for all } i \in I;$$

so, it is sufficient to find a fixed point for f.

For all $i \in I$, the set $B_i(a_{-i})$ is non empty since $\geq_i$ is continuous and A_i is compact; it is also convex since $\geq_i$ is quasi-concave on A_i. Moreover, f has a closed graph since each $\geq_i$ is continuous. So the hypothesis of Kakutani's theorem are satisfied and we can conclude that a Nash equilibrium exists.

□

3.2 Strategic Games of Incomplete Information

One frequently deals with situations in which some players do not have perfect information about other players. We call "strategic game of incomplete information" a strategic game in which some players do not know the other players' payoffs. More formally we give the following definition:

Definition 3.2.1. A *strategic game of incomplete information* or *Bayesian game* is defined by:

- a finite set $I = \{1, \ldots, N\}$;
- for all $i \in I$, a non empty set A_i;

- for all $i \in I$, a nonempty set Θ_i;
- a probability measure p over $\Theta = \Theta_1 \times \cdots \times \Theta_N$;
- for all $i \in I$, a function $u_i : A_1 \times \cdots \times A_N \times \Theta_1 \times \cdots \times \Theta_N \to \mathbb{R}$.

Let's explain this definition. As before, I represents the set of players and A_i the set of possible actions for the player i. The fact that players do not know the others players' payoffs is modelled by introducing for all i the type space Θ_i which represents player i's private information; the probability distribution p represents the prior belief regarding each player's type.

We can define the concept of Nash equilibrium for a Bayesian game.

Definition 3.2.2. Let $(I, (A_i), (\Theta_i), (u_i), p)$ a Bayesian game. Consider the strategic game of complete information defined as follows:

- $I = \{1, \cdots, N\}$;
- $A_i^\Theta = \{f : \Theta_i \to A_i\}$;
- $u_i^\Theta : A_1^\Theta \times \cdots \times A_N^\Theta \to \mathbb{R}$ defined as

$$u_i^\Theta(f_1, \cdots, f_N) = \sup_{\theta \in \Theta} u_i(f_1(\theta_1) \cdots f_N(\theta_N), \theta_1, \cdots \theta_N).$$

We say that $(f_1, \cdots, f_N) \in A_1^\Theta \times \cdots \times A_N^\Theta$ is a Nash equilibrium for the Bayesian game if it is a Nash equilibrium for the strategic game of complete information $(I, (A_i^\theta), (u_i^\theta))$.

Example 3.2.1. (*First price auctions as a Bayesian Game*) Consider a first price auction with N bidders. We can model this situation with a Bayesian game defined in the following way:

- $I = \{1, 2, \ldots, N\}$ is the set of players, that's to say the bidders;
- for all $i \in I$, $A_i = \mathbb{R}^+$ is the set of possible actions for the bidder i, that's to say the set of possible biddings;
- for all $i \in I$, the type set Θ_i is the interval $[v_{min}, v_{max}]$, that is the set of possible values for each bidder. We suppose that $v_{min} > 0$;

- $\mathbb{P}$ is the uniform probability over Θ;
- for all $i \in I$, a payoff function defined as follows:

$$u_i(a,v) = \begin{cases} \frac{v_i - P(a)}{m} & \text{if } a_j \leq a_i \text{ for all } j \neq i \text{ and } |\{j : a_j = a_i\}| = m \\ 0 & \text{otherwise} \end{cases},$$

  where $P(a)$ is the transfer price paid by the winner if the bid profile is a.

Let's analyse the Bayesian equilibrium of a first price auction. Let v_1 be the first player type and suppose the other bidders' values are independent; moreover suppose that the first player believes that the other bidders use strategies such as

$$\beta(v_i) = av_i,$$

for a fixed value a. If b is the price paid by first player if he wins, his expected payoff is

$$\begin{aligned}(v_1 - b)\mathbb{P}\{\text{ he wins }\} &= (v_1 - b)\mathbb{P}\{b > av_2, \ldots, b > av_N\} \\ &= (v_1 - b)\mathbb{P}\{b > av_2\}\mathbb{P}\{b > av_3\} \ldots \mathbb{P}\{b > av_N\} \\ &= (v_1 - b)\left(\frac{\frac{b}{a} - v_{min}}{v_{max - v_{min}}}\right)^{n-1}.\end{aligned}$$

Maximizing implies first derivative equal to zero, so we obtain the relation

$$-\left(\frac{\frac{b}{a} - v_{min}}{v_{max - v_{min}}}\right)^{n-1} + (n-1)\frac{v_1 - b}{a}\left(\frac{\frac{b}{a} - v_{min}}{v_{max - v_{min}}}\right)^{n-2} = 0$$

from which we obtain

$$b = \frac{(n-1)v_1}{1 + (n-1)(v_{max} - v_{min})},$$

which represents the Bayesian equilibrium of the game.

Chapter 4

Bayesian Mechanism Design: general theory

We may define mechanism design as the art of designing the rules of a game to achieve a specific outcome. As a matter of fact, mechanism design deals with a class of Bayesian games in which there is a player, called the principle, who chooses the payoff structure pursuing his own aim. The other players are called the agents.

4.1 Mechanisms and Direct Mechanisms

Definition 4.1.1. A *mechanism* is defined by:

- a finite set $I = \{1, \cdots, N\}$;
- for all $i \in I$, a set Θ_i;
- for all $i \in I$, a utility function

$$U_i : A \times \Theta \to \mathbb{R},$$

where $\Theta = \Theta_1 \times \cdots \times \Theta_N$, $A = A_1 \times \cdots \times A_N$ and $A_i = \{\sigma_i : \Theta_i \to \Theta_i\}$.

Let's explain this definition. We can think of I as the set of agents and of Θ_i as the set of agent i's private information, called the type. We denote by θ the type vector $(\theta_1, \cdots, \theta_N)$. The agents are asked to report their types and they can tell the truth or a lie: the set A_i represents the agent i's strategies, that's to say $\sigma_i(\theta_i)$ is the type that agent i declares if his real type is θ_i. Finally, the function $U_i(\sigma, \theta)$ represents the player i's outcome if θ is the real type vector and the agents choose respectively the strategies $(\sigma_1, \cdots, \sigma_N) = \sigma$. We can think of U_i as a function with this structure:

$$U_i(\sigma, \theta) = u_i(\sigma, \theta) - t_i$$

where u_i is agent i's profit and t_i is the transfer that he has to pay to attend the game.
We denote by θ_{-i} the vector type θ without the i-th element and Θ_{-i} the Cartesian product of the sets Θ_j leaving out Θ_i.

It is immediate by definition the following property.

Proposition 4.1.1. *If we put a probability measure over the set* $\Theta_1 \times \cdots \times \Theta_N$*, a mechanism defines a Bayesian Game.*

An important class of mechanisms are the direct mechanisms.

Definition 4.1.2. A *direct mechanism* is defined by

- a finite set $I = \{1, \cdots, N\}$;
- for all $i \in I$, a set Θ_i;
- a set X;
- a function $q : \Theta \to X$, called the decision rule;
- for all i, a function $t_i : \Theta \to \mathbb{R}$, called the payment rule;
- for all i, a function $u_i : X \times \Theta_i \to \mathbb{R}$.

Let's explain this definition: X is the set of possible outcomes; if the agents declare the type vector θ, $q(\theta)$ is the collectively chosen outcome and $t_i(\theta)$ is the transfer that agent i has to pay.

Now we are going to show the connection between mechanism and direct mechanism.

Proposition 4.1.2. *A direct mechanism defines a mechanism.*

Proof. The set of agents, the sets of types and the sets of strategies are the same. Then it is sufficient to define for all $i \in I$ the utility function U_i : we write

$$U_i(\sigma, \theta) = u_i(q(\sigma(\theta)), \theta_i) - t_i(\sigma(\theta)),$$

where $\sigma(\theta) = (\sigma_1(\theta_1), \cdots, \sigma_N(\theta_N))$.

□

Given a mechanism, we can always find a direct mechanism with equivalent payoff in which telling the truth represents an equilibrium of the game. More formally:

Theorem 4.1.3. *(Revelation Principle) For any mechanism Γ and any Nash equilibrium σ of it, there exists a direct mechanism Γ' and a Nash equilibrium σ' of it such that:*

1. *for all i, $\sigma'_i(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$;*
2. *for all i and for all $\theta \in \Theta$*

$$U_i(\sigma, \theta) = U'_i(\sigma', \theta)$$

where U_i, U'_i are the utility functions of Γ and Γ'.

Proof. Build Γ' as required by part (ii) of the proposition. We have to show that truth telling will be a Nash equilibriun in this direct mechanism. To see this, suppose it were not. Suppose that type θ_i prefers to report that her type is θ'_i for some type vector of the other agents θ_{-i}; then the same type θ_i would have preferred to deviate from σ_i, and to play the strategy that σ_i prescribes for θ'_i in Γ, for the strategy combination that the types θ_{-i} play in Γ'. Hence σ would not be a Nash equilibrium of Γ, which is impossible.

□

The revelation principle is very important since it establishes that we can always restrict our attention to direct mechanisms in which the agents report their types truthfully. We use a specific name for such a mechanism:

Definition 4.1.3. A direct mechanism is *dominant incentive compatible* if for all $\theta \in \Theta$, for all $i \in I$ and for all $\theta'_i \in \Theta_i$, we have

$$u_i(q(\theta), \theta_i) - t_i(\theta) \geq u_i(q(\theta'_i, \theta_{-i}), \theta_i) - t_i(\theta'_i, \theta_{-i}).$$

4.2 Implementing efficient decision rules

In this section we suppose to have a direct mechanism such that for all i there is a payoff function u_i defined on the Cartesian product $X \times \Theta_i$. Once we have fixed a decision rule q, our first aim is to find the set of transfer rules implementing q. More formally, we say:

Definition 4.2.1. Suppose we have set a decision rule q. We say that the payment rules $(t_1, \cdots, t_N)$ *implements* q if $(q, t_1, \cdots, t_N, u_1, \cdots, u_N)$ is a dominant incentive compatible mechanism. If there exist payment rules implementing q, we say that q is *implementable.*

Now we are going to deal with the problem of uniqueness of payment rules implementing a fixed decision rule. This result is due to Krishna and Maenner (2001).

