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# Growth of Corallium Rubrum: from experimental data towards mathematical models

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# Introduction

The flowering of mathematical ecology and population dynamics occurred during the first half of twentieth century: many nowadays well-known equations were estabilished, such as the Verhulst equation for the logistic model and the Lotka-Volterra system for the competition between two species.
In the last decades the mathematical interest in modelling biological structures and phenomena has had a further unusual development, mostly due to the increase of themes coming also from the medical field; in addition, new improvements of classical models appeared, in order to reproduce more specific situations.

In the present work we deal with some structured models for dynamics of two *Corallium rubrum* populations.

Red Coral (*Corallium rubrum* L 1758) is a *colonial anthozoan* endemic to the Mediterranean sea. Due to the high economical value of its carbonate skeleton (used in jewelry), this species has been harvested since ancient times. For this reason in the last two decades a reduction of the overall fishing yield by $\frac{2}{3}$ has been recorded [SA01]. Therefore in the last years two papers, [SBI07] and [SBI09], have dealt with the study of a red coral population located in Calafuria (LI, Italy) and its sensitivity to environmental factors. Later another demographic study [Vie09] concerning two different populations, located in Portofino (Ligurian coast, Italy) and Cap de Creus (Costa Brava, Spain), has revealed some differences between these populations and the old one.
The aim of the present study is to develop some more suitable models for the data coming from Portofino and Cap de Creus. These datasets reveal a high variability concerning the noticed growth rates of the colonies. Furthermore they appear not homogeneous (probably for the small quantity of samples) and present a lack of information at different levels. In particular for several colonies we do not have fertility parameters and there is no information about certain age classes: indeed, during the samples of the bigger colonies, the smaller ones got lost. Hence we used this data as much as we could and tried to follow the noticed variability towards the modelling process. Probably more time would be necessary in order to obtain a more significant dataset, but this would go beyond the bounds of this

work.

Let us describe the content of the present work. In the first chapter we present some biological and mathematical preliminaries to this work, in order to introduce the notations used and themes.

The second chapter is devoted to the presentation of the main part of the data and to its analysis, performed with R program. There are several types of data: some of them contain growth informations in term of the diameter or the circular annual crown, whereas others contain informations about reproduction (i.e. number of *planulae* produced by a colony). Concerning the growth of the basal section, we infer, analyzing a collection of few selected data, that is nearly linear with respect to the year. Furthermore we notice a peculiar variability: indeed we report some colonies (from both locations) that present a much stronger growth rate. However we construct an age distribution for the two populations using the entire datasets; in this way we try to obtain the survival parameters. Unfortunately these distributions appear affected by some noise, caused by a lack of data for certain age classes. Hence the obtained distributions seem not satisfactory and need to be fitted to obtain a survival function. On the other hand reproductive data present several lacks; hence we have to refer, when possible, to [Vie09].

In the third chapter some models are presented and analyzed; we begin with discrete ones that should be more suitable and were used in [SBI07] and [SBI09]: indeed reproduction of red coral occurs once a year in early summer and a discrete model with a step of one year reproduces satisfactorily this phenomenon. In this framework we improve the older models, since we take into account the high variability of the income data by adding another parameter, measuring the growth level of a colony within its age class.
We also construct a continuous model that is equivalent to a system of Volterra equations [Ian94]; also in this case we add a size parameter. We prove an existence and uniqueness result for solutions of integral equations, using a fixed point argument. The continuous model is examined by analyzing the Volterra equations: the behaviour of their solutions is connected to the Laplace transform of the integral kernel, via the Paley-Wiener theorem [PW33]. Thus we prove a stability result for the solutions of integral equations.
Finally we construct a simple spatial model, considering the spread of the planulae within a limited interval or a two dimensional disk: this leads to a discrete dynamical system containing a diffusion equation through the steps. We find a general solution of the diffusion equation (by means of Fourier and Fourier-Bessel series) with Neumann and Dirichlet boundary conditions and therefore the solution of the discrete system.

In the last chapter we perform some numerical computations concerning the discrete models defined above and the stability of its steady states. In this frame-

work the non-homogeneity of the data causes a loss of accuracy in the determination of survival parameters. Moreover the lack of complete and organic informations concerning the reproduction process led us to use *cutoff* function and some reproduction parameters from the old model concerning Calafuria populations. Hence the obtained results are not so explanatory. Finally we add two appendices to present some mathematical results and definitions used throughout in the thesis: in the first one we deal with two particular equations involving the Laplace transform and the locations of their solutions. Then we present several properties of the linear Volterra equations (and a special nonlinear one) and prove the Paley-Wiener theorem.
In the second appendix we show some simple results concerning the Bessel functions and, in particular, the Fourier-Bessel series.

# Chapter 1

# Preliminaries

In this chapter we introduce the red coral from a biological point of view. We say something about its harvesting and conservation, in particular with respect to the two sites from which our data were sampled [Vie09]. Then we give a brief mathematical introduction concerning discrete and continuous models for structured populations, just to fix some notations and display some basic results, used throughout in this work.

## 1.1 Red coral

This section is devoted to Corallium Rubrum: its biological features and its conservation, in particular we give several information about the legal regulation of fishing within the two analyzed locations and the actual state of affairs.

### 1.1.1 Biological introduction

Mediterranean red coral (Anthozoa, Octocorallia, Gorgonacea, Corallium rubrum, L. 1758) is an anthozoa endemic to the Mediterranean Sea. It is a long-lived, gonochoric, brooder species, whose larvae (*planulae*) do not travel very far from the parental colonies.
The bathymetric distribution of Corallium rubrum lies between 10 m (5 in caves) and 300 m depth, but recently this species has been recorded up to 800 m depth. Reproduction happens within a limited time interval in early summer, hence such populations can only decrease in number (due to mortality) between two different reproductive cycles.
For this species natural mortality is low, as there are few predators; a mortality source is represented by boring sponges, that drill the colonies and fill them up of holes. Coastal shallow water populations of Anthozoan are furthermore

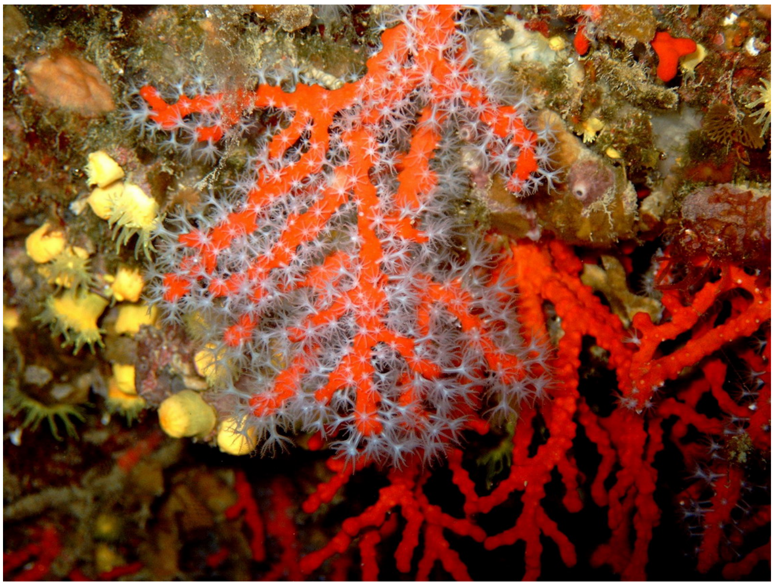


Figure 1.1: *Corallium Rubrum*.

vulnerable to chemical and physical changes of their habitat.

### Shallow and deeper population

According to an operational definition, not yet based on genetic differences, two different types of population can be distinguished according to the bathymetric depth range: shallow and deep populations. The first one, living in the bathymetric range between 20-60 m depth, is characterized by colonies at high densities but with small basal diameter. Due to the size and the frequent boring sponges infection, such populations have a limited economic value. Deeper populations, on the contrary, are characterized by sparse but larger colonies, thus with higher economic value.

The majority of papers describes shallow water populations, because these are easier to reach, and therefore to sample. Such inferior limit deals with SCUBA diving operative limits about air supply and therefore shallow water populations have been deeply harvested and some are still now, despite their lower intrinsic economic value. This overharvesting seems to cause also the actual dominance of small/young colonies in these populations. For this reason, nowadays, where harvesting still occurs management measures are necessary to plan harvesting limits, especially for shallow water populations.

Deep populations are generally distributed in the belt between 60 and 250 m depth where large colonies are sparsely distributed and less affected by boring sponges, thus acquiring an economic value up to 20 times higher with respect to

the other ones. Only recently *ROV* (Remote Operative Vehicle) surveys, carried out off Cap de Creus (or Sardinia), gave us some data about deep populations. Therefore despite preliminary studies have been recently carried out, data on deep colony biology such as colony growth, population age and sexual structure, age at first maturity, reproductive output, recruitment, natural mortality are still absent.
Similarly harvesting of deep dwelling colonies could be actually considered a selective technique. Legal authorised harvesters are professional divers who dive with mixed gases up to 120 m depth. Furthermore due to the effort required by harvesting activities, a preliminary ROV survey of the harvesting site is recently conducted. Harvesters, in this case, have not any interest to deplete indiscriminately deep populations and a selective long term fishing strategy is considered advantageous also for economic reasons.

### 1.1.2 Harvesting and conservation

Due to the high economic value of its carbonate skeleton, red coral is the Mediterranean most precious marine species and it has been harvested since the Neolithic period. C. rubrum is fished in the whole Mediterranean sea and more than 90 % of the harvest is manufectured in Italy; in particular, the red coral jewelry industry of Torre del Greco near Naples (Italy), is estimated to generate about two hundred million $ per year.
The European community legislation about the harvesting of red coral in Mediterranean sea states that it should be practiced using non-selective and non-destructive trawling gears (since 1994); furthermore the number of harvesters is regulated by licenses and yield is limited by harvest quotas and minimum size limits.
In practice, Corallium rubrum coastal shallow populations conservation is effective just in those sites falling within Marine Protected Areas (MPA) and management actions are usually not based on specificstudies. Likely as coastal shallow populations are actually considered of low economic value harvesting, intheory, they could not considered under threat. Knowledge of distribution and density of Corallium rubrum are mainly due to scientific papers but a real mapping lacks. However mass mortality events, strongly affecting *Octocorals* are more and more frequent in Mediterranean Sea [BMMS05] and these could represent an important mortality source, especially for shallow water coastal populations. In particular, recently (late summer 1999 and 2003) the red coral populations of Marseille and Calafuria have been affected by mass mortality, associated to an anomalous temperature increases in the Eastern Ligurian Sea and in Gulf of Lion. Assessment of the impact of such events, in terms of mortality, on long-lived species requires long-time data series collected before and after the event, but few studies on the long-term effects of mass mortality exists [CCS+08].

Hence although over-fishing could not be coupled with mass mortality, the co-occurence of these two mortality sources could dramatically depress population recovery. Indeed during the last two decades a reduction by $\frac{2}{3}$ of the overall Mediterranean fishing yield has been recorded [SA01], the risk is not an ecological extinction for this species; however an economical extinction is possible (i.e. a population of a slow growing species, with positive density dependence, can survive to over-fishing but cannot reach again a size/age structure and a density suitable for commercial harvesting). In this framework models suitable to project red coral population trends overtime could supply highly useful insights into population dynamics and suggestions for species conservation.

### 1.1.3 The sites of Portofino and Cap de Creus

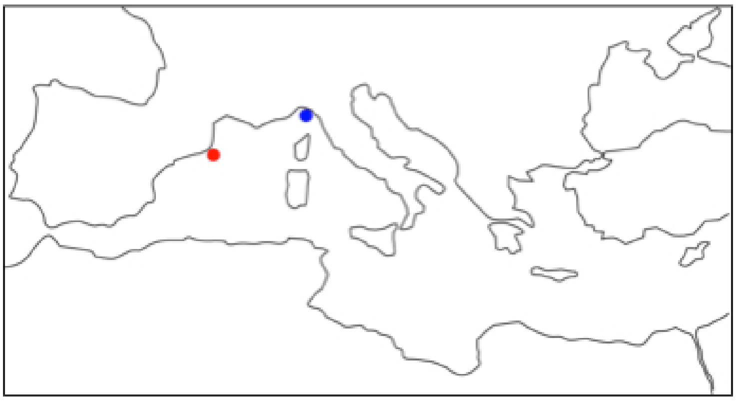


Figure 1.2: The two spots in the Mediterranean map identify the two sampling areas (blue spot: Portofino; red spot: Cap de Creus).

Our data come from two Mediterranean sites: Portofino (Eastern Ligurian Sea $44°18', 18$ N $09°12', 83$ E, Italy) and Cap de Creus (Costa Brava, North-Western Mediterranean; $42°29', 21$ N; $3°30', 18$E, Spain).
The MPA of Portofino consists of 13 km of coast and $3.46\text{km}^2$. This area has been included in the SPAMI (Specially Protected Areas of Mediterranean Importance) list in 2002 because of the representativeness of its coralligenous community. This community flourishes on the submerged cliff and on the rocks, while coralligenous platforms develop off the rocky bottoms, at a depth of 60 to 100 m.
This protected area has been established with the law of the Italian Environment Department (April 26th 1999) and includes the Municipalities of Camogli, Portofino, and S. Margherita Ligure (all belonging to the Province of Genova). The establishment of this MPA is provided by two national laws: the Legislation

regarding the defence of the sea (n° 979 of December 31st 1982) and the Outline Law on protected areas ((n° 394 of December 6th 1991). No harvesting activity is allowed in the whole, only professional fishing is regulated by the MPA management plan. The aims of the MPA Portofino are both the preservation of the sea biodiversity and biological resources and the promotion and the enhancement of the local economic activities, provided that they are compatible with the importance of naturalistic aspects and landscapes of the area.
The MPA of Cap de Creus, with its $30.5\text{km}^2$, falls within the Marine and Coastal Protected Area of Cap de Creus. It was established in 1999 according to the Regional Law n$^\circ 4 - 98$, March 1998, and declared SPAMI in 2001. Its Managed Body is constituted by the Public Institution "Natural Park of Cap de Creus" but a proper Management Plan still lacks. In the meantime harvesting and fishing activities are regulated by national and regional laws. In particular, the Spanish law gives limits on the harvested (in weight and maximum diameter).
Since 2005 C. rubrum harvesting has been totally banned in the three partial natural reserves (RNP) of the MPA of Cap de Creus National Park at any depth all over the year. This decision has been enforced due to a drastic reduction in biomass and density of Corallium rubrum in the Cap de Creus MPA (Ordinance 293/2005) caused by the past intensive harvesting. Nonetheless C. rubrum is still harvested all along the MPA coast because the lack of any surveillance. This management incongruence has serious negative consequences on the population structure of the shallow populations of Cap de Creus.

## 1.2 Mathematical introduction

This work is based on the theory concerning structured population dynamics. We will construct discrete and continuous models to explain the evolution of Portofino and Cap de Creus populations. Hence in this section we introduce two linear models (a discrete model and a continuous one), to become familiar with ideas and notations. Moreover we begin to analyze them presenting some basic results (without proof) that will be useful through the whole work. Finally we present a generalization of the Grönwall lemma, that will be useful in the sequel. Our main references are two books by M. Iannelli [Ian94] and J.M. Cushing [Cus98], upon which is based this introduction and a part of this work.
In the first section of this section we show a linear discrete model, defining the *Leslie matrix* and displaying two basic results. In the second section we present a continuous linear model and its reformulation as a Volterra integral equation.