Theorem 4.2.1. *(Revenue Equivalence Theorem) Suppose that for all i the set Θ_i is a convex subset of a finite dimensional Euclidean space, and that for all $\theta_{-i} \in \Theta_{-i}$, the function*

$$\theta_i \mapsto u_i(q(\theta_i, \theta_{-i}), \theta_i)$$

is convex. Let q be a decision rule. Assume that $(t_1, \cdots, t_N)$ implements q. Then any other payment rule $(t'_1, \cdots, t'_N)$ implements q if and only if for all i there exists a function

$$\tau_i : \Theta_{-i} \to \mathbb{R}$$

such that

$$t'_i(\theta) = t_i(\theta) + \tau_i(\theta_{-i})$$

for all $\theta \in \Theta$.

The proof of this theorem is based on another important result.

Theorem 4.2.2. *(Payoff Equivalence Theorem) Suppose that for all i, Θ_i is convex subset of $\mathbb{R}^n$ and for all $\theta_{-i} \in \Theta_{-i}$ the function*

$$\theta_i \mapsto u_i(q(\theta_i, \theta_{-i}), \theta_i)$$

is convex. Assume that the direct mechanism $(q, t_1, \cdots, t_N)$ is dominant incentive compatible. Then for all $\theta_i^1, \theta_i^2 \in \Theta_i$, for all smooth path α from θ_i^1 to θ_i^2, we have

$$u_i(q(\theta_i^1, \theta_{-i}), \theta_i^1) - t_i(\theta_i^1, \theta_{-i}) = u_i(q(\theta_i^2, \theta_{-i}), \theta_i^2) - t_i(\theta_i^2, \theta_{-i}) + \int_\alpha Q_i \cdot r ds,$$

where Q_i is any selection from $\partial u_i(q(\cdot, \theta_{-i}), \cdot)$.

Proof (of Theorem 4.2.2). Let $\theta_{-i} \in \Theta_{-i}$ and let $P_{\theta_{-i}}$ be a measurable selection from $\partial u_i(q(\cdot, \theta_{-i}), \cdot)$. By definition, we have

$$u_i(q(\theta_i, \theta_{-i}), \theta_i) \geq u_i(q(\theta_i', \theta_{-i}), \theta_i') + P_{\theta_{-i}}(\theta_i')(\theta_i - \theta_i')$$

for all $\theta_i, \theta_i' \in \Theta_i$. Since $(q, t_1, \cdots, t_N)$ is dominant incentive compatible, we can write

$$u_i(q(\theta), \theta_i) - t_i(\theta) = \sup_{\theta_i' \in \Theta_i} \left(u_i(q(\theta_i', \theta_{-i}), \theta_i) - t_i(\theta_i', \theta_{-i}) \right).$$

For all θ_i' and θ_{-i}, the function

$$\theta_i \mapsto u_i(q(\theta_i', \theta_{-i}), \theta_i) - t_i(\theta_i', \theta_{-i})$$

is convex since it is the difference between $u_i(q(\cdot, \theta_{-i}), \cdot)$, convex by hypothesis, and a constant. Since the upper bound of a set of convex function is convex, we can conclude that

$$U_{\theta_{-i}} : \theta_i \mapsto u_i(q(\theta), \theta_i) - t_i(\theta)$$

is convex for all θ_{-i}. In addition, for all $\theta_i, \theta_i' \in \Theta_i$,

$$\begin{aligned} U_{\theta_{-i}}(\theta_i) &\geq u_i(q(\theta_i', \theta_{-i}), \theta_i) - t_i(\theta_i', \theta_{-i}) \\ &\geq u_i(q(\theta_i', \theta_{-i}), \theta_i') + P_{\theta_{-i}}(\theta_i')(\theta_i - \theta_i') - t_i(\theta_i', \theta_{-i}) \\ &= U_{\theta_{-i}}(\theta_i') + P_{\theta_{-i}}(\theta_i')(\theta_i - \theta_i'), \end{aligned}$$

for incentive compatibility and the definition of $P_{\theta_{-i}}$; in other words, $P_{\theta_{-i}}$ is a selection from $\partial U_{\theta_{-i}}$.

Now it is sufficient to apply theorem 2.1.6 to obtain the thesis.

□

Proof. (of Theorem 4.2.1) Let's observe that the implication ($\Leftarrow$) is quite immediate. Since $(q, t_1, \cdots t_n)$ is dominant incentive compatible, we have

$$u_i(q(\theta), \theta_i) - t_i(\theta) \geq u_i(q(\theta'_i, \theta_{-i}), \theta_i) - t_i(\theta'_i, \theta_{-i}),$$

for all $\theta \in \Theta$, $i \in I$ and θ'_i. Suppose now that for all i we have

$$t'_i(\theta) = t_i(\theta) + \tau_i(\theta_{-i});$$

then

$$\begin{aligned} u_i(q(\theta), \theta_i) - t'_i(\theta) =& u_i(q(\theta), \theta_i) - t_i(\theta) - \tau_i(\theta_{-i}) \\ &\geq u_i(q(\theta'_i, \theta_{-i}), \theta_i) - t_i(\theta'_i, \theta_{-i}) - \tau_i(\theta_{-i}) \\ &= u_i(q(\theta'_i, \theta_{-i}), \theta_i) - t'_i(\theta'_i, \theta_{-i}), \end{aligned}$$

for all $\theta \in \Theta$, $i \in I$ and θ'_i, i.e. $(q, t'_1, \cdots, t'_n)$ is dominant incentive compatible.

Let's prove ($\Rightarrow$). Let θ_i be a fixed element of Θ_i and let θ_{-i} be an element of Θ_{-i}. Choose an element $\theta'_i \in \Theta_i$: we can join θ_i and θ'_i with a smooth path α, since Θ_i is convex. Both $(q, t_1, \cdots, t_n)$ and $(q, t'_1, \cdots, t'_n)$ are dominant incentive compatible direct mechanism, so, using 4.2.2, we can write

$$u_i(q(\theta_i, \theta_{-i}), \theta_{-i}) - t_i(\theta_i, \theta_{-i}) = u_i(q(\theta'_i, \theta_{-i}), \theta'_{-i}) - t_i(\theta'_i, \theta_{-i}) + \int_\alpha P_{\theta_{-i}} \cdot r ds,$$

$$u_i(q(\theta_i, \theta_{-i}), \theta_{-i}) - t'_i(\theta_i, \theta_{-i}) = u_i(q(\theta'_i, \theta_{-i}), \theta'_{-i}) - t'_i(\theta'_i, \theta_{-i}) + \int_\alpha P_{\theta_{-i}} \cdot r ds.$$

Subtracting the second equation from the first one, we obtain

$$t'_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i}) = t'_i(\theta'_i, \theta_{-i}) - t_i(\theta'_i, \theta_{-i}).$$

If we define

$$\tau_i(\theta_{-i}) = t'_i(\theta'_i, \theta_{-i}) - t_i(\theta'_i, \theta_{-i}),$$

we have

$$t'_i(\theta_i, \theta_{-i}) = t_i(\theta_i, \theta_{-i}) + \tau_i(\theta_{-i}),$$

which is the thesis.

□

Our investigation of the possibility of implementing a decision rule is focussed on efficient decision rules.

Definition 4.2.2. A decision rule q^* is called *efficient* if for all $\theta \in \Theta$ we have

$$\sum_{i=1}^{N} u_i(q^*(\theta), \theta_i) \geq \sum_{i=1}^{N} u_i(a, \theta_i),$$

for all $a \in X$.

We shall now introduce an important class of incentive compatible mechanism in which the decision rule is efficient.

Definition 4.2.3. A direct mechanism $(q, t_1, \cdots, t_N)$ is a *Vickrey-Clark-Groves mechanism (VCG)* if q is efficient and for all i there exists a function

$$\tau_i : \Theta_{-i} \to \mathbb{R}$$

such that

$$t_i(\theta) = -\sum_{j \neq i} u_j(q(\theta), \theta_j) + \tau_i(\theta_{-i})$$

for all $\theta \in \Theta$.

Proposition 4.2.3. *VCG mechanism are dominant incentive compatible.*

Proof. Suppose that agent i's type is θ_i and the reported type is θ'_i. Agent i's utility is given by

$$\begin{aligned}
u_i(q(\theta'_i, \theta_{-i}), \theta_i) - t_i(\theta'_i, \theta_{-i}) =& u_i(q(\theta'_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} u_j(q(\theta'_i, \theta_{-i}), \theta_j) - \tau_i(\theta_{-i}) \\
&= \sum_{j=1}^{N} u_j(q(\theta'_i, \theta_{-i}), \theta_j) - \tau_i(\theta_{-i}) \\
&\leq \sum_{j=1}^{N} u_j(q(\theta), \theta_j) - \tau_i(\theta_{-i}) \\
&= u_i(q(\theta), \theta_i) + \sum_{j \neq i} u_j(q(\theta), \theta_j) - \tau_i(\theta_{-i}) \\
&= u_i(q(\theta), \theta_i) - t_i(\theta),
\end{aligned}$$

which is agent i's utility if he declares his real type θ_i.

□

Under the hypotheses of the Revenue Equivalence Theorem, VCG mechanisms are the only dominant incentive compatible ones with q being efficient.

Corollary 4.2.4. *Suppose that for all i the set Θ_i is a convex subset of a finite dimensional Euclidean space. Suppose that for all $a \in X$, the function*

$$\theta_i \mapsto u_i(a, \theta_i)$$

is convex. Let $(q, t_1, \cdots, t_N)$ be a dominant incentive compatible mechanism and assume that q is efficient. Then $(q, t_1, \cdots, t_N)$ is a VCG mechanism.

Proof. Once we have fixed q, we can build a VCG mechanism with decision rule q : it is sufficient to write for all i

$$\tilde{t}_i(\theta) = -\sum_{j \neq i} u_j(q(\theta), \theta_j).$$

Then $(q, \tilde{t}_1, \cdots, \tilde{t}_N)$ is a VCG mechanism and for the previous proposition it is dominant incentive compatible. Since the hypotheses of the Revenue Equivalence Theorem are satisfied, if $(q, t_1, \cdots, t_N)$ is a dominant incentive compatible mechanism, for all i there exist a function

$$\tau_i : \Theta_{-i} \to \mathbb{R}$$

such that

$$t_i(\theta) = \tilde{t}_i(\theta) + \tau_i(\theta_{-i}) = -\sum_{j \neq i} u_j(q(\theta), \theta_j) + \tau_i(\theta_{-i});$$

in other words $(q, t_1, \cdots, t_N)$ is VCG.

□

4.3 Implementable decision rules

We are going to discuss the conditions under which a decision rule is implementable. We have seen that efficiency is a sufficient condition, but sometimes the designer could find some constraints for which the decision rule may not be efficient: we are looking for more general conditions.

Definition 4.3.1. A decision rule q is *weakly monotone* if for all i, for all $\theta_{-i} \in \Theta_{-i}$, for all θ_i^1 and $\theta_i^2 \in \Theta_i$, if

$$a^1 = q(\theta_i^1, \theta_{-i}) \text{ and } a^2 = q(\theta_i^2, \theta_{-i}),$$

then

$$u_i(a^1, \theta_i^1) - u_i(a^2, \theta_i^1) \geq u_i(a^1, \theta_i^2) - u_i(a^2, \theta_i^2).$$

Let's explain this definition. The value $u_i(a^1, \theta_i^1)$ is the payoff that player i receives for the alternative a^1 if he declares his real type θ_i^1, while $u_i(a^1, \theta_i^2)$ is his payoff if he falsely reports the type θ_i^2. Then we can interpret the difference on the left-hand side as his willingness to pay for alternative a^1 as opposed to alternative a^2 if his real type is θ_i^1. We can write the same considerations for the difference between $u_i(a^1, \theta_i^2)$ and $u_i(a^2, \theta_i^2)$, which hence is agent i's willingness to pay for alternative a^1 as opposed to alternative a^2 if his real type is θ_i^2. As a result, the weakly monotonicity can be thought of as a property for which agent i prefers to pay for a^1 instead of a^2 if his real type is θ_i^1.

The weak monotonicity is a necessary condition for a decision rule to be implementable.

Proposition 4.3.1. *Suppose that* $(q, t_1, \cdots, t_N)$ *is dominant incentive compatible. Then* q *is weakly monotone.*

Proof. Let $i \in I$, $\theta_{-i} \in \Theta_{-i}$ and $\theta_i^1, \theta_i^2 \in \Theta$. Let

$$a^1 = q(\theta_i^1, \theta_{-i}) \text{ and } a^2 = q(\theta_i^2, \theta_{-i}).$$

Since $(q, t_1, \cdots t_N)$ is dominant incentive compatible, we can write

$$u_i(a^1, \theta_i^1) - t_i(\theta_i^1, \theta_{-i}) \geq u_i(a^2, \theta_i^1) - t_i(\theta_i^2, \theta_{-i})$$

$$u_i(a^2, \theta_i^2) - t_i(\theta_i^2, \theta_{-i}) \geq u_i(a^1, \theta_i^2) - t_i(\theta_i^1, \theta_{-i}),$$

from which

$$u_i(a^1, \theta_i^1) - u_i(a^2, \theta_i^1) \geq t_i(\theta_i^1, \theta_{-i}) - t_i(\theta_i^2, \theta_{-i})$$

$$u_i(a^1, \theta_i^2) - u_i(a^2, \theta_i^2) \leq t_i(\theta_i^1, \theta_{-i}) - t_i(\theta_i^2, \theta_{-i});$$

hence

$$u_i(a^1, \theta_i^1) - u_i(a^2, \theta_i^1) \geq u_i(a^1, \theta_i^2) - u_i(a^2, \theta_i^2),$$

i.e. q is weakly monotone.

□

We shall now show that a stronger condition than weak monotonicity, called cyclic monotonicity, is necessary and sufficient for implementability.

Definition 4.3.2. A decision rule q is *cyclically monotone* if for all $i \in I$, for all $\theta_{-i} \in \Theta_{-i}$, for all $k \in \mathbb{N}, k \geq 2$, and for any sequence of types $\theta_i^1, \cdots, \theta_i^k \in \Theta_i$ such that

$$\theta_i^1 = \theta_i^k,$$

we have

$$\sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) \leq 0,$$

where $a^j = q(\theta_i^j, \theta_{-i})$ for all $j = 1, \cdots, k$.

Remark 4.3.3. If q is cyclically monotone then it is weakly monotone: it is sufficient to apply the definition of cyclic monotonicity with $k = 2$.

We have now the instruments to give the characterization of implementability. This theorem is due to Rochet (1987).

Theorem 4.3.2. *(Rochet Theorem) A decision rule is implementable if and only if it is cyclically monotone.*

Proof. Firstly suppose that q is implementable: there exist N payment rules $(t_1, \cdots, t_N)$ such that $(q, t_1, \cdots, t_N)$ is a dominant incentive compatible mechanism. Let $\theta_i^1, \cdots, \theta_i^k \in \Theta_i$ with $\theta_i^1 = \theta_i^k$. By hypothesis, we have for all $j = 1, \cdots, k - 1$,

$$u_i(a^j, \theta_i^{j+1}) - t_i(\theta_i^j, \theta_{-i}) \leq u_i(a^{j+1}, \theta_i^{j+1}) - t_i(\theta_i^{j+1}, \theta_{-i})$$

where

$$a^j = q(\theta_i^j, \theta_{-i}).$$

We can rewrite the previous inequality as

$$u_i(a^j, \theta_i^{j+1}) - u_i(a^{j+1}, \theta_i^{j+1}) \leq t_i(\theta_i^j, \theta_{-i}) - t_i(\theta_i^{j+1}, \theta_{-i}),$$

and, by summing on j, we obtain

$$\begin{aligned}\sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^{j+1}, \theta_i^{j+1})\right) &\leq \sum_{j=1}^{k-1} \left(t_i(\theta_i^j, \theta_{-i}) - t_i(\theta_i^{j+1}, \theta_{-i})\right) \\ &= t_i(\theta_i^1, \theta_{-i}) - t_i(\theta_i^k, \theta_{-i}) = 0,\end{aligned}$$

since $\theta_i^1 = \theta_i^k$; by splitting the left hand side and observing that

$$\sum_{j=1}^{k-1} u_i(a^{j+1}, \theta_i^{j+1}) = \sum_{j=2}^{k} u_i(a^j, \theta_i^j) = \sum_{j=1}^{k-1} u_i(a^j, \theta_i^j)$$

since $a^k = a^1$, we obtain

$$\sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j)\right) = \sum_{j=1}^{k-1} u_i(a^j, \theta_i^{j+1}) - \sum_{j=1}^{k-1} u_i(a^j, \theta_i^j) \leq 0,$$

i.e. q is cyclically monotone.

Suppose now that q is cyclically monotone: we want to define N payment rules implementing q. Suppose $\tilde{\theta}_i$ is a fixed element of Θ_i. For all $\theta_i \in \Theta_i$. consider the set

$$S(\theta_i) = \left\{(\theta_i^1, \cdots, \theta_i^k) : k \in \mathbb{N},\ k \geq 2,\ \theta_i^j \in \Theta_i,\ j = 1, \cdots k,\ \theta_i^1 = \tilde{\theta}_i,\ \theta_i^k = \theta_i\right\}.$$

For all i, let's define the function

$$V_i : \Theta \to \mathbb{R}$$

as follows:

$$V_i(\theta_i, \theta_{-i}) = \sup_{S(\theta_i)} \left\{ \sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) \right\}.$$

The next step is to show that V_i is well defined, that is the set

$$\left\{ \sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) : (\theta_i^1, \cdots, \theta_i^k) \in S(\theta_i) \right\}$$

is bounded above.

If $\theta_i = \tilde{\theta}_i$, then the cyclic monotonicity implies that for all $(\theta_i^1, \cdots, \theta_i^k) \in S(\theta_i)$,

$$\sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) \leq 0.$$

Hence

$$V_i(\tilde{\theta}_i, \theta_{-i}) \leq 0.$$

In addition $(\tilde{\theta}_i, \tilde{\theta}_i)$ is an element of $S(\theta)$: hence we can deduce that

$$V_i(\tilde{\theta}_i, \theta_{-i}) = 0$$

for all $\theta_{-i} \in \Theta_{-i}$.

Consider now the case $\theta_i \neq \tilde{\theta}_i$. Let $(\theta_i^1, \cdots, \theta_i^k) \in S(\theta_i)$; then $(\theta_i^1, \cdots, \theta_i^{k-1}, \tilde{\theta}_i) \in S(\tilde{\theta}_i)$. We have

$$0 = V_i(\tilde{\theta}_i, \theta_{-i}) \geq \sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) + u_i(a^k, \tilde{\theta}_i) - u_i(a^k, \theta_i);$$

thus

$$\sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) \leq u_i(a^k, \theta_i) - u_i(a^k, \tilde{\theta}_i),$$

and the set Γ is bounded above: we can conclude that V_i is well defined.