### 1.2.1 Discrete linear models

Suppose that individuals of a population are categorized into a finite number of classes $a_\dagger$ (i.e. chronological age) and let $u_i^n$ denote the number of individuals of class $i$ at time $n$. We have to define the inherent parameters of the model. Therefore we call $0 \leq \sigma_{i,j} \leq 1$ the fraction of $j$-class individuals expected to survive and move to class $i$ per unit of time and $b_{i,j} \geq 0$ the expected number of $i$-class offspring per $j$-class individual per unit of time. If only birth and death processes are allowed we have that, setting $\mathbf{u}^n = (u_1^n, \ldots, u_{a_\dagger}^n)$ :

$$\mathbf{u}^{n+1} = (T + F)\mathbf{u}^n = P\mathbf{u}^n \tag{1.1}$$

where $T$ and $F$ are the two $a_\dagger \times a_\dagger$ matrix containing survival and birth rate respectively. Since the model is linear, the matrix $P$ is constant and not depends on time (i.e. the index $n$). We observe that, when the structure is based on age, survival rates and birth rates take the form:

$$\sigma_{i,j} = 0 \text{ for } j \neq i - 1, \qquad \sigma_{i,i-1} = \sigma_i, \tag{1.2}$$

$$b_{i,j} = 0 \text{ for } i \neq 1, \qquad b_{1,j} = b_j; \tag{1.3}$$

therefore the matrix $P$ is a so-called Leslie matrix

$$P = \begin{bmatrix} f_1 & f_2 & \cdots & f_{a_\dagger} \\ \sigma_1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & \sigma_{a_\dagger - 1} & 0 \end{bmatrix}.$$

It is reasonable that, in order to analyze the behaviour of such populations, we have to check the eigenvalues and eigenvectors of the matrix $P$. Now we introduce two preliminary definitions and then present a basic result proved in [Cus98].

**Definition 1.2.1.** *Let $A$ be a square matrix; then $A$ is irreducible if it is not similar via a permutation to a block triangular matrix.*

From the Perron-Frobenius Theorem [Mey00], a non-negative irreducible matrix has a dominant positive algebraically simple eigenvalue $r$, with positive right and left eigenvectors. Moreover if $r$ is strictly dominant then we say that the matrix $A$ is *primitive*. Now, setting

$$|\mathbf{u}| = \sum_{i=1}^{a_\dagger} |u_i|,$$

we have the following theorem:

**Theorem 1.2.1.** *Suppose that the nonnegative matrix $P$ is irreducible and primitive. Let $r$ be its strictly dominant eigenvalue and let $\mathbf{v} > 0$ be an associated eigenvector. Let $\mathbf{u}^{n+1}$ be the solution of the linear matrix equation $\mathbf{u}^{n+1} = P\mathbf{u}^n$, with an initial state satisfying $\mathbf{u}(0) \geq 0$. Then:*

*i)* $\lim_{n\to\infty} |\mathbf{u}^n| = 0$ *if* $r < 1$ *and* $\lim_{n\to\infty} |\mathbf{u}^n| = \infty$ *if* $r > 1$.

*ii)* $\lim_{n\to\infty} \dfrac{\mathbf{u}^n}{|\mathbf{u}^n|} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$.

The above result connects the evolution of (1.1) to the dominant eigenvalue of the matrix $P$ (if it exists). Another important quantity, also from a biological point of view is $\mathcal{R}_0$, the so-called *basic reproduction number*. This is defined to be the expected number of offspring for every individual during its lifetime. For a Leslie model a formula for $\mathcal{R}_0$ is obtained by summing the products of the expected number of offspring $b_i$ for each age class and the probability to reaching that age class (i.e. $\prod_{j=1}^{i-1} \sigma_i$). Thus,

$$\mathcal{R}_0 = \sum_{i=1}^{a_\dagger} b_i \prod_{j=1}^{i-1} \sigma_i. \tag{1.4}$$

For a general matrix $P = T + F$ the basic reproduction number can be defined as follows:

**Definition 1.2.2.** *Let $P = T + F$. Suppose that $I - T$ is invertible and that $F(I - T)^{-1}$ has a positive, (algebraically) simple, strictly dominant eigenvalue $\mathcal{R}_0$ and an eigenvector $\mathbf{u} \geq 0$; then $\mathcal{R}_0$ is called the basic reproduction number for $P$.*

With the previous general definition, we can present another basic result that describes the behaviour of the system (1.1) with respect to the basic reproduction number.

**Theorem 1.2.2.** *Let $P = T + F$ a nonnegative matrix. Suppose that $P$ has a positive, simple, strictly dominant eigenvalue $r$. Assume further that $I - T$ is invertible and $F(I - T)^{-1}$ has a positive, simple, strictly dominant eigenvalue $\mathcal{R}_0$ with nonnegative eigenvector $\mathbf{u}$ such that $(I - T)^{-1}\mathbf{u} > 0$. Then:*

*i)* $r < 1$ *if and only if* $\mathcal{R}_0 < 1$;

*ii)* $r > 1$ *if and only if* $\mathcal{R}_0 > 1$.

The above Theorem is useful also for nonlinear models presented through this work (see chapter 3, §1): indeed it can be applied to the matrix of a linearized system, in order to investigate the stability of an equilibrium.

### 1.2.2 Continuous linear models

Here we deal with a simple linear continuous model to describe an age structured population. In this case we work with a density function, that expresses the number of individuals of age $a$ at time $t$:

$$p(a,t), \quad a \in [0, a_\dagger], \quad t \geq 0,$$

where $a_\dagger$ denotes again the maximum age of an individual, which we assume to be finite. Thus the total population at time $t$ is

$$P(t) = \int_0^{a_\dagger} p(a,t)da.$$

In this framework *fertility* and *mortality* are represented by two functions (depending on age $a$ only) $\beta(a)$ and $\mu(a)$. Now we can define the *survival probability* (i.e. the probability for an individual to survive to age $a$):

$$\Pi(a) = e^{-\int_0^a \mu(\sigma)d\sigma}, \quad a \in [0, a_\dagger].$$

Analogously to the discrete case we can also define the basic reproduction number as

$$\mathcal{R}_0 = \int_0^{a_\dagger} \beta(a)\Pi(a)da.$$

Now we derive the basic equations that describe the evolution of the above population. These equations arise as a consequence of the balance of births and deaths along time. Consider the function

$$N(a,t) = \int_0^a p(\sigma,t)d\sigma$$

which represents the number of individuals that, at time $t$, have age less then or equal to $a$. Then we have, for $h > 0$,

$$\begin{aligned} N(a+h, t+h) &= N(a,t) + \int_t^{t+h} B(s)ds \\ &\quad - \int_0^h \int_0^{a+s} \mu(\sigma)p(\sigma, t+s)d\sigma ds \end{aligned} \tag{1.5}$$

where $B(t) = \int_0^{a_\dagger} \beta(a,t)p(a,t)da$ is the *total birth rate.* Indeed, in (1.5), the second term on the right gives the number of newborns in the time interval $[t, t+h]$ : these have age less or equal to $h$, consequently they have to be included in $N(a+h, t+h)$. Moreover, since:

$$\int_0^{a+s} \mu(\sigma)p(\sigma, t+s)d\sigma$$

is the number of individuals who die at time $t+s$, with age less or equal to $a+s$, the third terms on the right of (1.5) gives the loss from the initial group of $N(a,t)$ individuals and from the newborns, through the time interval $[t, t+h]$. Now differentiating (1.5) with respect to $h$ and setting $h=0$ we obtain:

$$p(a,t) + \int_0^a p_t(\sigma,t)d\sigma = B(t) - \int_0^a \mu(\sigma)p(\sigma,t)d\sigma. \tag{1.6}$$

Now, setting $a=0$, we get:

$$p(0,t) = B(t)$$

and differentiating (1.6) with respect to $a$ :

$$p_t(a,t) + p_a(a,t) + \mu(a)p(a,t) = 0$$

Then the equations of the linear model for $a \in [0, a_\dagger]$ and $t \in [0, T]$ are the following:

$$\begin{cases} p_t(a,t) + p_a(a,t) + \mu(a)p(a,t) = 0 \\ p(0,t) = \int_0^{a_\dagger} \beta(\sigma)p(\sigma,t)d\sigma \\ p(a,0) = p_0(a) \end{cases} \tag{1.7}$$

Now we derive a different formulation of (1.7) where the unknown function is $B(t)$. Applying the method of characteristics to (1.7) we obtain:

$$p(a,t) = \begin{cases} p_0(a-t)\frac{\Pi(a)}{\Pi(a-t)} & \text{if } a \geq t \\ B(t-a)\Pi(a) & \text{if } a < t. \end{cases} \tag{1.8}$$

Now from the above formula we get a Volterra equation for $B(t)$ indeed inserting (1.8) into (1.7) we get that:

$$B(t) = F(t) + \int_0^t K(t-s)B(s)ds \tag{1.9}$$

where

$$F(t) = \int_t^\infty \beta(a)p_0(a-t)\frac{\Pi(a)}{\Pi(a-t)}da, \tag{1.10}$$

$$K(t) = \beta(t)\Pi(t), \tag{1.11}$$

where $t \geq 0$, and the functions $\beta, \Pi, p_0$ are extended by zero outside the interval $[0, a_\dagger]$. Equation (1.9) is known as the *renewal equation.*

The above procedure, that transforms the problem (1.7) into (1.9), is a standard argument within the study of continuous structured populations and we will

repeat it in chapter 3 in the case of some nonlinear models.
Concerning the existence and uniqueness of solutions of (1.9) we refer to [Ian94], Ch.I §4. The asymptotic behaviour of $B(t)$ can be analyzed through the Laplace transform of the kernel $K(t)$, in particular studying the roots of the equation $\hat{K}(\lambda) = 1$. This equation is analyzed in Appendix A §1, for a comprehensive treatment of the asymptotic behaviour we refer to [Ian94],Ch.I §5.

### 1.2.3 Grönwall lemma

In this section we prove the Grönwall lemma and its generalization; this results will be useful in the sequel.

**Lemma 1.2.1** (Grönwall). *Let I denote an interval of the real line of the form* $[a, \infty)$, $[a, b]$ *or* $[a, b)$ *with* $a < b$. *Let* $f, g, u$ *be real-valued functions defined on* $I$. *Suppose that* $g$ *and* $u$ *are continuous and* $f$ *is integrable on every closed and bounded subinterval of* $I$. *If*

$$u(t) \leq f(t) + \int_a^t g(s)u(s)ds, \quad \forall t \in I, \tag{1.12}$$

*then:*

*i) If* $g$ *is non-negative then*

$$u(t) \leq f(t) + \int_a^t f(s)g(s)e^{\int_s^t g(r)dr}ds, \quad \forall t \in I. \tag{1.13}$$

*ii) If moreover,* $f$ *is increasing, then*

$$u(t) \leq f(t)e^{\int_a^t g(s)ds}, \quad \forall t \in I \tag{1.14}$$

*Proof.* Concerning the first part, we define:

$$v(s) = e^{-\int_a^s g(r)dr} \int_a^s g(r)u(r)dr, \quad s \in I.$$

Differentiating $v(s)$, we obtain, for $s \in I$ :

$$\begin{aligned} v'(s) &= \left(u(s) - \int_a^s g(r)u(r)dr\right) g(s)e^{-\int_a^s g(r)dr} \\ &\leq f(s)g(s)e^{-\int_a^s g(r)dr} \end{aligned}$$

Since $g$ and the exponential are non-negative and $v(a) = 0$, integrating this inequality from $a$ to $t$ we obtain

$$v(t) \leq \int_a^t f(s)g(s)e^{-\int_a^s g(r)dr}ds.$$

Now, using the definition of $v$ and the previous inequality we obtain

$$\begin{aligned}
\int_a^t g(s)u(s)ds &= e^{\int_a^t g(r)dr}v(t) \\
&\leq e^{\int_a^t g(r)dr}\int_a^t f(s)g(s)e^{-\int_a^s g(r)dr}ds \\
&\leq \int_a^t f(s)g(s)e^{\int_s^t g(r)dr}ds.
\end{aligned}$$

Substituting this result in (1.12) we obtain (1.13).
If $f(s) \leq f(t)$ for $s < t$, then

$$\begin{aligned}
u(t) &\leq f(t) + \int_a^t f(s)g(s)e^{\int_s^t g(r)dr}ds \\
&\leq f(t)\left(1 - e^{\int_s^t g(r)dr})\Big|_{s=a}^{s=t}\right) \\
&= f(t)e^{\int_a^t g(s)ds},
\end{aligned}$$

where we have used that $\frac{d}{ds}e^{\int_s^t g(r)dr} = -g(s)e^{\int_s^t g(r)dr}$. □

The following result is taken from [Ama84] and generalize the Grönwall lemma to inequalities involving the convolution product of the analyzed function with a decreasing function:

**Lemma 1.2.2.** *Suppose that* $0 \leq t_0 \leq T$, *that* $b \in L^1([0,T], \mathbb{R}^+)$ *is a decreasing function and* $a, u \in C([0,T], \mathbb{R}^+)$. *Moreover, suppose that*

$$u(t) \leq a(t) + \int_{t_0}^t b(t-\tau)u(\tau)d\tau \quad \forall t \in [t_0, T]. \tag{1.15}$$

*Then there exists a constant* $\beta > 0$, *depending only on* $b$, *such that*

$$u(t) \leq 2a^*(t)e^{\beta(t-t_0)} \quad \textit{for } t_0 \leq t \leq T, \tag{1.16}$$

*where* $a^*(t) = \max_{t_0 \leq s \leq T} a(s)$.

*Proof.* Choose $\epsilon > 0$, such that $\int_0^\epsilon b(\tau)d\tau \leq \frac{1}{2}$ and let $u(t) \doteqdot 0$ for $t < t_0$. Then, for $t_0 \leq s \leq t \leq T$,

$$\begin{aligned} u(s) &\leq a(s) + \int_{t_0}^{s-\epsilon} b(s-\tau)u(\tau)d\tau + \int_{s-\epsilon}^{s} b(s-\tau)u(\tau)d\tau \\ &\leq a^*(s) + b(\epsilon)\int_{t_0}^{s-\epsilon} u(\tau)d\tau + \int_0^\epsilon b(r)u(s-r)dr \\ &\leq a^*(s) + b(\epsilon)\int_{t_0}^{s} u(\tau)d\tau + \frac{1}{2}u^*(s), \end{aligned}$$

where $u^*(t) = \max_{t_0 \leq s \leq T} u(s)$. Hence

$$u^*(t) \leq 2a^*(t) + \beta \int_0^t u^*(\tau)d\tau, \quad t_0 \leq t \leq T,$$

where $\beta = 2b(\epsilon)$. Now (1.16) follows from the previous lemma, since $a^*$ is increasing. □

# Chapter 2

# Data analysis and age distribution

In this chapter we present and analyze a significant part of the data contained in [Vie09].
Our purpose is to deduce from these data some ideas concerning the two populations of Portofino and Cap de Creus and possibly to estimate several significant parameters in order to construct a mathematical model.
For this study various type of data are relevant:

- *Growth data* (diameter, circular crown areas and age);

- *Reproductive data* (number of polyps and their fecundity).

Concerning the reproductive process, due to lack and non-homogeneity of available data, for these quantities we just consider the estimation done in [Vie09].
We deal instead with the growth estimation (for year) of the red corals within these two populations. In particular we analyze different types of data and construct a regression model to express the diameter growth with respect to a year.
Then we use a larger set of data containing only diameters of the colonies and try to obtain a distribution of the colonies through their age. Unfortunately, for the small quantity and non-homogeneity of the samples, the two obtained distributions seem not satisfactory and have to be fitted to a survival distribution.

## 2.1 Growth data

For this type of data we have three levels of accuracy:

- The *first* level consists of circular crown areas of colonies sections for every year in their lifespan: this dataset includes 23 colonies from Portofino and 10 from Cap de Creus.

- The *second* level consists of couples diameter-age relative to several colonies: the sampling consists of 75 colonies from Portofino and 44 from Cap de Creus.
- The *third* level consists of the only diameter relative to larger number of colonies: 472 and 143 respectively from Portofino and Cap de Creus.

Moreover we know the number of *recruits* (i.e. colonies one year old) photographed in the squares chosen for the samples.
Now we explain some information concerning the sampling procedure for our datasets: in each area (i.e. Portofino and Cap de Creus) 4 sites have been randomly chosen. In each site, 3 squares of 20 x 20 cm have been randomly chosen. For each square some pictures have been taken. Next, colonies have been selectively harvested with chisel. Then sampling squares have been again photographically sampled to estimate number of recruits.

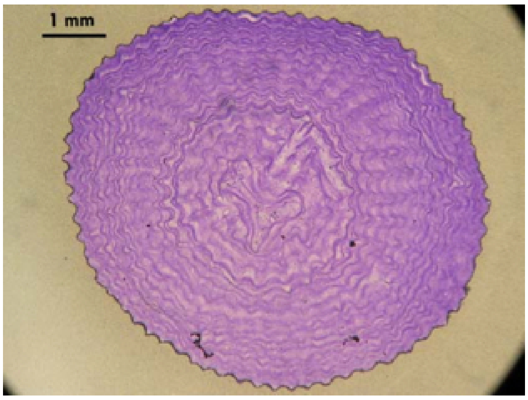



Figure 2.1: Basal section of a colony.