Now define for all i the payment rule

$$t_i(\theta) = u_i(q(\theta), \theta_i) - V_i(\theta);$$

we have to verify that $(q, t_1, \cdots, t_N)$ is a dominant incentive compatible mechanism, that is for all $\theta_i' \in \Theta_i$,

$$u_i(q(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta_i, \theta_{-i}) \geq u_i(q(\theta_i', \theta_{-i}); \theta_i) - t_i(\theta_i', \theta_{-i})$$

this is equivalent to

$$V_i(\theta_i, \theta_{-i}) \geq V_i(\theta_i', \theta_{-i}) + u_i(q(\theta_i', \theta_{-i}), \theta_i) - u_i(q(\theta_i', \theta_{-i}), \theta_i'),$$

by definition of t_i. Consider the set

$$\Gamma(\theta_i) \left\{ (\theta_i^1, \cdots, \theta_i^k) : k \in \mathbb{N},\, k \geq 3,\, \theta_i^1 = \tilde{\theta}_i,\, \theta_i^k = \theta_i,\, \theta_i^{k-1} = \theta_i' \right\} \subseteq S(\theta_i).$$

We have

$$\begin{aligned}
V_i(\theta_i, \theta_{-i}) &\geq \sup_{\Gamma(\theta_i)} \left\{ \sum_{j=1}^{k-1} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) \right\} \\
&= \sup_{\Gamma(\theta_i)} \left\{ \sum_{j=1}^{k-2} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) + u_i(a^{k-1}, \theta_i^k) - u_i(a^{k-1}, \theta_i^{k-1}) \right\} \\
&= \sup_{\Gamma(\theta_i)} \left\{ \sum_{j=1}^{k-2} \left(u_i(a^j, \theta_i^{j+1}) - u_i(a^j, \theta_i^j) \right) \right\} + u_i(q(\theta_i', \theta_{-i}), \theta_i) - u_i(q(\theta_i', \theta_{-i}), \theta_i') \\
&= V_i(\theta_i', \theta_{-i}) + u_i(q(\theta_i', \theta_{-i}), \theta_i) - u_i(q(\theta_i', \theta_{-i}), \theta_i').
\end{aligned}$$

□

Chapter 5

Screening: pricing a single indivisible good

The theory of screening is a first example of application of mechanism design to real problems. Moreover it offers an enlightening vision of the techniques used in the mechanism design, being a natural extension of it.
We start this chapter analysing the problem of pricing a good and modelling this situation with the instruments of mechanism design.

5.1 The problem

A seller wants to trade a good with the aim of maximizing his own profit. Suppose that there is just a potential buyer, whose utility function is given by

$$\theta - t;$$

the number $\theta > 0$ represents buyer's evaluation of the good and t is the transfer payment he will pay in case of trade.
We suppose that the value of θ is known by the buyer but not by the seller, who knows only a probability distribution over the possible values of θ : we denote the distribution function by F and we suppose that F has density f; we also suppose that the support of F is an interval such as $[\theta_1, \theta_2]$, on which f is strictly positive. We can think of θ as a random variable with

distribution F, whose realization can be seen by the buyer but not by the seller.
We want to find the procedure by means of which the seller can maximize his profit.

5.2 The model

We are going to model this problem with a mechanism in which there is only a player, the buyer, and the principle is the seller. To do it, we are going to use a suitable direct mechanism. According to the definition given in the previous chapter, we introduce the functions

$$q : [\theta_1, \theta_2] \to [0, 1],$$

and

$$t : [\theta_1, \theta_2] \to \mathbb{R}.$$

In this situation the set of possible outcomes is $[0, 1]$, which represents the probability under which the trade will occur. As already explained, the buyer is asked to report his type: if he reports θ, the seller commits to transferring the good with probability $q(\theta)$ and the buyer has to pay $t(\theta)$; we suppose that the payment is not conditional on the event that the buyer obtains the good, but the buyer has to pay it in every cases.
As previous stated, the buyer's strategies are the functions

$$\sigma : [\theta_1, \theta_2] \to [\theta_1, \theta_2];$$

the number $\sigma(\theta)$ is the type that the buyer declares if his real type is θ.

In this case the payoff function is

$$u(\sigma, \theta) = \theta - t(\sigma(\theta)).$$

For a fixed strategy σ, we consider the stochastic process defined as follows:

$$X_\sigma : \Omega \times [\theta_1, \theta_2] \to \mathbb{R}$$

such that

$$X_\sigma(\omega,\theta) = \begin{cases} \theta - t(\sigma(\theta)) & \text{with probability } q(\sigma(\theta)) \\ -t(\sigma(\theta)) & \text{with probability } 1 - q(\sigma(\theta)). \end{cases}$$

This stochastic process represents the possible payoffs for the buyer when he chooses the strategy σ : if the trade occurs (and it occurs with probability $q(\sigma(\theta))$), then his utility is $\theta - t(\sigma(\theta))$; if the trade does not occur (and it happens with probability $1 - q(\sigma(\theta))$), then he has to pay $t(\sigma(\theta))$.

The expected revenue for the buyer for fixed θ and σ is thus

$$\begin{aligned} \mathbb{E}\left[X_\sigma(\cdot,\theta)\right] &= (\theta - t(\sigma(\theta)))q(\sigma(\theta)) + (-t(\sigma(\theta)))(1 - q(\sigma(\theta))) \\ &= \theta q(\sigma(\theta)) - t(\sigma(\theta)). \end{aligned}$$

The buyer's expected utility is thus

$$u(\sigma,\theta) = \theta q(\sigma(\theta)) - t(\sigma(\theta)).$$

Thanks to revelation principle, we can restrict our attention to direct mechanisms in which the buyer reports his type truthfully: if we consider the case in which the buyer tell the truth, his expected utility is

$$u(\theta) = u(id,\theta) = \theta q(\theta) - t(\theta),$$

where *id* is the identity function.
As we know, a direct mechanism is *incentive compatible* if telling the truth is the optimal strategy, that's to say

$$u(\theta) \geq \theta q(\theta') - t(\theta'),$$

for all $\theta, \theta' \in [\theta_1, \theta_2]$.

Now we need a new definition.

Definition 5.2.1. A direct mechanism $(q, t_1, \cdots t_N, u_1, \cdots, u_N)$ is *individually rational* if

$$u_i(\theta) \geq 0,$$

for all $\theta \in [\theta_1, \theta_2]$, for all $i = 1, \cdots, N$.

In other words, in an individually rational direct mechanism the players participate in the game if they have a nonnegative utility.

Our aim is studying the incentive compatible mechanism maximizing seller's expected revenue: in fact, if the buyer reports his real type, it will be easier to find the optimal mechanism for the seller. In this way, our first step is studying the conditions under which a direct mechanism is incentive compatible. In the previous chapter we have introduced the concept of weak monotonicity as a necessary condition for implementability. In this particular case, weak monotonicity reduces to the concept of increasing function.

Proposition 5.2.1. *If the direct mechanism previously defined is incentive compatible, then q is increasing.*

Proof. Let $\theta, \theta' \in \theta \in [\theta_1, \theta_2]$, with $\theta > \theta'$. Since the mechanism is incentive compatible, we have

$$\theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta'),$$

$$\theta' q(\theta') - t(\theta') \geq \theta' q(\theta) - t(\theta).$$

Subtracting these inequalities, we obtain

$$(\theta - \theta')q(\theta) \geq (\theta - \theta')q(\theta'),$$

that is

$$q(\theta) \geq q(\theta').$$

□

Proposition 5.2.2. *If the direct mechanism previously defined is incentive compatible, then the payoff function is increasing and convex, thus it is differentiable except in at most countably many points. For all $\theta \in [\theta_1, \theta_2]$ in which u is differentiable, we have*

$$u'(\theta) = q(\theta).$$

Proof. By definition of incentive compatibility, we have

$$u(\theta) = \max_{\theta' \in [\theta_1, \theta_2]} (\theta q(\theta') - t(\theta')),$$

for all $\theta \in [\theta_1, \theta_2]$.

For a fixed value θ', the function

$$\theta \mapsto \theta q(\theta') - t(\theta')$$

is increasing and convex, since it is a straight line with positive slope. Thus u is increasing and convex, being the maximum of such functions; in particular u is differentiable except in at most countably many points. Let θ be a point of differentiability. Using incentive compatibility, we have

$$\lim_{h\to 0^+} \frac{u(\theta+h)-u(\theta)}{h} \geq \lim_{h\to 0^+} \frac{(\theta+h)q(\theta)-t(\theta)-(\theta q(\theta)-t(\theta))}{h} = q(\theta);$$

in the same way, we obtain

$$\lim_{h\to 0^-} \frac{u(\theta+h)-u(\theta)}{h} \leq \lim_{h\to 0^-} \frac{(\theta+h)q(\theta)-t(\theta)-(\theta q(\theta)-t(\theta))}{h} = q(\theta).$$

In other words, we have

$$u'(\theta) = q(\theta)$$

in each point of differentiability.

□

It is immediate the following result:

Proposition 5.2.3. *If the mechanism is direct and incentive compatible, then we have*

$$u(\theta) = u(\theta_1) + \int_{\theta_1}^{\theta} q(x)dx.$$

Now we have the tools to characterize an incentive compatible mechanism.

Theorem 5.2.4. *A direct mechanism (q,t) is incentive compatible if and only if*

- *q is increasing;*

- *for all* $\theta \in [\theta_1, \theta_2]$,

$$t(\theta) = t(\theta_1) + \theta q(\theta) - \theta_1 q(\theta_1) - \int_{\theta_1}^{\theta} q(x)dx.$$

Proof. We have already proved the necessity of these last conditions. Suppose now that they are satisfied: let us show that the mechanism is incentive compatible.
Notice that by hypothesis we have

$$\int_{\theta'}^{\theta} q(x)dx \geq \int_{\theta'}^{\theta} q(\theta')dx.$$

As a matter of fact, if $\theta > \theta'$, then

$$q(x) \geq q(\theta')$$

for all $x \in [\theta', \theta]$ and integrating we have the inequality; similarly, if $\theta < \theta'$, then

$$q(x) \leq q(\theta')$$

for all $x \in [\theta, \theta']$; thus

$$\int_{\theta'}^{\theta} q(x)dx = -\int_{\theta}^{\theta'} q(x)dx \geq -\int_{\theta}^{\theta'} q(\theta')dx = \int_{\theta'}^{\theta} q(\theta')dx.$$

Thanks to the second hypothesis, we can conclude that

$$u(\theta) - u(\theta') \geq (\theta - \theta')q(\theta'),$$

that is

$$u(\theta) \geq (\theta - \theta')q(\theta') + \theta' q(\theta') - t(\theta') = \theta q(\theta') - t(\theta').$$

In other words, the mechanism is incentive compatible.

□

We now need the following lemma.

Lemma 5.2.5. *Given an incentive compatible and rational direct mechanism* (q, t) *in which the function* q *maximizes the seller's expected revenue, we have*

$$u(\theta_1) = 0.$$

Proof. By definition of rationality, we have

$$\theta_1 q(\theta_1) - t(\theta_1) = u(\theta_1) \geq 0,$$

that is

$$\theta_1 q(\theta_1) \geq t(\theta_1).$$

If

$$t(\theta_1) < \theta_1 q(\theta_1),$$

then the seller could choose a mechanism with the same q but with higher $t(\theta_1)$, that is an initial price $t'(\theta_1)$ such that

$$t(\theta_1) < t'(\theta_1) \leq \theta_1 q(\theta_1).$$

The new transfer payments are determined by the previous proposition, that is

$$t'(\theta) = t'(\theta_1) + \theta q(\theta) - \theta_1 q(\theta_1) - \int_{\theta_1}^{\theta} q(x)dx > t(\theta);$$

thus the mechanism (q, t) does not maximize the seller's expected revenue, which is impossible.

□

At this point, we can say that in order to maximize the seller's expected revenue we have to find an increasing function $q : [\theta_1, \theta_2] \to [0, 1]$ maximizing the expected value of the function

$$t(\theta) = \theta q(\theta) - \int_{\theta_1}^{\theta} q(x)dx.$$

More formally, if we fix a probability measure μ over $[\theta_1, \theta_2]$, we want to maximize the function

$$q \mapsto \mathbb{E}\left[\theta q(\theta) - \int_{\theta_1}^{\theta} q(x)dx\right],$$

where the integral is calculated respect to μ. We can think of μ as the seller beliefs about the buyer's type.

5.3 Resolution

In order to find q, we are going to use an elegant argument of convex analysis. We denote with $\mathcal{F}$ the set of bounded and measurable functions $f : [\theta_1, \theta_2] \to \mathbb{R}$. Notice that $\mathcal{F}$ is a vector space with the usual operations. We give to this vector space the L^1 norm, that is

$$\|f\| = \int_{\theta_1}^{\theta^2} |f(x)|dx.$$

We define

$$\mathcal{M} = \{f \in \mathcal{F} : f \text{ is increasing}, f(x) \in [0, 1] \text{ for all } x\}.$$

We will look for a function $q \in \mathcal{M}$: it has to be increasing, as shown, and with values in $[0, 1]$, since $q(\theta)$ represents a probability for all θ. For this reason, we study the properties of the set $\mathcal{M}$.

Lemma 5.3.1. $\mathcal{M}$ *is compact and convex.*

Proof. Convex combinations of increasing functions are increasing: thus convexity is immediate. Regarding to compactness, it is an implication of Helly's compactness theorem. Indeed, thanks to it, for all sequence $\{f_n\} \in \mathcal{M}$, there exists a subsequence $\{f_{n_k}\}$ converging pointwise to a function f. Surely, f is increasing and with values in $[0, 1]$. Now we can apply the dominated convergence theorem to say that

$$\lim_{k\to\infty} \int_{\theta_1}^{\theta_2} |f_{n_k}(x) - f(x)|dx = 0,$$

that is the sequence $\{f_{n_k}\}$ converges to f in the norm L^1.

□

Notice that $t(\theta)$ is linear in q: we have to maximize a continuous linear function defined over a convex and compact set.

Using the Extreme Point Theorem, we look for the optimal q among the extreme points of $\mathcal{M}$: instead of maximizing t, we define another function $F : \mathcal{M} \to \mathbb{R}$ as

$$F(q) = \mathbb{E}\left[\theta q(\theta) - \int_{\theta_1}^{\theta} q(x)dx\right],$$

where the expected value is calculated under the probability measure μ, and try to maximize F.

In order to find the extreme points of $\mathcal{M}$ is mainly important, the following lemma is useful.

Lemma 5.3.2. *A function $q \in \mathcal{M}$ is an extreme point of $\mathcal{M}$ if and only if $q(\theta) \in \{0,1\}$ for almost all $\theta \in [\theta_1, \theta_2]$.*

Proof. Suppose that q has values in $\{0,1\}$: we have to show that it is an extreme point. For doing so, we choose a function $q' \in \mathcal{M}$ not identically zero: then there exists an element θ such that $q'(\theta) \neq 0$. Suppose that $q'(\theta) > 0$: if $q(\theta) = 0$, then

$$q(\theta) - q'(\theta) < 0,$$

so that $q - q' \notin \mathcal{M}$; if $q(\theta) = 1$, then

$$q(\theta) + q'(\theta) > 1,$$

and hence $q + q' \notin \mathcal{M}$. Hence both $q - q'$ and $q + q'$ cannot belong to $\mathcal{M}$. If $q'(\theta) < 0$, we can repeat a similar reasoning to arrive to the same conclusion. Hence, by definition, q is an extreme point of $\mathcal{M}$.

Conversely, consider a function $q \in \mathcal{M}$ such that there exists θ^* such that

$$q(\theta^*) \in (0,1).$$

We define a new function $q' \in \mathcal{M}$ in this way:

$$q'(\theta) = \begin{cases} q(\theta) & \text{if } q(\theta) \leq 1/2 \\ 1 - q(\theta) & \text{if } q(\theta) > 1/2. \end{cases}$$

Thus q' is a non zero element of $\mathcal{F}$ such that both $q + q'$ and $q - q'$ belong to $\mathcal{M}$. As a matter of fact,

$$q(\theta) + q'(\theta) = \begin{cases} 2q(\theta) & \text{if } q(\theta) \leq 1/2 \\ 1 & \text{if } q(\theta) > 1/2; \end{cases}$$

thus it is evidently increasing and with values in $[0,1]$;

$$q(\theta) - q'(\theta) = \begin{cases} 0 & \text{if } q(\theta) \leq 1/2 \\ 2q(\theta) - 1 & \text{if } q(\theta) > 1/2; \end{cases}$$

thus it is evidently increasing and with values in $[0,1]$.
Hence q cannot be an extreme point of $\mathcal{M}$. The lemma is proved.

□

We have proved that the seller can choose a non stochastic and monotone mechanism: there must exist a number $\theta^* \in [\theta_1, \theta_2]$ such that

$$q_{\theta^*}(\theta) = \begin{cases} 0 \text{ if } \theta < \theta^* \\ 1 \text{ if } \theta > \theta^*. \end{cases}$$

In this situation, the payment rule t_{θ^*} is of the form

$$t_{\theta^*}(\theta) = \begin{cases} 0 & \text{if } \theta < \theta^* \\ 1 - \theta + \theta^* & \text{if } \theta > \theta^*, \end{cases}$$

thanks to the relation

$$t(\theta) = q(\theta) - \int_{\theta_1}^{\theta} q(x)dx.$$

For this fixed value θ^*, the expected revenue is

$$\begin{aligned} F(q_{\theta^*}) &= \mathbb{E}\left[\theta q_{\theta^*}(\theta) - \int_{\theta_1}^{\theta} q_{\theta^*}(x)dx\right] \\ &= \int_{\theta^*}^{\theta_2} \theta d\mu - \int_{\theta^*}^{\theta_2}\int_{\theta^*}^{\theta} 1\, dx d\mu \\ &= \int_{\theta^*}^{\theta_2} \theta^* d\mu = \theta^* \mu([\theta^*, \theta_2]). \end{aligned}$$

Thus, in order to maximize it, the seller has to choose a value θ^* such that

$$\theta^* \in \text{argmax}_{\,[\theta_1,\theta_2]}(\theta^* \mu([\theta^*, \theta_2])).$$

We summarize what we have found:

Theorem 5.3.3. *An incentive compatible and individually rational direct mechanism maximizes the seller's expected revenue if there exists a number*

$$\theta^* \in \text{argmax}_{[\theta_1,\theta_2]}(\theta^* \mu([\theta^*, \theta_2]))$$

such that

$$q(\theta) = \begin{cases} 0 \ \textit{if} \ \theta < \theta^* \\ 1 \ \textit{if} \ \theta > \theta^*, \end{cases}$$

and

$$t(\theta) = \begin{cases} 0 & \textit{if} \ \theta < \theta^* \\ 1 - \theta + \theta^* & \textit{if} \ \theta > \theta^*. \end{cases}$$

Such a direct mechanism can be implemented by the seller simply quoting the price θ^* and the buyer either accepting or rejecting it. This is a very usual selling mechanism, but we have mathematically demonstrated that there is not a more efficient mechanism. Moreover we have an explicit formula for the calculus of the best price θ^*.

Chapter 6

Screening: non linear pricing

We now deal with a little bit more complicate situation: we are going to see that the resolution is interesting and not trivial.