Then harvested colonies have been cut at basal diameter, and these sections have been photographed under microscope (see figure 2.1). With these pictures one can estimate the age of the colonies by summing the number of the growth rings and adding 4 years to the number of growth rings counted [MGHP04], in this way the second level dataset has been obtained. Concerning the first level, for every colony, the area between each couple of consecutive growth rings has been appreciated using photographic programs. The third level of data consists of measures (in mm) of diameters of all the harvested colonies (measured by using a caliper).
A first preliminary remark concerning these two datasets is the following: there

no colonies with diameter less than 2 mm and moreover the ones with diameter less than 3 mm are heavily understimated. This is due to sampling problems: scraping the bigger colonies, also the little ones come off from the depth and get lost. This defect caused some problems in determination of age structures for the two populations, that appear empty in some age groups.

## 2.2 Analysis of the data

The aim of this section is to analyze the first two levels of data, and to estimate the medium growth rate of colonies diameter with respect to their age. We first analyze the first level of data to set up some ideas and then we study the second level to supply a model for the growth of diameters with respect to the age. All data analysis throughout this section are performed using R program.

### 2.2.1 Analysis of the first level of data

At this level every colony has its own dataset containing the areas of circular annual crowns for every year in life span (except the first 4 years).
For every colony we observed the values of circular annual crowns areas through years in a plot. As an example we take two colonies (one from Portofino and one from Cap de Creus) and perform two linear models of their growth.

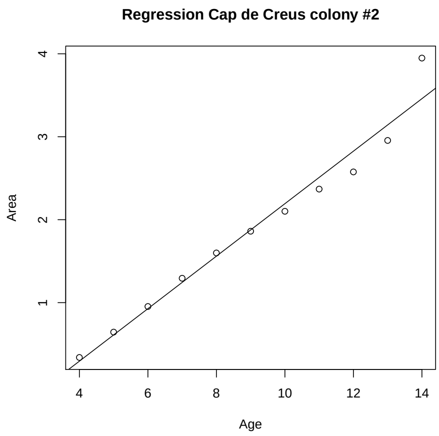
```
> cdc2<- read.table("cdc2.txt",header=T,dec=",")
> str(cdc2)
```

```
'data.frame':        11 obs. of  3 variables:
 $ Age  : int  4 5 6 7 8 9 10 11 12 13 ...
 $ Crown: num  0.339 0.306 0.309 0.34 0.305 ...
 $ Area : num  0.339 0.645 0.954 1.294 1.599 ...
```

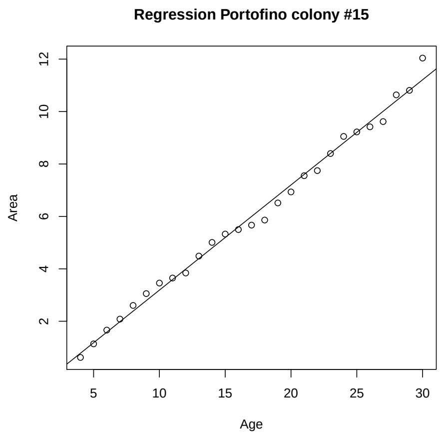
We see that the data, consisting in circular annual crown areas for every age, can be used to evaluate total areas. Now we perform a linear regression model for this colony and do the same thing with a Portofino colony.

```
> regression_cdc2=lm(Area ~ Age,data=cdc2)
> ptf15<- read.table("ptf15.txt",header=T,dec=",")
> regression_ptf15=lm(Area ~ Age,data=ptf15)
```

Then we draw two graphs for these regression models:




(a)




(b)

The observed data suggest that:

- colonies from two sites have a similar growth rate;
- the section growth is nearly linear with respect to the years.

From these observations we presume that for all colonies from both sites, growth can be linear with respect to the years, namely the circular annual crowns have the same area through the years. Hence we take the whole first level dataset and construct a linear model to see which variables influence circular annual crowns. Since colonies were sampled in the same year, the year of formation of all crowns can be deduced, too.

```
> crowns<-read.table("total_crowns.txt",header=T,dec=",")
> str(crowns)

'data.frame':        514 obs. of  5 variables:
 $ Sample: int  1 1 1 1 1 1 1 1 1 1 ...
 $ Year  : int  1 2 3 4 5 6 7 8 9 10 ...
 $ Age   : int  5 6 7 8 9 10 11 12 13 14 ...
 $ Crown : num  0.2 0.17 1.23 1.46 0.17 1.19 1.25 0.17 1.05 ...
 $ Site  : Factor w/ 2 levels "cdc","ptf": 2 2 2 2 2 2 2 2 2 ...

> summary(crowns)

     Sample            Year            Age             Crown
 Min.   :  1.00   Min.   : 1.00   Min.   : 5.00   Min.   :0.0200
 1st Qu.:  6.00   1st Qu.:20.00   1st Qu.: 8.00   1st Qu.:0.3025
```

```
 Median : 14.00   Median :26.00   Median :12.00   Median :0.6200
 Mean   : 29.34   Mean   :24.11   Mean   :13.89   Mean   :0.8601
 3rd Qu.: 21.75   3rd Qu.:30.00   3rd Qu.:18.00   3rd Qu.:1.2000
 Max.   :510.00   Max.   :33.00   Max.   :37.00   Max.   :5.2600
```

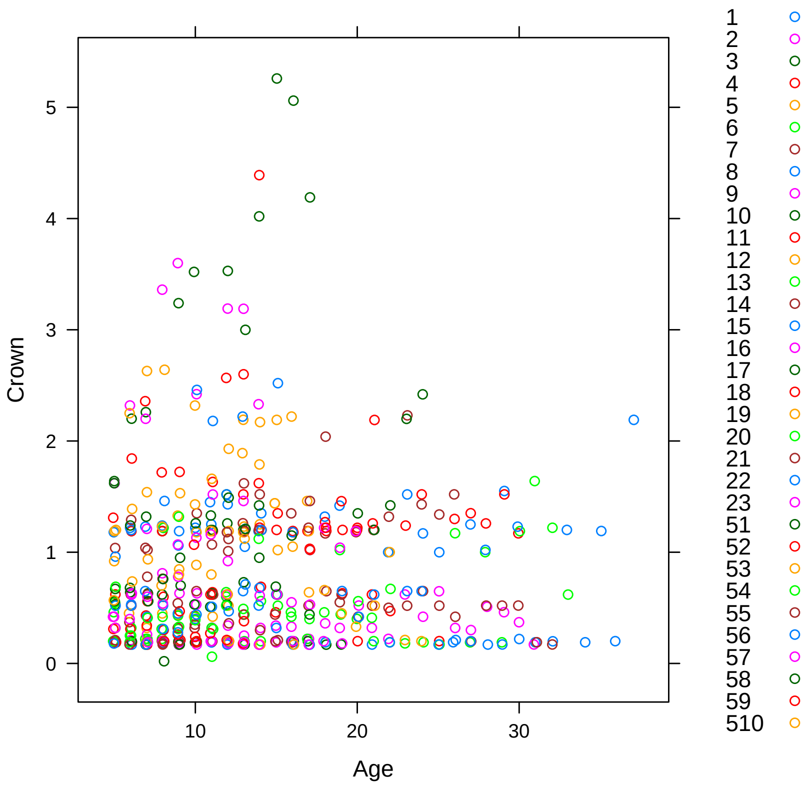



Figure 2.2: Circular annual crowns: Portofino (1-23), Cap de Creus (51-510)

Here we have some information about the dataset and a plot of all data (note that in the graph and in the next computations output the sampling codex for Cap de Creus colonies starts with 5). From the above graph we see that circular annual crowns seem to be nearly constant for every colony (but different for two different colonies). To realize this in more detail we present a first regression model, where circular annual crown is the regressand and sampled colonies with their ages are regressors.

```
> regression_crowns=lm(Crown~Age + factor(Sample), data=crowns)
> summary(regression_crowns)
```

```
Call:
lm(formula = Crown ~ Age + factor(Sample), data = crowns)

Residuals:
     Min       1Q   Median       3Q      Max
-1.90077 -0.30628 -0.00413  0.29003  2.16390

Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)        0.341939   0.114020   2.999 0.002850 **
Age                0.023356   0.003654   6.393 3.87e-10 ***
factor(Sample)2   -0.364577   0.126204  -2.889 0.004042 **
factor(Sample)3    2.501910   0.162804  15.368  < 2e-16 ***
factor(Sample)4    0.169708   0.127692   1.329 0.184466
factor(Sample)5    0.142679   0.162804   0.876 0.381260
factor(Sample)6   -0.239473   0.144587  -1.656 0.098323 .
factor(Sample)7    0.281491   0.134860   2.087 0.037389 *
factor(Sample)8    0.487175   0.179869   2.709 0.007000 **
factor(Sample)9    2.011175   0.179869  11.181  < 2e-16 ***
factor(Sample)10   0.101393   0.139335   0.728 0.467159
factor(Sample)11   0.112663   0.147567   0.763 0.445558
factor(Sample)12   1.301910   0.162804   7.997 9.63e-15 ***
factor(Sample)13  -0.287435   0.123536  -2.327 0.020395 *
factor(Sample)14  -0.345105   0.124821  -2.765 0.005915 **
factor(Sample)15  -0.312215   0.127692  -2.445 0.014841 *
factor(Sample)16  -0.262216   0.154415  -1.698 0.090132 .
factor(Sample)17   0.162298   0.187398   0.866 0.386889
factor(Sample)18   0.167369   0.147567   1.134 0.257281
factor(Sample)19   0.004893   0.139335   0.035 0.972003
factor(Sample)20   0.120566   0.238071   0.506 0.612788
factor(Sample)21   0.155891   0.158383   0.984 0.325482
factor(Sample)22   0.851152   0.167770   5.073 5.60e-07 ***
factor(Sample)23  -0.144519   0.150828  -0.958 0.338458
factor(Sample)51   0.074987   0.162804   0.461 0.645299
factor(Sample)52  -0.232825   0.179869  -1.294 0.196144
factor(Sample)53   0.251187   0.187398   1.340 0.180751
factor(Sample)54  -0.080825   0.179869  -0.449 0.653378
factor(Sample)55   0.433409   0.187398   2.313 0.021157 *
factor(Sample)56  -0.010825   0.179869  -0.060 0.952035
factor(Sample)57   0.255175   0.179869   1.419 0.156643
factor(Sample)58   0.690076   0.187398   3.682 0.000257 ***
```

```
factor(Sample)59   1.557175   0.179869   8.657  < 2e-16 ***
factor(Sample)510  0.880860   0.173403   5.080 5.42e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4845 on 480 degrees of freedom
Multiple R-squared: 0.6159,       Adjusted R-squared: 0.5895
F-statistic: 23.33 on 33 and 480 DF,  p-value: < 2.2e-16
```

From this computation we notice that dependence from the age is negligible with respect of that from the colony. Hence we construct a second regression model, where the total area (i.e. the sum of all circular annual crown areas) for every year is taken as regressand and the other variables (i.e. colonies and ages) are regressors.

```
> total_regression=lm(Area~Age + factor(Sample), data=total_areas)
> summary(total_regression)

Call:
lm(formula = Area ~ Age + factor(Sample), data = total_areas)

Residuals:
     Min       1Q   Median       3Q      Max
-12.0223  -1.1748  -0.1119   1.1182  19.1101

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)       -0.74138    0.60720  -1.221 0.222655
Age                0.70688    0.01909  37.037  < 2e-16 ***
factor(Sample)2   -5.56396    0.69333  -8.025 6.96e-15 ***
factor(Sample)3   11.10613    0.88072  12.610  < 2e-16 ***
factor(Sample)4   -0.70531    0.70113  -1.006 0.314910
factor(Sample)5   -2.86495    0.88072  -3.253 0.001217 **
factor(Sample)6   -4.82664    0.78863  -6.120 1.86e-09 ***
factor(Sample)7   -0.49993    0.73848  -0.677 0.498734
factor(Sample)8    0.53067    0.96448   0.550 0.582412
factor(Sample)9    8.40842    0.96448   8.718  < 2e-16 ***
factor(Sample)10  -2.11614    0.76163  -2.778 0.005662 **
factor(Sample)11  -2.05963    0.80386  -2.562 0.010686 *
factor(Sample)12   7.47684    0.88072   8.489 2.25e-16 ***
factor(Sample)13  -6.38949    0.67928  -9.406  < 2e-16 ***
factor(Sample)14  -5.75890    0.68606  -8.394 4.60e-16 ***
```

```
factor(Sample)15  -5.27920    0.70113  -7.530 2.31e-13 ***
factor(Sample)16  -4.62878    0.83862  -5.519 5.40e-08 ***
factor(Sample)17  -2.37066    1.00057  -2.369 0.018191 *
factor(Sample)18  -2.80035    0.80386  -3.484 0.000537 ***
factor(Sample)19  -1.07740    0.76163  -1.415 0.157795
factor(Sample)20  -1.20807    1.22833  -0.984 0.325825
factor(Sample)21  -2.21216    0.85861  -2.576 0.010260 *
factor(Sample)22   2.12661    0.90536   2.349 0.019207 *
factor(Sample)23  -3.54635    0.82046  -4.322 1.85e-05 ***
factor(Sample)51  -1.88528    0.88072  -2.141 0.032775 *
factor(Sample)52  -3.77120    0.96448  -3.910 0.000105 ***
factor(Sample)53  -1.29796    1.00057  -1.297 0.195138
factor(Sample)54  -2.74771    0.96448  -2.849 0.004563 **
factor(Sample)55  -0.62430    1.00057  -0.624 0.532944
factor(Sample)56  -1.98520    0.96448  -2.058 0.040065 *
factor(Sample)57  -1.55262    0.96448  -1.610 0.108056
factor(Sample)58   1.72518    1.00057   1.724 0.085275 .
factor(Sample)59   5.30378    0.96448   5.499 6.03e-08 ***
factor(Sample)510  2.14935    0.93305   2.304 0.021646 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.708 on 513 degrees of freedom
Multiple R-squared: 0.8225,       Adjusted R-squared: 0.8111
F-statistic: 72.02 on 33 and 513 DF,  p-value: < 2.2e-16
```

Looking at the importance of factors contained in these models and others of this type, it emerged that:

1. the area of circular annual crowns of a colony strictly depend on itself, in particular several colonies (4 out of 23 in Portofino and 3 out of 10 in Cap de Creus) had a higher growth rate;

2. the growth rate is furthermore affected by other situational factors (i.e. some years are characterized by an higher generalized growth rate);

3. the growth rate is affected by environmental factors (i.e. average growth rate of Portofino colonies is slightly bigger);

4. without specific environmental or situational factors the growth of section areas is nearly linear with respect to age for colonies from both sites.

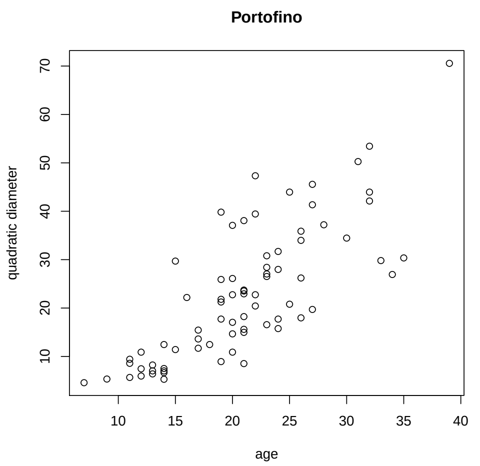
These considerations (the first in particular) will be a guideline in the construction of our models for these populations.
By the way, we conclude that the growth rate of section area is nearly linear with respect to the years, and similarly for colonies from both sites.