6.1 The problem

We consider the situation in which a monopolist wants to sell an infinitely divisible good to one potential buyer. We can think of the trade of sugar, for example: it is a more common situation than the one studied in the previous chapter and we will see that additional elements will be used to achieve the solution.
We assume that production costs are linear, and that there exists a constant $c > 0$ such that if the monopolist produces a quantity q of the good, he will have to invest cq.
We assume that the buyer's utility when trading a quantity q and paying t to the seller is given by

$$\theta\nu(q) - t,$$

where θ reflects how much the buyer values the good and $\nu(q)$ depends on how much quantity of good is sold. The product $\theta\nu(q)$ can be interpreted as the buyer's willingness to pay for the quantity q.
The seller's utility is thus given by

$$t - cq.$$

The parameter θ can take any value in the set $[\theta_1, \theta_2]$. It is known by the buyer but not by the seller, whose beliefs regarding θ are described by a distribution function F with density f on the interval $[\theta_1, \theta_2]$; we assume that f is strictly positive over $[\theta_1, \theta_2]$. We suppose that the following properties are satisfied:

1. $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\nu(0) = 0$; ν is twice differentiable, strictly increasing and strictly concave;

2. $\theta_2 \nu'(0) > c$;

3. $\lim_{q \to +\infty} \theta_2 \nu'(q) < c$.

Let's explain the reasons for these assumptions. Because $\nu(0) = 0$, buyer's utility when buying nothing and paying nothing is zero, as one can expected. The second assumption, that is $\theta_2 \nu'(0) > c$, ensures that the seller and the buyer have an incentive to trade: indeed, since

$$\theta_2 \nu(0) = 0 = c \cdot 0,$$

if

$$\theta_2 \nu'(0) > c,$$

then

$$\theta_2 \nu(q) > cq$$

in a neighbourhood of 0. Thus, choosing a payment t such that

$$\theta_2 \nu(q) \geq t \geq cq,$$

we obtain

$$t - cq \geq 0$$

and

$$\theta_2 \nu(q) - t \geq 0,$$

i.e. both the seller's and buyer's utility are positive.

Regarding the third assumption, it implies that the quantity the seller supplies to the buyer must be bounded for all possible types of the buyer, which is a natural condition. Indeed, by this assumption we have

$$\theta_2 \nu'(q) < c$$

definitively, hence the following relation

$$\theta_2\nu(q) < cq$$

is definitively satisfied. On the other hand, we know that

$$\theta_2\nu(q) > cq$$

in a neighbourhood of 0: thus it is well defined

$$\bar{q} = \sup\{q \in \mathbb{R}^+ : \theta_2\nu(q) \geq cq\}.$$

The buyer and the seller have a stake in trading for all $q < \bar{q}$; indeed their utilities have to be positive, that is

$$t - cq \geq 0$$

and

$$\theta_2\nu(q) - t \geq 0;$$

thus we have

$$\theta_2\nu(q) \geq t \geq cq,$$

which holds true for every $q < \bar{q}$, by definition of $\bar{q}$.

Thus the quantity to be sold has to be less than $\bar{q}$.

Our aim is maximizing seller's revenue: we seek to determine optimal selling procedures for him. The revelation principle holds and we can restrict our attention to direct mechanisms. In the following section we are going to model this situation.

6.2 The model

We consider a direct mechanism in which there is only an agent, the buyer, and the seller plays the role of the principal. Thus we define the following functions:

$$q : [\theta_1, \theta_2] \to \mathbb{R}^+,$$

and

$$t : [\theta_1, \theta_2] \to \mathbb{R}^+.$$

The interpretation is the following: the buyer reports the type θ and the seller commits to selling the quantity $q(\theta)$ with the payment $t(\theta)$. Notice that in this case $q(\theta)$ is a quantity, whereas previously it was a probability.

As already stated, we can suppose that the buyer reports his type truthfully. We want to study incentive compatibility and individual rationality, making our mechanism realistic. In this way, we give the following results. Their proofs are similar to those showed in the previous chapter and we omit the details.

Proposition 6.2.1. *A direct mechanism (q, t) is incentive compatible if and only if the following facts are verified:*

- *q is increasing;*
- *for every $\theta \in [\theta_1, \theta_2]$ we have*

$$t(\theta) = t(\theta_1) - \theta_1 \nu(q(\theta_1)) + \theta \nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx.$$

Proposition 6.2.2. *An incentive compatible mechanism is individually rational if and only if*

$$t(\theta_1) \leq \theta_1 \nu(q(\theta_1)).$$

The seller has to choose a direct mechanism satisfying the hypotheses of these two propositions. Moreover, we have this lemma.

Lemma 6.2.3. *Given an incentive compatible and rational direct mechanism (q, t) in which the function q maximizes the seller's expected revenue, we have*

$$t(\theta_1) = \nu(q(\theta_1)).$$

Under these hypotheses, the payment rule and decision rule are thus linked by this relation:

$$t(\theta) = \theta \nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx.$$

We have to find the proper q maximizing the expected value of $t(\theta)$; more formally we want to maximize the function

$$
\begin{aligned}
q \to & \mathbb{E}\left[\theta\nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx - cq(\theta)\right] \\
&= \int_{\theta_1}^{\theta_2} \left[\theta\nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx - cq(\theta)\right] f(\theta)d\theta,
\end{aligned}
$$

over the set

$$
\mathcal{N} = \left\{q : [\theta_1, \theta_2] \to \mathbb{R}^+ : q \text{ is increasing } \right\}.
$$

6.3 Resolution

We now depart from the line of argument we followed in the previous chapter: we now want to maximize a function which is not linear in q, since q enters the non linear function ν. As already stated, if the seller chooses the function q, his expected payoff is given by

$$
\begin{aligned}
&\int_{\theta_1}^{\theta_2} \left[\theta\nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx - cq(\theta)\right] f(\theta)d\theta \\
&\quad = \int_{\theta_1}^{\theta_2} \theta\nu(q(\theta))f(\theta)d\theta - \int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta} \nu(q(x))dxf(\theta)d\theta \\
&\quad - \int_{\theta_1}^{\theta_2} cq(\theta)f(\theta)d\theta.
\end{aligned}
$$

Focussing on the double integral of the previous expression, we have

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta} \nu(q(x))dxf(\theta)d\theta &= \int_{\theta_1}^{\theta_2}\int_{\theta_1}^{\theta} \nu(q(x))f(\theta)dxd\theta \\
&= \int_{\theta_1}^{\theta_2}\int_{x}^{\theta_2} \nu(q(x))f(\theta)d\theta dx,
\end{aligned}
$$

where we have changed the order of integration using Fubini's theorem; thus this integral is equal to

$$\int_{\theta_1}^{\theta_2} \nu(q(x)) \int_x^{\theta_2} f(\theta)d\theta dx = \int_{\theta_1}^{\theta_2} \nu(q(x))(1-F(x))dx$$
$$= \int_{\theta_1}^{\theta_2} \nu(q(\theta))(1-F(\theta))d\theta,$$

where, as already defined,

$$F(x) = \int_{\theta_1}^{x} f(\theta)d\theta.$$

Thus, the seller's expected payoff is given by

$$\int_{\theta_1}^{\theta_2} \left[\theta\nu(q(\theta)) - cq(\theta)\right] f(\theta)d\theta - \int_{\theta_1}^{\theta_2} \nu(q(\theta))(1-F(\theta))d\theta$$
$$= \int_{\theta_1}^{\theta_2} \left[\theta\nu(q(\theta)) - cq(\theta)\right] f(\theta)d\theta - \int_{\theta_1}^{\theta_2} \nu(q(\theta))\frac{1-F(\theta)}{f(\theta)}f(\theta)d\theta$$
$$= \int_{\theta_1}^{\theta_2} \left[\nu(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - cq(\theta)\right] f(\theta)d\theta.$$

We ignore momentarily that q has to be increasing. Without this property, the problem has an easy solution: we can choose $q(\theta)$ for each θ separately to maximize the expression

$$\nu(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - cq(\theta).$$

For doing so, for each fixed θ we consider the derivative in q and we study

$$\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - c = 0,$$

that is

$$\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) = c.$$

If

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \leq 0,$$

there is obviously no solution, since $\nu' > 0$ and $c > 0$. In this case the optimal choice is

$$q(\theta) = 0;$$

indeed the function

$$\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) - c$$

is strictly negative, thus its maximum is in 0.

Consider now the case in which

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) > 0.$$

If

$$\nu'(0)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \leq c,$$

the optimal choice is again

$$q(\theta) = 0;$$

indeed ν' is decreasing by hypothesis, thus $\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)$ is decreasing too: if in 0 it is less than c, it will not ever reach c.

If

$$\nu'(0)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) > c,$$

then there is a unique solution to

$$\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) = c,$$

and this stationary point is the optimal choice of $q(\theta)$.

We have now found the optimal q that the seller has to choose without asking that it is increasing: we now introduce an additional hypothesis implying this property of q.

We assume that

$$\theta - \frac{1-F(\theta)}{f(\theta)}$$

is increasing in θ.

Proposition 6.3.1. *Assuming that*

$$\theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing in θ, the function q previously defined is increasing.

Proof. By hypothesis,

$$\nu'(q(\theta)) \left(\theta - \frac{1 - F(\theta)}{f(\theta)} \right)$$

is increasing in θ for every q. The optimal q is the intersection point of that expression with c or zero. It is easy then to see that the optimal q is increasing in θ.

□

We now investigate the conditions under which $\theta - \frac{1-F(\theta)}{f(\theta)}$ is increasing. A sufficient condition is that

$$\frac{f(\theta)}{1 - F(\theta)}$$

is increasing. This condition is often called "increasing hazard rate" condition. To explain this name, we can think of $F(\theta)$ as the probability that an individual dies before time θ; thus $1 - F(\theta)$ is the probability that he survives until θ: $\frac{f(\theta)}{1-F(\theta)}$ is the density of the conditional probability of dying at θ if the individual has survived until θ. The increasing hazard rate condition can be thought as the assumption that this conditional probability of dying is increasing in θ.

Definition 6.3.1. We call *regular* the distributions F such that

$$\theta - \frac{1 - F(\theta)}{f(\theta)}$$

is increasing in θ.

We now summarize the analysis of this section in the following result.

Theorem 6.3.2. *If F is regular, then the expected profit maximizing choice q is described as follows:*

- *if*
$$\nu'(0)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \leq c$$
then
$$q(\theta) = 0;$$

- *if*
$$\nu'(0)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) > c$$
then
$$\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) = c.$$

The profit maximizing payment rule is given by:

$$t(\theta) = \theta\nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx.$$

6.4 A numerical example

We conclude this chapter with a numerical example. Suppose $[\theta_1, \theta_2] = [0, 1]$; put $c = 1, \nu(q) = \sqrt{q}$, and suppose that θ is a uniformly distributed random variable over $[0, 1]$, that is

$$F(\theta) = \theta,$$

and

$$f(\theta) = 1.$$

Checking the regularity of F, we obtain

$$\theta - \frac{1-F(\theta)}{f(\theta)} = \theta - \frac{1-\theta}{1} = 2\theta - 1,$$

which is increasing in θ. Thus theorem 6.3.2 applies, and we can compute the optimal q. We have

$$\lim_{q\to 0} \nu'(q)\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) \leq c$$

if and only if

$$\theta - \frac{1-F(\theta)}{f(\theta)} \leq 0,$$

that is

$$\theta \leq \frac{1}{2};$$

in this case

$$q(\theta) = 0.$$

On the other hand if $\theta > 1/2$, the optimal q is given by the equation

$$\nu'(q(\theta))\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) = c,$$

that is

$$\frac{1}{2\sqrt{q}}(2\theta - 1) = 1,$$

from which we get

$$q = \left(\theta - \frac{1}{2}\right)^2.$$

Consider now the payment rule. If $\theta \leq 1/2$, then $q(x) = 0$ for all $0 \leq x \leq \theta$; since $\nu(0) = 0$, we have

$$t(\theta) = \theta\nu(0) - \int_0^\theta \nu(0)dx = 0.$$

If $\theta > 1/2$, the payment rule is

$$\begin{aligned} t(\theta) &= \theta\nu(q(\theta)) - \int_{\theta_1}^{\theta} \nu(q(x))dx \\ &= \theta\left(\theta - \frac{1}{2}\right) - \int_{1/2}^{\theta}\left(x - \frac{1}{2}\right)dx \\ &= \theta\left(\theta - \frac{1}{2}\right) - \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_{1/2}^{\theta} \\ &= \theta\left(\theta - \frac{1}{2}\right) - \left(\frac{1}{2}\theta^2 - \frac{1}{2}\theta - \frac{1}{8} + \frac{1}{4}\right) \\ &= \frac{1}{2}\theta^2 - \frac{1}{8}. \end{aligned}$$

Since

$$q = \left(\theta - \frac{1}{2}\right)^2,$$

we have

$$\theta = \sqrt{q} + \frac{1}{2};$$

thus the payment rule is given by

$$t(\theta) = \frac{1}{2}\theta^2 - \frac{1}{8} = \frac{1}{2}\left(\sqrt{q} + \frac{1}{2}\right)^2 - \frac{1}{8} = \frac{1}{2}q + \frac{1}{2}\sqrt{q}.$$

Now we summarize what we have obtained, focussing on the optimal strategy for the buyer. The seller offers to the buyer the possibility of trading a quantity $q \in [0, \frac{1}{4}]$; thus buyer's type, given by the relation

$$\theta = \sqrt{q} + \frac{1}{2},$$

is a number in $[\frac{1}{2}, 1]$. Hence the payment rule is

$$t(q) = \frac{1}{2}q + \frac{1}{2}\sqrt{q}.$$

The price per unit is

$$\frac{t}{q} = \frac{\frac{1}{2}q + \frac{1}{2}\sqrt{q}}{q} = \frac{1}{2} + \frac{1}{2\sqrt{q}},$$

which is decreasing in q, as we could expect.

Chapter 7

Social Choice Function: the case of elections

In this chapter we analyze the role of the function q, the decision rule of a mechanism design. There is a very interesting mathematical theory studying the role of this important function in many contexts: we are going to analyze its application in the political elections. It is a very amazing and unexpected application of mechanism design, showing how this theory is relevant in real life.

7.1 The problem

Suppose we have a finite set of alternatives K and a set I of individuals who have to make a collective decision. To be clearer, we need these definitions.

Definition 7.1.1. A binary relation $\geq$ over a set K is a *partial order* if it is:

- reflexive, that is $a \geq a$;
- antisymmetric, that is if $a \geq b$ and $b \geq a$ then $a = b$;
- transitive, that is if $a \geq b$ and $b \geq c$ then $a \geq c$.

Definition 7.1.2. A partial order $\geq$ is *total* if for all $a, b \in K$ we have

$$a \geq b \text{ or } b \geq a.$$

More specifically each individual has a personal total order on the set K: we want to define a function q called *social choice function* which gives a collective total order on K once the single orders are established.

We can model this situation with a direct mechanism: for each $i \in I$, the type set Θ_i is a set of total orders over the set K; the set X of possible outcomes is a set of total orders over K too. The decision rule

$$q : \Theta \to X$$

is the social choice function we have to find. The functions t_i and u_i will depend on the particular situation we will deal with.

We proceed with a practical example. Consider the case of elections: we are interested in finding an electoral law.
We have three candidates, x, y and z, and a set of n voters. We put

$$K = \{x, y, z\};$$

the voters can express their favourite candidate and a second preference: this automatically induces a total order over the set K for each elector. If we denote by X and Θ_i for all i the sets of possible total orders over K, our aim is finding the function

$$q : \Theta_1 \times \cdots \times \Theta_n \to X,$$

describing the voting law.
For a fixed $\theta_i \in \Theta_i$, we write

$$xP_{\theta_i}y$$

to indicate that x is strictly preferred to y in the total order θ_i.
We denote by

$$P_\theta$$

the preference relation determined by the total order $q(\theta)$.

7.2 Properties of the Social Choice Function

In this section we indicate the properties that an electoral law is expected to have in order to be democratic.

- **Property 1: completeness.**
For each $\theta \in \Theta = \Theta_1 \times \cdots \times \Theta_n$ there is one and just one $P \in X$ such that
$$q(\theta) = P.$$
In other words, q has to be a well defined function.

- **Property 2: citizens' sovereignty.**
For all $x, y \in K$ there exists a $\theta \in \Theta$ such that
$$xP_\theta y.$$
This means that for each pair of candidates x, y there is at least one citizen for which the candidate x is preferred to y.

- **Property 3: positive correlation.**
For fixed $x, y \in K$, suppose that, for a given θ,
$$xP_\theta y;$$
then the same holds for every θ' such that
$$xP_{\theta'_i} y$$
if
$$xP_{\theta_i} y$$
for all i. In other words, if for θ the candidate x is preferred to y, then for all θ' deferring from the previous just because in some θ'_i the candidate x has improved his position, in the resulting new total order the candidate x is already preferred to y.

- **Property 4: independence of irrelevant alternatives.**
Suppose that for fixed $x, y \in K$ and $\theta \in \Theta$ we have
$$xP_\theta y.$$
Consider now a $\theta' \in \Theta$ such that for all $i \in I$, if
$$xP_{\theta_i} y,$$
then
$$xP_{\theta'_i} y,$$

and if

$$yP_{\theta_i}x,$$

then

$$yP_{\theta'_i}x.$$

Then we have

$$xP_{\theta'}y.$$

This property implies that if, for a fixed θ, x is preferred to y, then the same holds for every θ' such that for each total order θ'_i the preference between x and y is equal to the preference expressed by θ_i.

- **Property 5: non-dictatorship.**
  There does not exist an elector i such that, for all θ, if

  $$xP_{\theta_i}y,$$

  then

  $$xP_{\theta}y.$$

  In other words, an elector cannot determine on his own the collective total order.

Notice that these five properties are very realistic: we expect that all these axioms are satisfied when an election takes place.

7.3 Arrow's Impossibility Theorem

In this section we study this important theorem, which expresses the paradox that democratic elections are impossible.

Theorem 7.3.1. *If K has more than three elements, it is impossible that a function*

$$q : \Theta \to X$$

satisfies the five properties stated before.

Before proving this theorem, we proceed with a preliminary result.

Proposition 7.3.2. *(Pareto property) If the properties 2,3,4 are satisfied, for all θ such that*

$$xP_{\theta_i}y$$

for all $i = 1, \cdots, n$, we have

$$xP_{\theta}y.$$

Proof. Let $\theta \in \Theta$ such that

$$xP_{\theta_i}y$$

for all $i = 1, \cdots, n$. Thanks to the second property, there exists θ' such that

$$xP_{\theta'}y.$$

Suppose that there exists i such that

$$yP_{\theta'_i}x.$$

Then we change this total order by putting x as the most favourite element of K; we obtain a new total order in which

$$xP_{\theta^*_i}y$$

and thanks to the third property

$$xP_{\theta'}y.$$

In this way, we build a $\theta^* \in \Theta$ such that

$$xP_{\theta^*_i}y$$

for all i, and thanks to the third property,

$$xP_{\theta^*}y.$$

Since

$$xP_{\theta_i}y$$

for all i too, applying the property 4, we obtain

$$xP_{\theta}y.$$

$\square$

We now need some preliminary definitions and considerations.