### 2.2.2 Analysis of the second level and growth rate

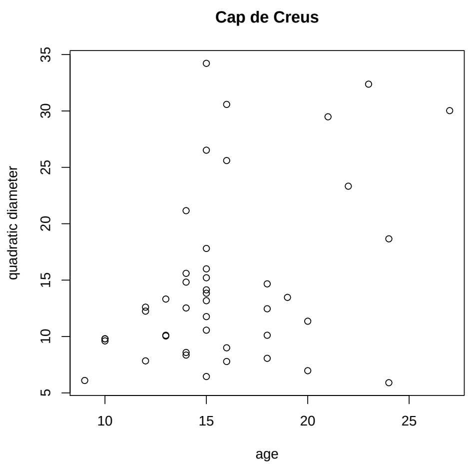
From the analysis performed on the first level of data, we used the second level to determine the growth rate of diameters with respect to the age. We consider the formula to calculate the area of a disk from its diameter, and suppose that circular annual crowns have a constant section through the years. Hence we can express the age of a colony as a function of quadratic diameter: in practice, if we call $A_4$ the area of the organic matter settled in the first four years of life, we have:

$$\frac{\pi}{4}d^2 = A_4 + \sigma(age - 4)$$

where $d$ is section diameter. Therefore is reasonable to construct another linear regression model to express the age of colonies from quadratic diameter.
These are two datasets (one pertaining to Portofino and the other to Cap de Creus) in a graph:




(a)




(b)

we see that the Cap de Creus dataset consists in few non-homogeneous points; on the contrary, the dataset from Portofino looks better. Hence to construct the regression model we use both datasets:

```
> total_level2<-read.table("total_level2.txt",header=T,dec=",")
> str(total_level2)
```

```
'data.frame':        119 obs. of  3 variables:
 $ Diameter: num  2.47 3.13 3.13 3.1 2.8 3.5 3.55 3.17 3.65 ...
 $ Age     : int  9 10 10 10 12 12 12 13 13 13 ...
 $ Site    : Factor w/ 2 levels "cdc","ptf": 1 1 1 1 1 1 1 1 ...

> total_level2$Quadratic_diameter=(total_level2$Diameter)^2
> summary(regression_quadratic_diameter)

Call:
lm(formula = Quadratic_diameter ~ Age, data = total_level2)

Residuals:
     Min       1Q   Median       3Q      Max
-21.4176  -5.2325  -0.0047   3.4423  23.0073

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -8.6225     2.4558  -3.511 0.000635 ***
Age           1.4977     0.1224  12.239  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 8.32 on 117 degrees of freedom
Multiple R-squared: 0.5614,       Adjusted R-squared: 0.5577
F-statistic: 149.8 on 1 and 117 DF,  p-value: < 2.2e-16
```

The regression model just produced (see figure 2.3)is what we need to generate, from the third level of data, an age distribution for the two populations. We observe that:

1. as for the first level, we obtain a good fit using datasets from both sites, hence growth rates are similar;

2. even this second level of data includes many colonies that had grown more than the others, as in the previous dataset (this phenomenon is mainly frequent within Portofino colonies).

Therefore we have compared the first level data with results obtained from second. For this reason we have rescaled the second level growth rate by a factor $\frac{\pi}{4}$, and took the average circular annual crown area within the first level. We have seen that the second level growth rate is slightly bigger; this result was expected, since first level measures concerned the areas of circular crowns between rings

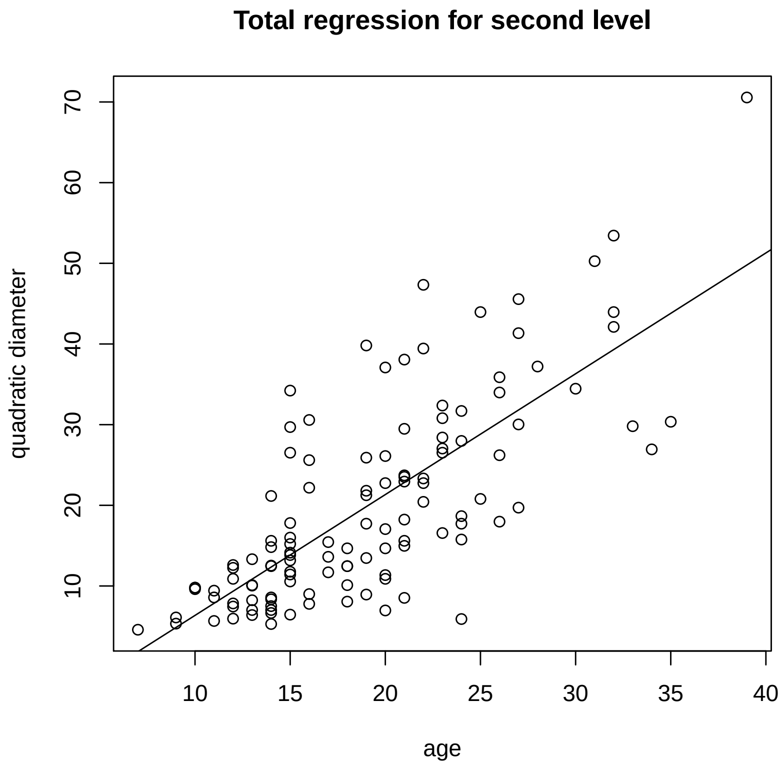



Figure 2.3

of inorganic matter: these rings have a surface, not included in these data, so that the area of the sections is slightly smaller than as stated within the analysis of the first level. Therefore these datasets are consistent and it is reasonable to use the second one to estimate the growth rate of colonies from both sites.

## 2.3 Age distribution

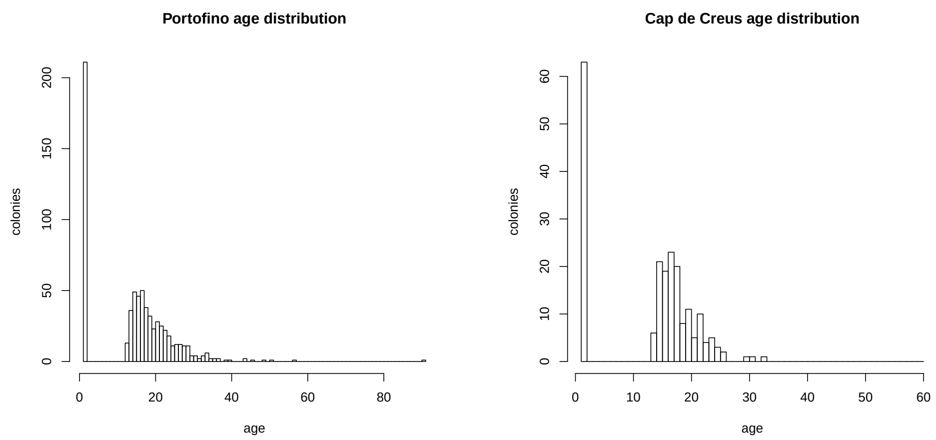
We used the regression model described in the previous section to appreciate the age of colonies within the third level of data. So we have obtained an age distribution for Cap de Creus and Portofino populations:

```
> ptf_diameters=read.table("ptf_diameters.txt",header=T,dec=",")
> ptf_diameters$Quadratic_diameter=(ptf_diameters$Diameter)^2
> str(ptf_diameters)
```

```
'data.frame':        472 obs. of  2 variables:
 $ Diameter          : num  1.15 1.25 1.48 1.53 1.64 ...
 $ Quadratic_diameter: num  1.32 1.56 2.19 2.34 2.69 ...

> ptf_diameters$Age=predict(regression_age,ptf_diameters)
```

In an analogous way we treat Cap de Creus data. Hence we add the *recruits*(211 for Portofino and 63 for Cap de Creus) to these two distributions, obtaining these histograms:




(a) (b)

Figure 2.4: Age distributions

We notice that both distributions present a lack of information pertaining to the first age/diameter classes (except for the recruits, which have been measured in a different way). Furthermore the two distributions do not follow exactly the expected structure (this is due to the non-homogeneneity of data that caused a loss of accuracy in the regression model). Therefore we have to fit them to a survival function to appreciate the survival rates from an age to the following one.

# Chapter 3

# Mathematical models

In this chapter we introduce several mathematical models to analyze the red coral populations of Portofino and Cap de Creus.
In the first section we deal with discrete models, in the second we define a continuous model and in the third we construct a diffusion equation that models the spread of the planulae among the surrounding field.

## 3.1 Discrete models

This chapter is devoted to introduce two appropriate discrete models in order to study a Corallium Rubrum population. Several works have described these populations through discrete models, in particular the red coral settlement of Calafuria (LI) (see [SBI07] and [SBI09]).
From a biological point of view, since the reproduction period occurs within a limited time interval in early summer, during the rest of the year such populations can only decrease in number. Therefore a discrete model with one year as discrete step appears to be more realistic to describe the evolution of such type of populations.
In the first part of this section we deal with the usual discrete model, which is also used in [SBI07]: this is a quasi-linear model which contains a *cutoff* function to rule survival of planulae.
In the second section we improve the previous model, in order to follow the variability observed in our datasets, as inspected during the second chapter. Then we analyze this model, stating a stability result for trivial equilibrium and referring, for further analysis, to J.M. Cushing [Cus98] and to numerical computations performed in chapter 4.

### 3.1.1 First discrete model

We consider a population structured in $a_\dagger$ age groups, with survival and birth rate ($\sigma_i$ and $b_i$ respectively) depending only on the age. However the model is nonlinear, due to the presence of a cutoff function, which determines the survival rate of the *planulae* that become *recruits* (actual members of the population) as a function of weighted population density. Here and in the subsequent models, we always take a cutoff function of the form:

$$S_0(x) = \frac{\alpha}{x + \beta e^{-\gamma x}}, \tag{3.1}$$

where $\alpha, \beta, \gamma$ are positive parameters; the choice of such type of functions is justified by empirical motivations (see [SBI07]). The argument of $S_0$ is a weighted sum of the population elements:

$$U = \sum_{i=1}^{a_\dagger} u_i \omega_i \tag{3.2}$$

where $\omega_i$, $i = 1, \dots, a_\dagger$, are positive constants. Now for $n \in \mathbb{N}$, which specifies the $n$-th year in the timeline, we write the equations for this model:

$$\begin{cases} u_1^{n+1} = S_0(U^n) \displaystyle\sum_{i=1}^{a_\dagger} b_i u_i^n \\ u_i^{n+1} = \sigma_{i-1} u_{i-1}^n, \quad i = 2, \dots a_\dagger \\ U^n = \displaystyle\sum_{i=1}^{a_\dagger} u_i^n \omega_i. \end{cases} \tag{3.3}$$

Looking for the equilibria of the model we obtain the following equations

$$\begin{cases} u_1^* = S_0(U^*) \sum_{i=1}^{a_\dagger} b_i u_i^* \\ u_i^* = \sigma_{i-1} u_{i-1}^*, \quad i = 2, \dots a_\dagger, \\ U^* = \sum_{i=1}^{a_\dagger} u_i^* \omega_i. \end{cases} \tag{3.4}$$

setting $\Pi_i = \prod_{j=1}^{i-1} \sigma_j$ and inserting the equations on the second line of (3.4) in the first line we obtain:

$$S_0(U^*) = \frac{1}{\mathcal{R}_0}, \tag{3.5}$$

where $\mathcal{R}_0 = \sum_{i=1}^{a_\dagger} b_i \Pi_i$ is the *basic reproduction number* of the population and represents the average number of planulae produced by a single element (in our case a *colony*) during its life-span.

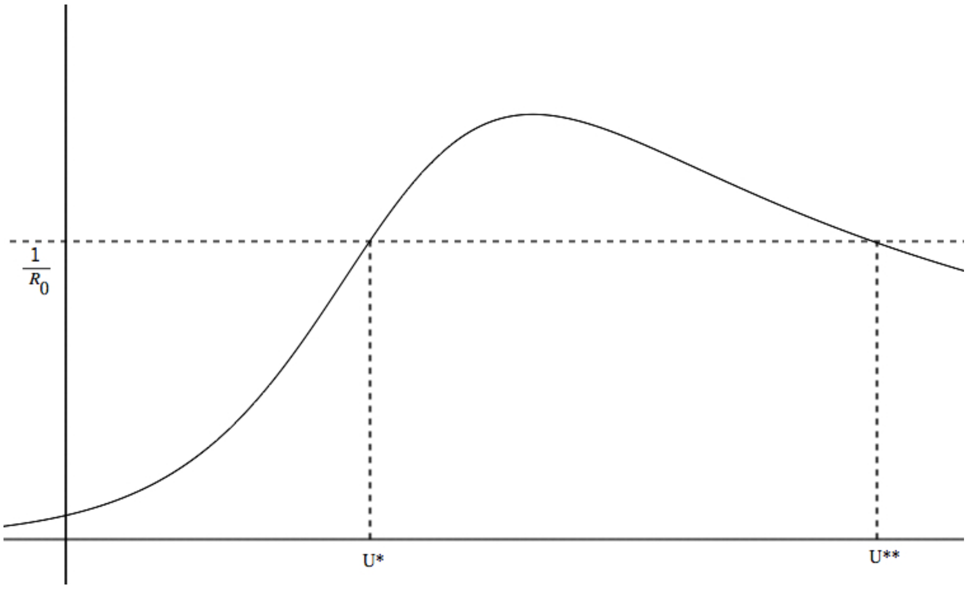



Figure 3.1: Solutions of equation (3.5)

A cutoff function of the form (3.1) leads to three different cases (see figure 3.1) depending on the different relations of $\mathcal{R}_0$ with respect to the parameters $\frac{\alpha}{\beta} = S_0(0)$ and $V_0$ (maximum value reached by the function $S_0$). In particular, we find:

(i.) one nontrivial equilibrium for $\mathcal{R}_0 > \frac{1}{S_0(0)}$;

(ii.) two nontrivial equilibria for $V_0 < \mathcal{R}_0 < \frac{1}{S_0(0)}$;

(iii.) no nontrivial equilibria for $\mathcal{R}_0 < V_0$.

### Linearization and stability of equilibria

Now we want to investigate whether the detected nontrivial and trivial (i.e. $u_i^* = 0$ for $i = 1, \ldots, a_\dagger$) equilibria are *stable*. As done in chapter 1 we consider in $\mathbb{R}^{a_\dagger}$ the norm $|\mathbf{u}| = \sum_{i=1}^{a_\dagger} |u_i|$ and give the following definition:

**Definition 3.1.1.** *Let $\mathbf{u}^* = (u_1^*, \ldots, u_{a_\dagger}^*)$ be an equilibrium for the system (3.3); we say that $\mathbf{u}^*$ is stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $\mathbf{u}^0$ satisfies:*

$$|\mathbf{u}^0 - \mathbf{u}^*| = \sum_{i=1}^{a_\dagger} |u_i^0 - u_i^*| \leq \delta,$$

*then we have:*

$$|\mathbf{u}^n - \mathbf{u}^*| \leq \epsilon \quad \forall n \in \mathbb{N}^+.$$

*The equilibrium is asymptotically stable if it is stable and $\delta$ can be chosen such that, if $|\mathbf{u}^0 - u^*| \leq \delta$, then*

$$\lim_{n\to\infty} |\mathbf{u}^n - \mathbf{u}^*| = 0.$$

*Finally, the equilibrium is unstable if it is not stable.*

Now we perform a linearization of the system (3.3): setting $v_i^n = u_i^n - u_i^*$, (with $u_i^*$ that satisfying (3.4) for $i = 1, \ldots, a_\dagger$) and omitting the second order or higher terms we get:

$$\begin{cases} v_1^{n+1} = S_0(U^*) \sum_{i=1}^{a_\dagger} b_i v_i^n + \mathcal{R}_0 S_0'(U^*) u_1^* \sum_{i=1}^{a_\dagger} \omega_i v_i^n \\ v_i^{n+1} = \sigma_{i-1} v_{i-1}^n, \qquad i = 2, \ldots a_\dagger. \end{cases} \tag{3.6}$$

Concerning the stability of equilibria, we will refer to some classical results that are discussed in [Cus98], where one can find more details.
Before presenting a stability result we define another matrix type in addition to those defined in chapter 1.