Definition 7.3.1. A set $D \subseteq I$ is *decisive for the alternative* x *with respect to the alternative* y if and only if for all θ if

$$xP_{\theta_i}y$$

for all $i \in D$, then

$$xP_{\theta}y.$$

Definition 7.3.2. We say that $i \in I$ is a *dictator for the alternative* x *with respect to the alternative* y if and only if the set $\{i\}$ is decisive for the alternative x with respect to the alternative y.
We say that $i \in I$ is a *dictator* if for all $x, y \in K$ he is a dictator for the alternative x respect to alternative y.

Notice that the property of non-dictatorship states that there no exist dictators.

We now suppose that the five properties are satisfied. It is quite obvious the following fact.

Proposition 7.3.3. *For all* x, y *there exists a decisive set for the alternative* x *with respect to the alternative* y*.*

Proof. Thanks to Pareto property, the set I of all individuals is a decisive set for the alternative x with respect to the alternative y.

□

Proposition 7.3.4. *A set* D *is decisive for the alternative* x *with respect to the alternative* y *if and only if the following property is satisfied: for all* θ *such that for all* $i \in D$ *we have*

$$xP_{\theta_i}y,$$

and for all $i \notin D$*, we have*

$$yP_{\theta_i}x,$$

then

$$xP_{\theta}y.$$

Proof. Suppose that D is decisive for the alternative x with respect to the alternative y: thus if

$$xP_{\theta_i}y,$$

for all $i \in D$, by definition we have

$$xP_{\theta}y.$$

Suppose now that the property in the lemma is satisfied: thus we have

$$xP_{\theta}y,$$

for all θ such that

$$xP_{\theta_i}y$$

if $i \in D$, and

$$yP_{\theta_i}x$$

if $i \notin D$. Notice that θ of this kind exists thanks to property 2.
Consider θ' such that

$$xP_{\theta'_i}y$$

for all $i \in D$. We want to show that

$$xP_{\theta'}y.$$

If for all $i \notin D$ we have

$$yP_{\theta'_i}x,$$

thus we have the thesis by hypothesis. If there exists $i \notin D$ such that

$$xP_{\theta'_i}y,$$

we can consider the corresponding θ_i: we have

$$yP_{\theta_i}x.$$

We can improve the position of x in this total order so that we obtain a new θ^* for which

$$xP_{\theta^*_i}y.$$

Notice that thanks to property 3, we still have

$$xP_{\theta^*}y.$$

In this way, we can build a θ^* such that

$$xP_{\theta^*}y,$$

and for all i the related positions of x and y in the total orders θ'_i and θ^*_i are the same. Then, by property 4, we have

$$xP_{\theta'}y.$$

□

We introduce the following notation.

Definition 7.3.3. A set D is decisive for (x, y) if it is decisive for the alternative x with respect to the alternative y.

We now prove an important result from which the Arrow theorem easily follows.

Proposition 7.3.5. *If a set D is decisive for the alternative x with respect to the alternative y , then it is decisive for any other couple of alternatives.*

Proof. We analyze the different cases separately.

Let's prove that D is decisive for (x, z) with $z \neq x, y$. Let θ such that

- for all $i \in D$, $xP_{\theta_i}z$;
- for all $i \notin D$, $zP_{\theta_i}x$.

We have to show that

$$xP_{\theta}y.$$

Using property 4, we can modify the position of y without changing the related positions of x and z. We obtain a new θ such that:

- for all $i \in D$, $xP_{\theta_i}y$ and $yP_{\theta_i}z$;
- for all $i \notin D$, $yP_{\theta_i}z$ and $zP_{\theta_i}x$.

Since D is decisive for (x, y) then

$$xP_\theta y.$$

Since for all i we have

$$yP_{\theta_i}z,$$

by Pareto property we have

$$yP_\theta z.$$

Thus by the transitive property of a total order, we obtain

$$xP_\theta z.$$

Applying the previous proposition, we can conclude that D is decisive for (x, z).

Consider now $z, z' \neq x$. We want to show that D is decisive for (z, z'). Reasoning as before, we can suppose that θ satisfies the following properties:

- for all $i \in D$, $zP_{\theta_i}x$ and $xP_{\theta_i}z'$ (so that $zP_{\theta_i}z'$);
- for all $i \notin D$, $z'P_{\theta_i}z$ and $zP_{\theta_i}x$ (so that $z'P_{\theta_i}x$).

For what we have already shown, D is decisive for (x, z'), thus

$$xP_\theta z'.$$

By Pareto property, we have

$$zP_\theta x,$$

thus, by transitivity,

$$zP_\theta z'.$$

Thus D is decisive for (z, z').

Finally, consider $z \neq x$. We want to show that D is decisive for (z, x). Let z' another alternative; reasoning as before, we can consider a θ such that

- for all $i \in D$, $zP_{\theta_i}z'$ and $z'P_{\theta_i}x$ (so that $zP_{\theta_i}x$);
- for all $i \notin D$, $z'P_{\theta_i}x$ and $xP_{\theta_i}z$ (so that $z'P_{\theta_i}z$).

For what we have already shown, D is decisive for (z, z'), thus

$$zP_\theta z';$$

by Pareto property, we also have

$$z'P_\theta x,$$

thus, by transitivity, we have

$$zP_\theta x.$$

D is decisive for (z, x).
Summarizing, D is decisive for the couples $(x, z), (z, z'), (z, x)$ for all $z, z' \neq x$. Thus D is decisive for all couples.

□

Corollary 7.3.6. *If an individual is a dictator for a couple of alternatives (x, y), then he is a dictator.*

We now have the tools to prove Arrow's theorem.

Proof (of Arrow's Impossibility Theorem). We suppose that there exists a function

$$q : \Theta \to X$$

with the five properties stated in the previous section. Consider a couple of alternatives (x, y). As already observed, the set I is decisive for (x, y). Thus there exists a minimal decisive set, that is a set D which is decisive for (x, y) and such that its proper sets are not decisive for (x, y). Notice that a decisive set of one element is a dictator, for the previous corollary. Thus, since property 5 states that there does not exist a dictator, the set D has more than two elements. Moreover, D is decisive for all couples. We can divide D in two disjoint subsets D' and D''. Fix three elements x, y, z of K. Consider a θ such that

- for all $i \in D'$, $xP_{\theta_i}y$ and $yP_{\theta_i}z$;
- for all $i \in D''$, $zP_{\theta_i}x$ and $xP_{\theta_i}y$;
- for all $i \notin D$, $yP_{\theta_i}z$ and $zP_{\theta_i}x$;

Notice that

$$xP_\theta y,$$

since D is decisive for (x, y).
Moreover it cannot be

$$zP_\theta y,$$

since D'' can not be decisive for (z, y) and there exists a θ' such that

$$zP_{\theta'_i}y$$

for all $i \in D''$,

$$yP_{\theta'_i}z$$

for all $i \notin D''$ and

$$yP_{\theta'}z.$$

Thus by transitivity we have

$$zP_{\theta_i}y \text{ for all } i \in D'',$$

and

$$yP_{\theta_i}z \text{ for all } i \in D' \cup D^c = (D'')^c.$$

We conclude that the related positions of z and y are the same in θ_i and θ'_i for all i, and by property 4 we have

$$yP_\theta z.$$

If

$$yP_\theta z,$$

then

$$xP_\theta z.$$

Repeating the previous argument, we obtain

$$zP_\theta x,$$

which is impossible.
Thus

$$y = z,$$

so that

$$xP_\theta z,$$

which is impossible as already stated. In particular the couple (y, z) cannot be ordered by $q(\theta)$, which is impossible since $q(\theta)$ is a total order.

□

It seems that democratic elections are not possible: in fact we have shown that it is impossible to find a social choice function satisfying the previous properties.
A question immediately arises: is the majority rule, on which many electoral systems are based, democratic?
To answer we need to formally express the concept of majority rule. Before doing so, we have to sobstitute the set X of total orders over K with a larger set.

Let Y be the set of binary relations P over K such that:

- if xPy then not yPx;
- if xPy then not yIx;
- if xIy then not xPy and also not yPx;
- for all x and y, xPy or yPx or xIy.

The relation xIy indicates that x and y are indifferent.

We now consider social choice functions defined as follows:

$$q : \Theta \to Y$$

where Θ is the same set previously defined. We can now define the concept of majority rule.

Definition 7.3.4. A social choice function

$$q : \Theta \to Y$$

is a majority rule if and only if for all $x, y \in K$ we have

$$xP_\theta y$$

if and only if the set

$$\{i \in I : xP_{\theta_i}y\}$$

has more elements than the set

$$\{i \in I : yP_{\theta_i}x\}.$$

Why have we changed the set X with Y? To answer, consider the following example.
We have three candidates, a, b, c, and three electors, X, Y, Z. Consider the following orders:

$$X : a > b > c;$$
$$Y : b > c > a;$$
$$Z : c > a > b.$$

It is clear that if we use the majority rule, it is impossible for us to give a global total order over $\{a, b, c\}$, since two electors prefer a to b, two electors prefer b to c and two electors prefer c to a: the majority preference does not respect the transitivity property.
This situation is known as the "voting paradox" and shows how the majority rule cannot define a transitive relation over K: in other words, a majority rule does not always give a total order over the set of alternatives, thus we cannot define it with values in X, but we need a larger set, the set Y. We can see that the condition which is not satisfied by a majority rule is thus the property of completeness: however dropping this condition and enforcing the others, it is possible to arrive to a set of conditions satisfied by majority rule alone.

Chapter 8

Conclusions

"The theory of mechanism design currently plays a major role in many areas of economics and in parts of political science, and has led to many fruitful applications. Its domain of application has expanded in recent years, due to globalization and growing internet trade, phenomena that impose new demands on old institutions."

This thesis shows how Mechanism Design, this new field of applications of Game Theory, is very interesting from a theoretical point of view and has many applications in Economics and Political Science. Moreover, it plays an important role in many parts of Computer Science, Cryptography and Auction Theory.
We have seen the basic elements of Mechanism Design, but there is an amazing world of theory and applications starting from here.

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