**Definition 3.1.2.** *Let $P$ be a square matrix; we say that $P$ is hyperbolic if all its eigenvalues $\zeta$ satisfy $|\zeta| \neq 1$.*

Now let $\mathbf{u}^*$ be the equilibrium and let $\mathbf{v}^n = (v_1^n, \ldots, v_{a_\dagger}^n)$ be the solution of (3.6). As we know the transition matrix $P(\mathbf{u}^*)$ associated to (3.6), i.e. such that (3.6) can be written as $\mathbf{v}^{n+1} = P(\mathbf{u}^*)\mathbf{v}^n$, is a Leslie matrix (see chapter 1). Then we have the following result:

**Theorem 3.1.1.** *Let $P(\mathbf{u}^*)$ be irreducible, primitive and hyperbolic, and let $r > 0$ be its strictly dominant eigenvalue. Then*

*(i.) if $r < 1$, then $\mathbf{u}^* = 0$ is (locally asymptotically) stable;*

*(ii.) if $r > 1$, then $\mathbf{u}^* = 0$ is unstable.*

Hence we can determine stability of an equilibrium from the eigenvalues of the linearized system. The system (3.6) for trivial equilibrium, has the form:

$$\begin{cases} v_1^{n+1} = S_0(0) \sum_{i=1}^{a_\dagger} b_i v_i^n \\ v_i^{n+1} = \sigma_{i-1} v_{i-1}^n, \qquad i = 2, \ldots a_\dagger. \end{cases} \tag{3.7}$$

In this case we can apply theorems 3.1.1 and 1.2.2 obtaining that:

**Proposition 3.1.1.** *The trivial equilibrium* $\mathbf{u}^* \equiv 0$ *is asymptotically stable if*

$$S_0(0)\mathcal{R}_0 = S_0(0) \sum_{i=1}^{a_\dagger} \Pi_i b_i < 1,$$

*and is unstable if*

$$S_0(0)\mathcal{R}_0 > 1.$$

Concerning the nontrivial equilibria, analytic results for stability are not available in general, due to the more complicated form of the matrix $P(\mathbf{u}^*)$ (since it contains negative terms if $S_0'(U^*) < 0$). To treat this problem numerical computation are usually performed for every particular case. In our situation we can however observe that, if $S_0'(U^*) > 0$, the associated equilibrium is unstable. Indeed applying again Theorems 3.1.1 and 1.2.2 we obtain that

$$\xi = \sum_{i=1}^{n} \Pi_i (b_i S_0(U^*) + \mathcal{R}_0 S_0'(U^*) \omega_i u_1^*) > 1,$$

so that $r > 1$.

### 3.1.2 Discrete model with crown distribution

Now we deal with a generalization of the previous model, that seems to be more suitable with our data: indeed, as stated in previous chapter, Cap de Creus and Portofino populations present a high variability, i.e. several bigger colonies with respect to the rest of the population; this implies that some members produce more planulae and occupy a larger part of the ecological niche.
The age structure of this model is the same (with $a_\dagger$ age groups) as before, but we add a distribution parameter $\omega$ that takes into account the average growth rate of colonies (i.e. identifies the average circular crown built every year within the growth process). Therefore we insert a function $g(\omega)$ expressing this distribution among recruits and consider the birth and survival rates $b_i(\omega)$, $\sigma_i(\omega)$ as functions of $\omega$. Hence the model (3.3) is modified in this way:

$$\begin{cases} u_1^{n+1}(\omega) = g(\omega) S_0(U^n) \displaystyle\sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega') u_i^n(\omega') d\omega' \\ u_i^{n+1}(\omega) = \sigma_{i-1}(\omega) u_{i-1}^n(\omega), \qquad i = 2, \dots a_\dagger \\ U^n = \displaystyle\sum_{i=1}^{a_\dagger} \int_0^\infty u_i^n(\omega) H_i(\omega) d\omega. \end{cases} \tag{3.8}$$

Now we introduce some assumptions, that will be valid throughout this section, concerning the functions $H_i(\omega), \sigma_i(\omega)$ and $g(\omega)$, for $i = 1, \dots a_\dagger$ :

$$\|g\|_1 = 1; \tag{3.9}$$

$$\|\sigma_i\|_1, \|\sigma_i\|_\infty < 1; \tag{3.10}$$

$$\|H_i\|_1 < \infty. \tag{3.11}$$

These assumptions seem reasonable: indeed $g$ represents a distribution on the positive half-line, the quantities $\sigma_i$ represent survival rates and the two assumptions mean that this rate is less than 1 for every age and growth level. On the other hand the quantities $H_i$ are positive weights.

### Steady states and their stability

Here we find the nontrivial equilibria of (3.8), i.e. stationary distributions $u_i^*(\omega) \neq 0$ for every age group: from (3.8) we obtain the following system

$$\begin{cases} u_1^*(\omega) = g(\omega) S_0(U^*) \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega') u_i^*(\omega') d\omega' \\ u_i^*(\omega) = \sigma_{i-1}(\omega) u_{i-1}^*(\omega) \\ U^* = \sum_{i=1}^{a_\dagger} \int_0^\infty u_i^*(\omega) H_i(\omega) d\omega. \end{cases} \tag{3.12}$$

Then, as in the previous model, we put $\Pi_i(\omega) = \prod_{j=1}^{i-1} \sigma_j(\omega)$ and from the second equation of (3.12) we obtain

$$u_i^*(\omega) = \prod_{j=1}^{i-1} \sigma_j(\omega) u_1^*(\omega) = \Pi_i(\omega) u_1^*, \qquad i = 2, \dots a_\dagger. \tag{3.13}$$

Now we consider $X^* = \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega') u_i^*(\omega') d\omega'$ and insert (3.13) and the first equation of (3.12) in the last expression. So we get that

$$1 = \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega) \Pi_i(\omega) g(\omega) d\omega S_0(U^*).$$

Now we define

$$\mathcal{R}_0 = \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega) \Pi_i(\omega) g(\omega) d\omega \tag{3.14}$$

the *basic reproduction number* (which can be interpreted as usual), and obtain, for the nontrivial equilibria, the same equation reached in the previous section,

$\mathcal{R}_0 S_0(U^*) = 1$. Starting from this equation we can repeat the arguments stated above about existence of the equilibria with a cutoff function of type (3.1).
Now we deal with the *stability* of the detected equilibria. In defining this concept, we rephrase Definition 3.1.1, but using the $L^1$-norm over $(0, \infty)$ instead of the $\mathbb{R}^{a_\dagger}$ norm. Then we linearize equation (3.8) at the equilibrium: setting

$$W_i^n(\omega) = u_i^n(\omega) - u_i^*(\omega).$$

and omitting the second order or higher terms we get the following system:

$$\begin{cases} W_1^{n+1}(\omega) &=& g(\omega)S_0(U^*)\displaystyle\sum_{i=1}^{a_\dagger}\int_0^\infty b_i(\omega')W_i^n(\omega')d\omega' \\ && +g(\omega)S_0'(U^*)\displaystyle\sum_{i=1}^{a_\dagger}\int_0^\infty b_i(\omega')u_i^*(\omega')d\omega(U^n - U^*) \\ W_i^{n+1}(\omega) &=& \sigma_{i-1}(\omega)W_{i-1}^n(\omega), \qquad i = 2, \dots a_\dagger \end{cases} \tag{3.15}$$

where $U^n - U^* = \displaystyle\sum_{i=1}^{a_\dagger}\int_0^\infty W_i^n(\omega)H_i(\omega)d\omega$.
Concerning the trivial steady state the system (3.15) become:

$$\begin{cases} W_1^{n+1}(\omega) = g(\omega)S_0(0)\sum_{i=1}^{a_\dagger}\int_0^\infty b_i(\omega')W_i^n(\omega')d\omega' \\ W_i^{n+1}(\omega) = \sigma_{i-1}(\omega)W_{i-1}^n(\omega), \quad i = 2, \dots a_\dagger. \end{cases} \tag{3.16}$$

In this case we prove a stability result:

**Proposition 3.1.2.** *The trivial equilibrium* $\mathbf{u}^* \equiv 0$ *is asymptotically stable if* $S_0(0)\mathcal{R}_0 < 1$, *and unstable if* $S_0(0)\mathcal{R}_0 > 1$.

*Proof.* We start by setting in (3.16) $\nu^n = \frac{W_1^n(\omega)}{g(\omega)}$; note that $\nu^n \in \mathbb{R}^+$ is a constant with respect to $\omega$. Suppose first that $S_0(0)\mathcal{R}_0 < 1$; we want to prove that

$$\lim_{n\to\infty} \nu^n = 0. \tag{3.17}$$

This is sufficient for our first statement: indeed, from (3.9) and (3.10), $\|W_1^n\|_1 = \nu^n$ and, from Hölder inequality, (if $n > a_\dagger$)

$$\|W_i^n\|_1 = \|\Pi_i W_1^{n-i+1}\|_1 = \|\Pi_i g\|_1 \nu^{n-i+1} < \nu^{n-i+1}.$$

Hence, if (3.17) holds, we have $\|W_i^n\|_1 \to 0$ as $n \to \infty$ for every $i = 2 \dots a_\dagger$. From the first equation of (3.16) we obtain a recursive relation for $\nu^n$ :

$$\begin{aligned}
\nu^{n+1} &= S_0(0) \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega) W_i^n(\omega) d\omega \\
&= S_0(0) \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega) W_1^{n-i+1}(\omega) \Pi_i(\omega) d\omega \\
&= S_0(0) \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega) g(\omega) \Pi_i(\omega) d\omega \nu^{n-i+1}.
\end{aligned}$$

Now, considering (3.14), let $\delta$ such that $S_0(0)\mathcal{R}_0 \le \delta < 1$ and set

$$M_n = \max_{i=1,\dots,a_\dagger} \nu^{n-i+1}.$$

We have (assuming that $n > a_\dagger$)

$$\nu^{n+1} \le \delta M_n. \tag{3.18}$$

From this estimate our first statement easily follows: indeed let $\epsilon > 0$ and $M = \max_{i=1,\dots,a_\dagger} \nu^i$; if we choose $\bar{n}$ such that $\delta^{\bar{n}} \le \dfrac{\epsilon}{M}$, we obtain that $\nu^n \le \epsilon$, for every $n > (a_\dagger + 1)\bar{n}$.
To prove the second part, we proceed in the same way: from the recursive relation for $\nu^n$ we give a backward estimate based on $m_n = \min_{i=1,\dots,a_\dagger} \nu^{n-i+1}$, proving that

$$\lim_{n\to\infty} \nu^n = \infty.$$

This implies our second statement. □

As before we cannot state generical results concerning the stability of non-trivial steady states. However we refer to the next section for a useful reformulation of problem (3.8) in the form (3.3).

### Further analysis

Here we state some observations that may help to handle (3.8) in more detail. Consider (3.8) and suppose $n > a_\dagger$, as done in the proof of the previous proposition: we set

$$v^{n+1} = \frac{u_1^{n+1}(\omega)}{g(\omega)} = \sum_{i=1}^{a_\dagger} \int_0^\infty b_i(\omega') u_i^n(\omega') d\omega' S_0(U^n). \tag{3.19}$$

Now from the second equation of (3.8) we get, for all $i = 2, \dots, a_\dagger$

$$u_i^{n+1}(\omega) = \Pi_i(\omega)u_1^{n-i+1}(\omega) = \Pi_i(\omega)g(\omega)v^{n-i+1}.$$

Hence we set $\alpha_i(\omega) = \Pi_i(\omega)g(\omega)$ and $w_i^{n+1} = v^{n-i+1}$, obtaining that

$$\frac{u_i^{n+1}(\omega)}{\alpha_i(\omega)} = w_i^{n+1}, \tag{3.20}$$

which is a constant.
Therefore inserting (3.20) into (3.8) we obtain a similar system which does not depend on $\omega$ :

$$\begin{cases} w_1^{n+1} = S_0(V^n)\displaystyle\sum_{i=1}^{a_\dagger} h_i w_i^n \\ w_i^{n+1} = w_{i-1}^n, \quad i = 2, \dots a_\dagger. \\ V^n = \displaystyle\sum_{i=1}^{a_\dagger} a_i w_i^n, \end{cases} \tag{3.21}$$

where $w_1^{n+1} = v^{n+1}$, $h_i = \int_0^\infty b_i(\omega)\alpha_i(\omega)d\omega$ and $a_i = \int_0^\infty H_i(\omega)\alpha_i(\omega)d\omega$. Now we can linearize the system and apply theorems 3.1.1 and 1.2.2 to investigate the stability of nontrivial equilibria from the eigenvalues of the associated matrix. For further development we refer to chapter 4, where we use this characterization and compute numerically the eigenvalues of the linearized transition matrix (of both Portofino and Cap de Creus populations), to see whether nontrivial equilibria are stable.

## 3.2 Continuous model

In this section we present a continuous model to describe the evolution of a red coral population.
The model is nonlinear and is based on the general nonlinear continuous one displayed in the book of M. Iannelli [Ian94]. Moreover it shows strong analogies with the second discrete model presented in the previous section: indeed it contains a cutoff function (here named $\phi$) with a weighted population distribution as argument. Furthermore we have added another distribution variable $\omega$ that takes into account the average growth rate of a colony (as suggested in the second chapter).
In the first part of the section we define the continuous model and provide several hypothesis that we assume throughout in this section. Then we prove existence and uniqueness of the solution for a slight generalization of the displayed model. Finally we deal with stationary solutions of the system and their stability. In this framework we use some results proved in Appendix A, such as the Paley-Wiener Theorem.

### 3.2.1 Definition and hypotheses

We consider, as in the discrete case, a population structured with respect to the age parameter $a \in [0, a_\dagger)$, where $a_\dagger < \infty$, and there is a growth distribution parameter $\omega \in (0, \infty)$; $t$ is the time variable.
The model is the following:

$$\begin{cases} p_t(a,\omega,t) + p_a(a,\omega,t) + \mu(a,\omega)p(a,\omega,t) = 0 \\ p(0,\omega,t) = g(\omega)\displaystyle\int_0^{a_\dagger}\int_0^{\infty} \beta(\sigma,\eta)\phi(U(t))p(\sigma,\eta,t)d\eta d\sigma \\ p(a,\omega,0) = p_0(a,\omega) \\ U(t) = \displaystyle\int_0^{a_\dagger}\int_0^{\infty} p(a,\omega,t)H(a,\omega)d\omega da. \end{cases} \tag{3.22}$$

Now we introduce some related assumptions: we suppose that $\beta, \mu, p_0$ are continuous functions and $\beta \in L^1((0,a_\dagger) \times (0,\infty))$, $\mu \in L^1_{loc}([0,a_\dagger) \times (0,\infty))$, $p_0 \in L^1((0,a_\dagger) \times (0,\infty))$ and $H \in L^\infty((0,a_\dagger) \times (0,\infty))$. Furthermore we assume that $g \in L^1(0,\infty)$ with norm equal to $1$, and

$$0 \le \beta(a,\omega) \le \beta_+ \text{ a.e. in } [0,a_\dagger] \times (0,\infty); \tag{3.23}$$

$$\mu, H \ge 0 \text{ a.e. in } [0,a_\dagger] \times (0,\infty); \tag{3.24}$$

$$\int_0^{a_\dagger}\int_0^{\infty}\mu(a,\omega)d\omega da = \infty. \tag{3.25}$$

Finally we suppose that $\phi$ is continuosly differentiable in $[0,\infty)$. Now we set $\Pi(a,\omega) = e^{-\int_0^a \mu(\sigma,\omega)d\sigma}$ and we note that, as seen in the first chapter, problem (3.22) can be integrated along the characteristics, obtaining:

$$p(a,\omega,t) = \begin{cases} p_0(a-t,\omega)\frac{\Pi(a,\omega)}{\Pi(a-t,\omega)} & \text{if } a \geq t \\ b(t-a,\omega)\Pi(a,\omega) & \text{if } a < t. \end{cases} \tag{3.26}$$

where $b(t,\omega)$ is the solution of the following Volterra integral equation:

$$b(t,\omega) = g(\omega)\phi(U(t))\left[F(t) + \int_0^t\int_0^{\infty} K(t-\sigma,\eta)b(\sigma,\eta)d\eta d\sigma\right] \tag{3.27}$$

where $K(\sigma,\omega) = \beta(\sigma,\omega)\Pi(\sigma,\omega)$ and

$$F(t) = \int_t^{\infty}\int_0^{\infty}\beta(\sigma,\eta)p_0(\sigma-t,\eta)\frac{\Pi(\sigma,\eta)}{\Pi(\sigma-t,\eta)}d\sigma d\eta.$$

The integral equation can be uniquely solved , as shown in the next section.

### 3.2.2 Existence and uniqueness

Here we prove the existence and uniqueness of the solution of a slight generalization of (3.22). The proof generalizes a result proved in [Ian94], Ch.III §2 and is based on the fixed point argument used there.
We consider a more general definition of fertility $\beta$ and mortality $\mu$: indeed they now depend also on $n$ quantities that represent the status of the population, moreover the fertility depends on another distribution variable. Now in place of (3.22) we consider, for $\omega \in (0,\infty)$, $a \in [0,a_\dagger]$ and $t \in [0,T]$, the following problem:

$$\begin{cases} p_t(a,\omega,t) + p_a(a,\omega,t) + \mu(a,\omega;S_1(t),\ldots,S_n(t))p(a,\omega,t) = 0 \\ p(0,\omega,t) = g(\omega)\int_0^{a_\dagger}\int_0^{\infty}\beta(\sigma,\eta,\omega;S_1(t),\ldots,S_n(t))p(\sigma,\eta,t)d\eta d\sigma \\ p(a,\omega,0) = p_0(a,\omega) \\ S_i(t) = \int_0^{a_\dagger}\int_0^{\infty}H_i(a,\omega)p(a,\omega,t)d\omega da \qquad i=1,\ldots,n. \end{cases} \tag{3.28}$$

Observe that for $n = 1$, $S(t) = U(t)$ and $\beta(\sigma,\omega,\eta;U(t)) = \beta(\sigma,\eta)\phi(U(t))$, $\mu(a,\omega;U(t)) = \mu(a,t)$ we get (3.22).

Now we generalize the previous assumptions: we suppose that $\beta, \mu, p_0$ are continuous functions. Moreover we suppose that for every fixed $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\beta(\cdot,\cdot,\cdot,\mathbf{x}) \in L^1((0,a_\dagger)\times(0,\infty)^2)$ and satisfies (3.23), $\mu(\cdot,\cdot,\mathbf{x}) \in L^1_{loc}((0,a_\dagger)\times(0,\infty))$ and satisfies (3.24) and (3.25). Moreover we assume that for each $M > 0$ there exists $H(M) > 0$ such that, if $|x_i|, |\tilde{x}_i| \le M$ for $i = 1, \ldots, n$, then

$$
\begin{aligned}
|\beta(a,\eta,\omega;\mathbf{x}) - \beta(a,\eta,\omega;\tilde{\mathbf{x}})| &\le H(M)\sum_{i=1}^{n}|x_i - \tilde{x}_i| \\
|\mu(a,\omega;\mathbf{x}) - \mu(a,\omega;\tilde{\mathbf{x}})| &\le H(M)\sum_{i=1}^{n}|x_i - \tilde{x}_i|.
\end{aligned}
\tag{3.29}
$$

Furthermore we suppose that $H_i$ belongs to $L^\infty((0,a_\dagger)\times(0,\infty))$ for $i = 1, \ldots, n$ and that, since $g$ is a distribution function, $|g|_1 = 1$. For convenience of notation we use $|\cdot|_1$ to indicate the $L^1$-norm over the interval $(0,\infty)$ and $\|\cdot\|_1$ to indicate the $L^1$-norm over $(0,a_\dagger)\times(0,\infty)$.
Then we set

$$
\Pi(a,t,x,\omega;\mathbf{S}) = \exp\left[-\int_0^x \mu(a-\sigma,\omega;S_1(t-\sigma),\ldots,S_n(t-\sigma))d\sigma\right] \tag{3.30}
$$

and proceed as in the previous case: integrating (3.28), along the characteristics we get:

$$
p(a,\omega,t) = \begin{cases} p_0(a-t,\omega)\Pi(a,t,t,\omega;\mathbf{S}) & \text{if } a \ge t \\ b(t-a,\omega;\mathbf{S})\Pi(a,t,a,\omega;\mathbf{S}) & \text{if } a < t \end{cases} \tag{3.31}
$$

where $b(t,\omega;\mathbf{S})$ is the solution of the following Volterra equation:

$$
\begin{aligned}
b(t,\omega;\mathbf{S}) &= g(\omega)F(t,\omega;\mathbf{S}) \\
&\quad + g(\omega)\int_0^t\int_0^\infty K(t,t-\sigma,\eta,\omega;\mathbf{S})b(\sigma,\eta;\mathbf{S})d\eta d\sigma
\end{aligned}
\tag{3.32}
$$

where

$$
F(t,\omega;\mathbf{S}) = \int_t^{a_\dagger}\int_0^\infty \beta(\sigma,\eta,\omega;\mathbf{S}(t))p_0(\sigma-t,\eta)\Pi(\sigma,t,t,\eta;\mathbf{S})d\eta d\sigma \tag{3.33}
$$

and

$$
K(t,\sigma,\eta,\omega;\mathbf{S}) = \beta(\sigma,\eta,\omega;\mathbf{S}(t))\Pi(\sigma,t,\sigma,\eta;\mathbf{S}). \tag{3.34}
$$

We observe that if $\mathbf{S} \in C([0,T],\mathbb{R}^n)$ then, by (3.29),(3.33) and (3.34) it easily follows that both $F(\cdot,\omega;\mathbf{S})$ and $K(\cdot,\cdot,\eta,\omega;\mathbf{S})$ are continuous; hence the integral equation (3.32) is uniquely solvable. Moreover, it can be easily shown that, any

$p(\cdot,\cdot,t)$ satisfying (3.31) must belong to $C([0,T],L^1[(0,a_\dagger)\times(0,\infty)])$. Notice that (3.31) is not a definition of $p$, since the function $\mathbf{S}$ contains $p$ itself (see (3.28)). Hence, we must *solve* (3.31), i.e. find a function $p$, with a suitable regularity, satisfying (3.31).
From (3.29), assuming that $\mathbf{S}(t),\tilde{\mathbf{S}}(t)\in C([0,T],\mathbb{R}^n)$ with $|S_i(t)|,|\tilde{S}_i(t)|\le M$, $i=1,\dots,n$ and $t\in[0,T]$, we get:

$$|\beta(a,\eta,\omega;\mathbf{S}(t))-\beta(a,\eta,\omega;\tilde{\mathbf{S}}(t))|\le H(M)\sum_{i=1}^{n}|S_i(t)-\tilde{S}_i(t))| \tag{3.35}$$

$$\begin{aligned}&|\Pi(a,t,x,\omega;\mathbf{S})-\Pi(a,t,x,\omega;\tilde{\mathbf{S}})|\le\\ &\quad\le\left|\int_0^x\mu(a-\sigma,\omega;\mathbf{S}(t-\sigma))-\mu(a-\sigma,\omega;\tilde{\mathbf{S}}(t-\sigma))d\sigma\right|\\ &\quad\le H(M)\sum_{i=1}^{n}\int_{t-x}^{t}|S_i(\sigma)-\tilde{S}_i(\sigma)|d\sigma.\end{aligned} \tag{3.36}$$

Now we can prove a preliminary estimate about $b(t,\omega;\mathbf{S})$ :

**Lemma 3.2.1.** *Let* $\mathbf{S},\tilde{\mathbf{S}}\in C([0,T],\mathbb{R}^n)$ *with* $|S_i(t)|,|\tilde{S}_i(t)|\le M$, $i=1,\dots,n$ *and* $t\in[0,T]$ *and let* $b$ *the solution of (3.32). Then:*

$$|b(t,\cdot;\mathbf{S})|_1\le\beta_+e^{\beta_+t}\|p_0\|_1 \tag{3.37}$$

*and there exists* $L(M)>0$ *such that:*

$$\begin{aligned}&|b(t,\cdot;\mathbf{S})-b(t,\cdot;\tilde{\mathbf{S}})|_1\le\\ &\quad\le L(M)\|p_0\|_1\sum_{i=1}^{n}\left[|S_i(t)-\tilde{S}_i(t)|+\int_0^t|S_i(\sigma)-\tilde{S}_i(\sigma)|d\sigma\right]\end{aligned} \tag{3.38}$$

*Proof.* For the first part we apply (3.23) and (3.30) to (3.33), obtaining that

$$F(t,\omega;\mathbf{S})\le\beta_+\int_t^\infty\int_0^\infty p_0(\sigma-t,\eta)d\eta d\sigma\le\beta_+\|p_0\|_1,$$

whereas by (3.34) and (3.23) we get

$$\int_0^t\int_0^\infty K(t,t-\sigma,\eta,\omega;\mathbf{S})b(\sigma,\eta;\mathbf{S})d\omega d\eta\le\beta_+\int_0^t\int_0^\infty b(\sigma,\eta;\mathbf{S})d\eta d\sigma.$$

Hence by (3.32), integrating with respect to $\omega$ in $(0,\infty)$, we get:

$$\int_0^\infty b(t,\omega;\mathbf{S})d\omega\le\beta_+\|p_0\|_1+\beta_+\int_0^t\int_0^\infty b(\sigma,\omega;\mathbf{S})d\omega d\sigma.$$

Hence by Grönwall inequality (lemma 1.2.1 chapter 1 §2) we obtain (3.37).
To prove (3.38) we have to work a little more: by (3.32) we have

$$|b(t,\cdot;\mathbf{S}-b(t,\cdot;\tilde{\mathbf{S}})|_1 \le I + II,$$

where

$$I = |F(t,\omega;\mathbf{S}) - F(t,\omega;\tilde{\mathbf{S}})|,$$

$$II = \int_0^t \int_0^\infty |K(t,t-\sigma,\eta,\omega;\mathbf{S})b(\sigma,\eta;\mathbf{S}) - K(t,t-\sigma,\eta,\omega;\tilde{\mathbf{S}})b(\sigma,\eta;\tilde{\mathbf{S}})|d\eta d\sigma.$$

Now examining the two terms, by (3.35) and (3.36) we see that:

$$\begin{aligned} I &\le \int_t^\infty \int_0^\infty |\beta(\sigma,\eta,\omega;\mathbf{S}) - \beta(\sigma,\eta,\omega;\tilde{\mathbf{S}})|p_0(\sigma-t,\eta)d\eta d\sigma \\ &\quad + \beta_+ \int_0^\infty \int_0^\infty |\Pi(\sigma+t,t,t,\omega;\mathbf{S}) - \Pi(\sigma+t,t,t,\omega;\tilde{\mathbf{S}})|p_0(\sigma,\omega)d\omega d\sigma \\ &\le \|p_0\|_1 H(M) \sum_{i=1}^n \left[ |S_i(t) - \tilde{S}_i(t)| + \beta_+ \int_0^t |S_i(\sigma) - \tilde{S}_i(\sigma)|d\sigma \right], \end{aligned}$$

and by (3.37)

$$\begin{aligned} II &\le \int_0^t \int_0^\infty |\beta(\sigma,\eta,\omega;\mathbf{S}) - \beta(\sigma,\eta,\omega;\tilde{\mathbf{S}})|b(t-\sigma,\eta;\mathbf{S})|d\eta d\sigma \\ &\quad + \beta_+ \int_0^t \int_0^\infty |\Pi(\sigma,t,\sigma,\omega;\mathbf{S}) - \Pi(\sigma,t,\sigma,\omega;\tilde{\mathbf{S}})|b(t-\sigma,\omega;\mathbf{S})|d\omega d\sigma \\ &\quad + \beta_+ \int_0^t \int_0^\infty |b(\sigma,\omega;\mathbf{S}) - b(\sigma,\omega;\tilde{\mathbf{S}})||d\omega d\sigma \\ &\le \beta_+ \|p_0\|_1 H(M) \int_0^t e^{\beta_+\sigma} d\sigma \left[ \sum_{i=1}^n |S_i(t) - \tilde{S}_i(t)| \right] \\ &\quad + \beta_+^2 \|p_0\|_1 H(M) \int_0^t e^{\beta_+\sigma} d\sigma \sum_{i=1}^n \int_0^t |S_i(r) - \tilde{S}_i(r)|dr \\ &\quad + \beta_+ \int_0^t \int_0^\infty |b(\sigma,\omega;\mathbf{S}) - b(\sigma,\omega;\tilde{\mathbf{S}})|d\omega d\sigma. \end{aligned}$$

Then summing the two terms obtained and integrating over $[0,\infty)$ we have that:

$$\begin{aligned} &\int_0^\infty |b(t,\omega;\mathbf{S}) - b(t,\omega;\tilde{\mathbf{S}})|d\omega \\ \le\ & 2H(M)(1+\beta_+)e^{b_+T}\|p_0\|_1 \sum_{i=1}^n \left[ |S_i(t) - \tilde{S}_i(t)| + \int_0^t |S_i(\sigma) - \tilde{S}_i(\sigma)|d\sigma \right] \\ & + \beta_+ \int_0^t \int_0^\infty |b(\sigma,\omega;\mathbf{S}) - b(\sigma,\omega;\tilde{\mathbf{S}})|d\omega d\sigma. \end{aligned}$$

Using lemma 1.2.2, chapter 1 §2, we obtain (3.38) with $L(M) = 2H(M)(1 + \beta_+)e^{2\beta_+ T}$. □

Now, we consider the space $E = C([0,T], L^1[(0,a_\dagger)\times(0,\infty)])$ and the closed subset

$$\mathcal{K} = \{q \in E | q \geq 0, \|q(\cdot,\cdot,t)\|_1 \leq M\}. \tag{3.39}$$

For $q \in \mathcal{K}$ we set $\mathbf{Q}(t) = (Q_1(t), \ldots, Q_n(t))$, with

$$Q_i(t) = \int_0^{a_\dagger}\int_0^\infty H_i(a,\omega)q(a,\omega,t)d\omega da \tag{3.40}$$

and define, for a fixed $p_0 \in L^1[(0,a_\dagger) \times (0,\infty)]$, a map $\mathcal{T}_{p_0} : \mathcal{K} \subset E \to E$ by:

$$(\mathcal{T}_{p_0}q)(a,t,\omega) \doteqdot \begin{cases} p_0(a-t,\omega)\Pi(a,t,t,\omega;\mathbf{Q}(t)) & \text{if } a \geq t \\ b(t-a,\omega;\mathbf{Q}(t))\Pi(a,t,a,\omega;\mathbf{Q}(t)) & \text{if } a < t. \end{cases} \tag{3.41}$$

The other functions ($\beta$, $\mu$, etc.) are supposed to be assigned. Note the analogy between this definition and (3.31): now we finally obtain the desired solution as a fixed point of the defined map.

**Lemma 3.2.2.** *Let $\mathcal{K}$ be defined in (3.39), let $p_0 \in L^1[(0,a_\dagger) \times (0,\infty)]$ be a fixed initial datum and take the constant $M$ in (3.29) such that:*

$$M > e^{\beta_+ T}\|p_0\|_1 \tag{3.42}$$

*then*

1. *The operator $\mathcal{T}_{p_0}$ defined in (3.41) maps $\mathcal{K}$ into itself;*
2. *for $q, \tilde{q} \in \mathcal{K}, t \in [0,T]$, we have:*

$$\begin{aligned} &\|(\mathcal{T}_{p_0}q)(\cdot,\cdot,t) - (\mathcal{T}_{p_0}\tilde{q})(\cdot,\cdot,t)\|_1 \leq \\ &\quad \leq C(M,T)\int_0^t \|q(\cdot,\cdot,\sigma) - \tilde{q}(\cdot,\cdot,\sigma)\|_1 d\sigma. \end{aligned} \tag{3.43}$$

*Proof.* For the first point, let $q \in \mathcal{K}$; as $(\mathcal{T}_{p_0}q) \geq 0$, from (3.37) and (3.42) we have:

$$\begin{aligned} \|(\mathcal{T}_{p_0}q)(\cdot,\cdot,t)\|_1 &= \int_0^t\int_0^\infty b(t-a,\omega;\mathbf{Q})\Pi(a,t,a,\omega;\mathbf{Q})d\omega da \\ &\quad + \int_t^{a_\dagger}\int_0^\infty p_0(a-t,\omega)\Pi(a,t,t,\omega;\mathbf{Q})d\omega da \\ &\leq \int_0^t\int_0^\infty b(a,\omega;\mathbf{Q})d\omega da + \|p_0\|_1 \leq e^{\beta_+ T}\|p_0\|_1 < M. \end{aligned}$$

For the second point, let $q, \tilde{q} \in \mathcal{K}$; then

$$Q_i(t) = \int_0^{a_\dagger} \int_0^\infty H_i(a,\omega) q(a,t,\omega) d\omega da \leq H_+ \|q(\cdot,\cdot,t)\|_1 \leq H_+ M$$

where $H_+ = \max_{i=1,\dots,n} \|H_i\|_\infty$. Hence

$$\|(\mathcal{T}_{p_0} q)(\cdot,\cdot,t) - (\mathcal{T}_{p_0} \tilde{q})(\cdot,\cdot,t)\|_1 \leq I + II + III$$

where

$$\begin{aligned}
I &= \int_0^t \int_0^\infty |b(t-a,\omega;\mathbf{Q}) - b(t-a,\omega;\tilde{\mathbf{Q}})| \Pi(a,t,a,\omega;\mathbf{Q}) d\omega da, \\
II &= \int_0^t \int_0^\infty b(t-a,\omega;\tilde{\mathbf{Q}}) |\Pi(a,t,a,\omega;\mathbf{Q}) - \Pi(a,t,a,\omega;\tilde{\mathbf{Q}})| d\omega da, \\
III &= \int_t^{a_\dagger} \int_0^\infty p_0(a-t,\omega) |\Pi(a,t,t,\omega;\mathbf{Q}) - \Pi(a,t,t,\omega;\tilde{\mathbf{Q}})| d\omega da.
\end{aligned}$$

Now, using (3.38), we have:

$$\begin{aligned}
I &\leq L(H_+M)\|p_0\|_1 \sum_{i=1}^n \int_0^t \left[ |Q_i(a) - \tilde{Q}_i(a)| da + \int_0^a |Q_i(\sigma) - \tilde{Q}_i(\sigma)| d\sigma \right] da \\
&\leq L(H_+M)\|p_0\|_1 (1+T) \left( \sum_{i=1}^n \int_0^t |Q_i(\sigma) - \tilde{Q}_i(\sigma)| d\sigma \right).
\end{aligned}$$

Using (3.36) and (3.37):

$$\begin{aligned}
II &\leq \beta_+ \|p_0\|_1 H(H_+M) \sum_{i=1}^n \int_0^t e^{a\beta_+} \int_a^t |Q_i(\sigma) - \tilde{Q}_i(\sigma)| d\sigma da \\
&= \|p_0\|_1 H(H_+M)(e^{\beta_+ t} - 1) \sum_{i=1}^n \int_0^t |Q_i(\sigma) - \tilde{Q}_i(\sigma)| d\sigma,
\end{aligned}$$

and finally

$$III \leq \|p_0\|_1 H(H_+M) \sum_{i=1}^n \int_0^t |Q_i(\sigma) - \tilde{Q}_i(\sigma)| d\sigma.$$

Summing all terms and using (3.40) we have:

$$\begin{aligned}
&\|(\mathcal{T}_{p_0} q)(\cdot,\cdot,t) - (\mathcal{T}_{p_0} \tilde{q})(\cdot,\cdot,t)\|_1 \leq \\
&\leq \|p_0\|_1 [(1+T)L(H_+M) + e^{\beta_+ T} H(H_+M)] \sum_{i=1}^n \int_0^t |Q_i(\sigma) - \tilde{Q}_i(\sigma)| d\sigma \\
&\leq nH_+ \|p_0\|_1 [(1+T)L(H_+M) + e^{\beta_+ T} H(H_+M)] \int_0^t \|q(\cdot,\cdot,\sigma) - \tilde{q}(\cdot,\cdot,\sigma)\|_1 d\sigma
\end{aligned}$$

□

Before giving the existence result we need another observation, which we express as a Lemma:

**Lemma 3.2.3.** *Let $\mathcal{T}_{p_0}$ be the map defined by (3.41). Then for every integer $N > 0$,*

$$\|\mathcal{T}_{p_0}^N q - \mathcal{T}_{p_0}^N \tilde{q}\|_E \leq \frac{C(M,T)^N T^N}{N!} \|q - \tilde{q}\|_E. \tag{3.44}$$

*Proof.* Starting from (3.43), we prove by induction that for all $n > 0$:

$$\|(\mathcal{T}_{p_0}^n q)(\cdot,\cdot,t) - (\mathcal{T}_{p_0}^n \tilde{q})(\cdot,\cdot,t)\|_1 \leq C(M,T)^n \int_0^t \frac{(t-r)^{n-1}}{(n-1)!} \|q(\cdot,\cdot,t) - \tilde{q}(\cdot,\cdot,t)\|_1 dr.$$

this estimate, by integration, leads to the thesis.
Iterating (3.43) twice we obtain that:

$$\begin{aligned}
\|(\mathcal{T}^2 q)(\cdot,\cdot,t) - (\mathcal{T}^2 \tilde{q})(\cdot,\cdot,t)\|_1 &\leq C(M,T)^2 \int_0^t \int_0^\sigma \|q(\cdot,\cdot,r) - \tilde{q})(\cdot,\cdot,r)\|_1 dr d\sigma \\
&\leq C(M,T)^2 \int_0^t (t-r)\|q(\cdot,\cdot,r) - \tilde{q})(\cdot,\cdot,r)\|_1 dr
\end{aligned}$$

The inductive step is similar:

$$\begin{aligned}
&\|(\mathcal{T}^{n+1} q)(\cdot,\cdot,t) - (\mathcal{T}^{n+1} \tilde{q})(\cdot,\cdot,t)\|_1 \leq \\
&\leq C(M,T)^{n+1} \int_0^t \int_0^\sigma \frac{(\sigma-r)^{n-1}}{(n-1)!} \|q(\cdot,\cdot,r) - \tilde{q})(\cdot,\cdot,r)\|_1 dr d\sigma \\
&= C(M,T)^{n+1} \int_0^t \|q(\cdot,\cdot,r) - \tilde{q})(\cdot,\cdot,r)\|_1 dr \int_0^{t-r} \frac{s^{n-1}}{(n-1)!} ds.
\end{aligned}$$

The proof is complete. □

Now we can prove the expected result:

**Theorem 3.2.1.** *Let $p_0 \in L^1[(0,a_\dagger) \times (0,\infty)]$ and $M$ be as in (3.42); then there is one and only one $p \in \mathcal{K}$ verifying (3.31).*
*Moreover we have, for $p$ and $\tilde{p}$ coming from two different initial conditions $p_0$ and $\tilde{p}_0$:*

1. $\|p(\cdot,\cdot,t)\|_1 \leq e^{\beta_+ t}\|p_0\|_1$;
2. $\|p(\cdot,\cdot,t) - \tilde{p}(\cdot,\cdot,t)\|_1 \leq e^{C(M,T)t}\|p_0 - \tilde{p}_0\|_1$;
3. $\lim_{h\to 0} \frac{1}{h}\left[p(a+h,\omega,t+h) - p(a,\omega,t)\right] = -\mu(a,\omega;\mathbf{S})p(a,\omega,t)$
   $a.e. \quad in \quad [0,a_\dagger] \times [0,\infty) \times \mathbb{R}_+.$

*Proof.* From (3.44) we get that for $N$ sufficiently large $\mathcal{T}_{p_0}^N$ is a contraction, so that it admits a *unique* fixed point. From (3.42) we obtain also the first property. To prove the second property we use (3.43):

$$\begin{aligned}
&\|p(\cdot,\cdot,t)-\tilde{p}(\cdot,\cdot,t)\|_1 \leq \\
&\qquad \leq \|(\mathcal{T}_{p_0}p)(\cdot,\cdot,t)-(\mathcal{T}_{p_0}\tilde{p})(\cdot,\cdot,t)\|_1+\|(\mathcal{T}_{p_0}\tilde{p})(\cdot,\cdot,t)-(\mathcal{T}_{\tilde{p}_0}\tilde{p})(\cdot,\cdot,t)\|_1 \\
&\qquad \leq C(M,T)\int_0^t |p(\cdot,\cdot,\sigma)-\tilde{p}(\cdot,\cdot,\sigma)|_1 d\sigma+\|p_0-\tilde{p}_0\|_1;
\end{aligned}$$

now the thesis follow from Grönwall inequality. The proof of $3.$ follows by a straightforward direct study of (3.31), using (3.29), which we omit for brevity. □

### 3.2.3 Stationary solutions

In this section we look for stationary solutions of (3.22), i.e. $p(a,\omega,t)=p^*(a,\omega)$. Namely such solutions must satisfy, for $a\in[0,a_\dagger]$ and $\omega\in(0,\infty)$, the system:

$$\begin{cases}
p_a^*(a,\omega)+\mu(a,\omega)p^*(a,\omega)=0 \\
p^*(0,\omega)=g(\omega)\displaystyle\int_0^{a_\dagger}\int_0^\infty \beta(\sigma,\eta)\phi(U^*)p^*(\sigma,\eta)d\eta d\sigma \\
U^*=\displaystyle\int_0^{a_\dagger}\int_0^\infty p^*(a,\omega)H(a,\omega)d\omega da
\end{cases} \tag{3.45}$$

Obviously we have the trivial solution, i.e. $p^*(a,\omega)\equiv 0$; on the other hand nontrivial solutions have the form:

$$p^*(a,\omega)=\Pi(a,\omega)p^*(0,\omega), \tag{3.46}$$

now, as in the discrete model, we set

$$\eta^*=\frac{p(0,\omega)}{g(\omega)} \tag{3.47}$$

so that, from (3.46) and second equation of (3.45) we get:

$$\eta^*=\phi(U^*)\eta^*\int_0^{a_\dagger}\int_0^\infty \beta(a,\omega)g(\omega)\Pi(a,\omega)d\omega da.$$

Now setting

$$\mathcal{R}_0=\int_0^{a_\dagger}\int_0^\infty \beta(a,\omega)g(\omega)\Pi(a,\omega)d\omega da, \tag{3.48}$$

we get, for the nontrivial stationary solutions the same equation obtained in the discrete model

$$\phi(U^*)\mathcal{R}_0 = 1. \tag{3.49}$$

From (3.46),(3.47) and the third equation of (3.45) we get the other necessary equation to identify the stationary solutions:

$$\eta^* = \frac{U^*}{\int_0^{a_\dagger}\int_0^\infty H(a,\omega)g(\omega)\Pi(a,\omega)d\omega da}. \tag{3.50}$$

From the above observations we note the strict analogy with the discrete case, indeed using a cutoff function of the form (3.1), we can repeat the same arguments concerning the number of stationary solutions.

### 3.2.4 Stability of stationary solutions

In this section we investigate the behaviour of solutions of (3.22) when the initial conditions are close to a stationary distribution. First of all we define the concept of stability of an equilibrium in this context:

**Definition 3.2.1.** *A stationary solution $p^*(a,\omega)$ is stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $p_0(a,\omega)$ satisfies*

$$\|p_0(\cdot,\cdot) - p^*(\cdot,\cdot)\|_1 \leq \delta,$$

*then the corresponding solution $p(\cdot,\cdot,t)$ satisfies:*

$$\|p(\cdot,\cdot,t) - p^*(\cdot,\cdot)\|_1 < \epsilon \quad \forall t \geq 0.$$

*The solution is asymptotically stable if it is stable and we can choose $\delta$ such that:*

$$\lim_{t\to\infty} \|p(\cdot,\cdot,t) - p^*(\cdot,\cdot)\|_1 = 0.$$

*Finally $p^*(a,\omega)$ is unstable if it is not stable.*

Now we deal with the linearization procedure for the Volterra equation (3.27). With the above notations we define for $t \geq 0$

$$\begin{aligned} S(t) &= \eta(t) - \eta^* \\ W(t) &= U(t) - U^* \\ q_0(a,\omega) &= p_0(a,\omega) - p^*(a,\omega) \end{aligned}$$

where

$$\eta(t) = \frac{p(0,\omega,t)}{g(\omega)}.$$

Now we express the equation (3.27) near a stationary solution, as a linear integral equation plus a nonlinear term, with the form (A.27) (see Appendix A). Using (3.26) we obtain:

$$S(t) = \int_0^t S(a)M(t-a)da + \mathcal{R}_0\eta^*\phi'(U^*)W(t) + \mathcal{P}_S[S(\cdot), W(\cdot), q_0(\cdot)](t) \quad (3.51)$$

$$W(t) = \int_0^t S(a)N(t-a)da + \mathcal{P}_W[S(\cdot), W(\cdot), q_0(\cdot)](t) \quad (3.52)$$

where $\mathcal{P}_S, \mathcal{P}_W$ are the nonlinear terms and

$$N(a) = \int_0^\infty \beta(a,\omega)g(\omega)H(a,\omega)d\omega;$$

$$M(a) = \int_0^\infty \beta(a,\omega)g(\omega)\Pi(a,\omega)\phi(U^*)d\omega.$$

We observe that $\mathcal{P}[S(\cdot), W(\cdot), q_0(\cdot)](t) = (\mathcal{P}_S, \mathcal{P}_W)(t)$ satisfies the following assumptions:

$$\mathcal{P}[0,0,0] = 0$$

$$\|\mathcal{P}(0,0,q_0)\|_\infty \leq B\|q_0\|_1$$

where $B$ is a postive constant. Moreover there exists $\xi(s)$ with $\lim_{s\to 0}\xi(s) = 0$, such that, for $\|S\|_\infty, \|\tilde{S}\|_\infty, \|W\|_\infty, \|\tilde{W}\|_\infty, \|q_0\|_1 < s$

$$\|\mathcal{P}[S, W, q_0] - \mathcal{P}[\tilde{S}, \tilde{W}, q_0]\|_\infty \leq \xi(s)[\|S - \tilde{S}\|_\infty + \|W - \tilde{W}\|_\infty].$$

Note that this conditions are similar to ,(A.28)-(A.30). Then we can apply Theorem A.2.4 and the Paley-Wiener Theorem (Theorem A.2.3), which lead to the following characteristic equation:

$$\det(I - \alpha - \hat{A}(\lambda)) = 0 \quad \forall \Re\lambda \geq 0, \quad (3.53)$$

where

$$A(t) = \begin{bmatrix} M(t) & 0 \\ N(t) & 0 \end{bmatrix}, \quad (3.54)$$

and

$$\alpha = \begin{bmatrix} 0 & \mathcal{R}_0\eta^*\phi'(U^*) \\ 0 & 0 \end{bmatrix}. \quad (3.55)$$

Now by Theorem A.2.4 we have the following result:

**Theorem 3.2.2.** *Let $p^*(a,\omega)$ be a solution of (3.45). If (3.53) has only roots with negative real part, then $p^*$ is asymptotically stable.*

*Proof.* Let $(\eta^*, U^*)$ constant solutions of (3.49) and (3.50), associated to the stationary solution $p^*(a, \omega)$. Then by Theorem A.2.4 for every $\epsilon > 0$ there exists $\eta > 0$ such that, if

$$\|p_0 - p^*\|_1 = \|q_0\|_1 \leq \eta \tag{3.56}$$

then

$$\|S\|_\infty, \|W\|_\infty < \epsilon, \text{ and } \lim_{t\to\infty} S(t) = \lim_{t\to\infty} W(t) = 0. \tag{3.57}$$

Hence we have:

$$\begin{aligned} \sup_{t\in[0,a_\dagger]} \|p(\cdot,\cdot,t) - p^*(\cdot,\cdot)\|_1 &\leq \|p_0 - p^*\|_1 + a_\dagger \sup_{t\geq 0} |S(t)| \\ \sup_{t>a_\dagger} \|p(\cdot,\cdot,t) - p^*(\cdot,\cdot)\|_1 &\leq a_\dagger \sup_{t\geq 0} |S(t)|. \end{aligned}$$

Then we get

$$\|p(\cdot,\cdot,t) - p^*(\cdot,\cdot)\|_1 \leq \eta + a_\dagger\epsilon \quad \forall t \geq 0$$

and

$$\lim_{t\to\infty} \|p(\cdot,\cdot,t) - p^*(\cdot,\cdot)\|_1 = 0.$$

□

In our specific case, equation (3.53) leads to:

$$\hat{M}(\lambda) + \mathcal{R}_0\eta^*\phi'(U^*)\hat{N}(\lambda) = 1. \tag{3.58}$$

Concerning the trivial solution, we see that (3.58) takes the form (A.2), (see Appendix A) and we have:

**Proposition 3.2.1.** *The trivial equilibrium* $p^*(a) \equiv 0$ *is asymptotically stable if* $\mathcal{R}_0\phi(0) < 1$ *and is unstable if* $\mathcal{R}_0\phi(0) > 1$.

*Proof.* From Proposition A.1.1, if $\int_0^\infty K(a)da < 1$ then all the roots of $\hat{K}(\lambda) = 1$ have negative real part, but this integral is equal to $\mathcal{R}_0\phi(0)$. The unstable case is analog. □

The condition for stability or unstability of the trivial equilibrium is the same as in the discrete case. Observe that we can also treat the equation (3.58) in its general form: indeed applying Theorem A.4 we state that, if $\phi'(U^*) < 0$, for certain values of $\mathcal{R}_0$ and $\eta^*$ the nontrivial equilibrium is asymptotically stable.

## 3.3 Modelling the diffusion of planulae above a limited space

According to the reproductive cycle of Corallium Rubrum, fecundated larvae issued from the feminine gonads will spread among the surrounding environment for several days. If they attach in the ground they become a colony in about a month. Moreover, during these days they do not travel very far from parental colonies; hence we can suppose that do not leave a settlement. Hence we can study the diffusion process together with the dynamical system that rule the evolution of a population.

This chapter is devoted to improve the older models considering the diffusion of the larvae. In particular we consider the first discrete model (3.3) and introduce, for every year, a continuous component of diffusion within a limited space. In the first section we consider the diffusion problem in a compact one-dimensional interval; in the second one we work in a 2-dimensional disk. Differently from the previous chapters, here we name $a$ the age of the individuals and $m$ the year in the timeline.

### 3.3.1 Diffusion on an interval

We start with (3.3) and express the recruits as a diffusion integral of the planulae within the surrounding space. Hence we fix $L \in ]0, \infty[$ and set

$$\begin{cases} u_1^{m+1}(x) = \displaystyle\int_0^\infty p^m(t,x) S_0(U(x))dt, & x \in [0, L] \\ u_a^{m+1}(x) = \displaystyle\sum_{a=1}^{a_\dagger} \sigma_{a-1} u_{a-1}^m(x), & a = 2 \dots a_\dagger \end{cases} \tag{3.59}$$

where $p^m(t,x)$ represents the number of planulae in position $x$ at the year $m$ and time $t$; whereas $U, \sigma$ are analogous to the basic model. Now $p$ satisfies the following equation:

$$\begin{cases} p_t^m = p_{xx}^m - \mu p^m - S_0(U(x))p^m & t \in [0, T], x \in [0, L] \\ p^m(0,x) = \displaystyle\sum_{a=1}^{a_\dagger} b_a u_a^m(x) & x \in [0, L] \end{cases} \tag{3.60}$$

where $\mu$ is the death rate of planulae that is supposed to be constant. We observe that:

- the integral (3.59) is done between $0$ and $\infty$, despite it reflects only a one year diffusion of planulae: this is not a problem, since we suppose the mortality be enough strong to extinguish the planulae in a short time;

- in the first equation of (3.60) we can suppose that $S_0(U(x)) = 0$: indeed this term is negligible with respect to the other terms of the equation due to the high mortality of the planulae;
- we have to add boundary conditions to (3.60) that can be of Dirichlet type, i.e. $p^m(t,0) = p^m(t,L) = 0$, or of Neumann type, i.e. $\frac{d}{dx}p^m(t,0) = \frac{d}{dx}p^m(t,L) = 0$.

### Dirichlet boundary condition

We start calculating the solutions of (3.60) and then we replace them in (3.59). From now we omit the index $m$ for convenience of notation.
We apply the *separation of variables* looking for a solution of the form $p(t,x) = T(t)X(x)$, then we plug it into (3.60) obtaining two ordinary differential equations, one for $X(x)$ and the other for $T(t)$ with solutions:

$$T(t) = e^{-(\lambda+\mu)t}T(0),$$
$$X(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x)$$

where $A, B, \lambda$ are real constants with $\lambda < 0$. Then we impose the initial and boundary conditions and obtain (using the expansion in Fourier series) that $\lambda = -n^2$, so that

$$p(t,x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) e^{-(\frac{n^2\pi^2}{L^2}+\mu)t}, \tag{3.61}$$

where

$$c_n = \sum_{a=1}^{a_\dagger} b_a \gamma_{n,a}$$

with $\gamma_{n,a}$ coefficients of the sine expansions of $u_a(x)$, $a = 1, \ldots, a_\dagger$. Now we insert (3.61) into (3.59), integrate with respect to $t$ ( reintroducing the index $m$) and obtain:

$$\begin{cases} u_1^{m+1}(x) = S_0(U^m(x)) \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{L^2}\right)^{-1} \left(\sum_{a=1}^{a_\dagger} \gamma_{n,a}^m b_a\right) \sin\left(\frac{n\pi}{L}x\right) \\ u_a^{m+1}(x) = \sigma_{a-1} \sum_{n=1}^{\infty} \gamma_{n,a}^m b_a \sin(\frac{n\pi}{L}x). \end{cases} \tag{3.62}$$

### Neumann boundary condition

Now we consider the problems (3.59) and (3.60) with Neumann boundary conditions ( i.e. $\frac{d}{dx}p^m(t,0) = \frac{d}{dx}p^m(t,L) = 0$). We procede as in Dirichlet case, obtaining for (3.60) the following solution:

$$p(t,x) = \sum_{n=1}^{\infty} d_n \cos\left(\frac{n\pi}{L}x\right) e^{-(\frac{n^2\pi^2}{L^2}+\mu)t}; \tag{3.63}$$

again

$$d_n = \sum_{a=1}^{a_\dagger} b_a \delta_{n,a}$$

where $\delta_{n,a}$ are coefficients of the cosine expansions of $u_a(x)$. Again substituting in (3.59) we get:

$$\begin{cases} u_1^{m+1}(x) = S_0(U^m(x)) \sum_{n=1}^{\infty} \left(\frac{n^2\pi^2}{L^2}\right)^{-1} \left(\sum_{a=1}^{a_\dagger} \delta_{n,a}^m b_a\right) \sin\left(\frac{n\pi}{L}x\right) \\ u_a^{m+1}(x) = \sigma_{a-1} \sum_{n=1}^{\infty} \delta_{n,a}^m b_a \sin(\frac{n\pi}{L}x). \end{cases} \tag{3.64}$$

### 3.3.2 Diffusion on a disk

In this section we study the diffusion on a disk $D$ of unitary radius. The problem can be presented in the same way than the one dimensional case, indeed the equations for $u_i^m$ are the same of (3.59) and that one describing the evolution of $p^m$ is analogous:

$$\begin{cases} p_t^m(t,x,y) = (\Delta_{xy} - \mu)p^m(t,x,y) \quad t \in [0,T], (x,y) \in D \\ p^m(0,x,y) = \sum_{a=1}^{a_\dagger} b_a u_a^m(x,y) \qquad (x,y) \in D \end{cases} \tag{3.65}$$

with

$$p^m(t,x,y) = 0 \qquad \forall (x,y) \in \partial D, \quad \forall t \in [0,T] \tag{3.66}$$

for Dirichlet boundary conditions, and

$$\frac{\partial}{\partial \nu} p^m(t,x,y) = 0 \qquad \forall (x,y) :\in \partial D, \quad \forall t \in [0,T] \tag{3.67}$$

for Neumann boundary conditions ($\nu$ is the unit outward normal vector, i.e. $\nu(x,y) = (x,y) \quad \forall (x,y) \in \partial D$ ).

### Dirichlet boundary condition

As in one dimensional case we use the *separation of variables*. However we first transform problem (3.65) in polar coordinates, obtaining the following one, for $(r,\theta) \in [0,1] \times [0,2\pi[$ and $t \in [0,T]$:

$$\begin{cases} \dfrac{\partial p}{\partial t}(t,r,\theta) = \dfrac{1}{r^2}\dfrac{\partial^2 p}{\partial \theta^2}(t,r,\theta) + \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial p}{\partial r}(t,r,\theta)\right) - \mu p(t,r,\theta) \\ p(0,r,\theta) = \displaystyle\sum_{a=1}^{a_\dagger} b_a u_a^m(r,\theta) \\ p(t,1,\theta) = 0. \end{cases} \tag{3.68}$$

We look for a solution of the form $p(t,r,\theta) = X(r)Y(\theta)T(t)$, and plug it into (3.68) to transform our PDE in three ODE. Namely we have:

$$\frac{T'(t)}{T(t)} = \frac{1}{r^2}\frac{Y''(\theta)}{Y(\theta)} + \frac{1}{r}\frac{X'(r)}{X(r)} + \frac{X''(r)}{X(r)} - \mu.$$

Now we put

$$\frac{1}{r^2}\frac{Y''(\theta)}{Y(\theta)} + \frac{1}{r}\frac{X'(r)}{X(r)} + \frac{X''(r)}{X(r)} = -\lambda$$

where $\lambda$ is a constant and obtain that $T(t) = e^{-(\lambda+\mu)t}T(0)$. Putting

$$\frac{Y''(\theta)}{Y(\theta)} = -\beta$$

where $\beta$ is another constant, we get the two remaining ODE (that are both of Sturm-Liouville type):

$$Y''(\theta) + \beta Y(\theta) = 0, \tag{3.69}$$

$$r[rX'(r)]' + (\lambda r^2 - \beta)X(r) = 0, \tag{3.70}$$

where $\lambda$ and $\beta$ are supposed given. To solve (3.69) we need to impose suitable boundary conditions. As $\theta$ represents an angle, we must assume $2\pi-$periodicity of the function $Y$ :

$$Y(0) = Y(2\pi).$$

For this problem we obtain two types of solutions:

- the linear one: $Y(\theta) = A + B\theta$, corresponding to $\beta = 0$;
- the families of indipendent solutions generated by $Y_{n,1} = \cos n\theta$ and $Y_{n,2} = \sin n\theta$, corresponding to $\beta = n^2$.

Concerning (3.70), we replace $\beta$ with $n^2$, and transform this equation in the usually Sturm-Liouville form:

$$[rX']' - \frac{n^2}{r}X + \lambda r X = 0 \tag{3.71}$$

Note that $r = 0$ is a singular point for the differential equation. Therefore we require as boundary condition the boundedness of $X$ as $r \to 0$, and the condition $X(1) = 0$, as requested by (3.68). Suppose first that $\lambda > 0$. Then the change of variables $s = r\sqrt{\lambda}$ transforms (3.71) into the Bessel equation of order $n$, (B.7)

$$s^2 X'' + sX' + (s^2 - n^2)X = 0. \tag{3.72}$$

Hence the solution of (3.72) can be expressed as a linear combination of Bessel functions of first and second type and order $n$ :

$$X(r) = AJ_n(s) + BY_n(s) = AJ_n(\sqrt{\lambda}r) + BY_n(\sqrt{\lambda}r).$$

But $Y_n$ is unbounded as $r \to 0$, so that $B = 0$ and we use only the Bessel functions of first type $J_n$. If we impose the other boundary condition, we obtain that $J_n(\sqrt{\lambda}) = 0$, that is $\sqrt{\lambda}$ must be a root of $J_n$. Now $J_n$ has an infinite number of positive zeros, which we will denote by $z_k^n$, so we put $\lambda_k^n = (z_k^n)^2$. Then for each $n$ the set of the eigenvalues of (3.72) is $\{\lambda_i^n\}_{i\in\mathbb{N}^+}$ and the eigenfunctions are $\{\phi_i^n\}_{i\in\mathbb{N}^+}$, where $\phi_k^n(r) = J_n(z_k^n r)$.
Now we deal with the problem of satisfying the initial condition: we start with a solution of this form:

$$\begin{aligned} p(t,r,\theta) \quad = \quad & \sum_{k=1}^{\infty} c_k^0 \phi_k^0(r) e^{-((z_k^0)^2+\mu)t} \\ & + \sum_{n,k=1}^{\infty} \phi_k^n(r)[c_k^n \cos n\theta + d_k^n \sin n\theta] e^{-((z_k^n)^2+\mu)t} \end{aligned} \tag{3.73}$$

where we have to determine the $c_k^n, d_k^n$ in order to satisfy the second equation of (3.68). That is equivalent, for every $a = 1, \ldots, a_\dagger$ to write $u_a(r,\theta)$ as a double Fourier-Bessel series:

$$u_a(r,\theta) = \sum_{k=1}^{\infty} \alpha_{k,a}^0 \phi_k^0(r) + \sum_{n,k=1}^{\infty} \phi_k^n(r)[\alpha_{k,a}^n \cos n\theta + \gamma_{k,a}^n \sin n\theta]. \tag{3.74}$$

From Theorem B.2.3 in Appendix B we have that if $u_a(r,\theta) \in L^2([0,2\pi] \times [0,1], rdrd\theta)$ then it can be expanded in Fourier-Bessel series. Hence we consider $u_a(r,\theta)$ as a function of $\theta$ with $r$ fixed and writing it as a Fourier series:

$$u_a(r,\theta) = \eta_{0,a}(r) + \sum_{n=1}^{\infty}[\eta_{n,a}(r)\cos n\theta + \xi_{n,a} \sin n\theta]$$

where

$$\begin{aligned}\eta_{0,a}(r) &= \frac{1}{2\pi}\int_0^{2\pi} u_a(r,\theta)d\theta,\\ \eta_{n,a}(r) &= \frac{1}{\pi}\int_0^{2\pi} u_a(r,\theta)\cos n\theta d\theta,\\ \xi_{n,a}(r) &= \frac{1}{\pi}\int_0^{2\pi} u_a(r,\theta)\sin n\theta d\theta.\end{aligned} \tag{3.75}$$

Then we expand the $\eta_{n,a}$, $\xi_{n,a}$ in Fourier-Bessel series:

$$\begin{aligned}\eta_{0,a}(r) &= \sum_{k=1}^{\infty}\eta_k^{0,a}\phi_k^0(r), \qquad \eta_{n,a}(r) = \sum_{k=1}^{\infty}\eta_k^{n,a}\phi_k^n(r),\\ \xi_{n,a}(r) &= \sum_{k=1}^{\infty}\xi_k^{n,a}\phi_k^n(r)\end{aligned} \tag{3.76}$$

where

$$\begin{aligned}\eta_k^{0,a} &= \frac{\int_0^1 \eta_{0,a}(r)\phi_k^0(r)rdr}{\int_0^1 (\phi_k^0(r))^2 rdr}, \qquad \eta_k^{n,a} = \frac{\int_0^1 \eta_{n,a}(r)\phi_k^0(r)rdr}{\int_0^1 (\phi_k^n(r))^2 rdr},\\ \xi_k^{n,a} &= \frac{\int_0^1 \xi_{n,a}(r)\phi_k^0(r)rdr}{\int_0^1 (\phi_k^n(r))^2 rdr}.\end{aligned} \tag{3.77}$$

Then substituting (3.75) into (3.77) we obtain the coefficients for (3.74)

$$\begin{aligned}\alpha_{k,a}^0 &= \frac{\int_0^1\int_0^{2\pi} u_a(r,\theta)\phi_k^0(r)rd\theta dr}{\int_0^1\int_0^{2\pi}(\phi_k^0(r))^2 rdr},\\ \alpha_{k,a}^n &= \frac{\int_0^1\int_0^{2\pi} u_a(r,\theta)\phi_k^n(r)\cos n\theta rd\theta dr}{\int_0^1\int_0^{2\pi}(\phi_k^n(r))^2 rdr},\\ \gamma_{k,a}^n &= \frac{\int_0^1\int_0^{2\pi} u_a(r,\theta)\phi_k^n(r)\sin n\theta rd\theta dr}{\int_0^1\int_0^{2\pi}(\phi_k^n(r))^2 rdr}.\end{aligned}$$

Similarly we obtain the expansions for all functions $u_a$ within the initial conditions. Therefore substituting them in (3.73), we obtain finally:

$$
\begin{aligned}
p(t,r,\theta) &= \sum_{k=1}^{\infty}\Big(\sum_{a=1}^{a_\dagger}\alpha_{k,a}^{0}b_a\Big)\phi_k^0(r)e^{-((z_k^0)^2+\mu)t} \\
&+ \sum_{n,k=1}^{\infty}\left[\Big(\sum_{a=1}^{a_\dagger}\alpha_{k,a}^{n}b_a\Big)\cos n\theta + \Big(\sum_{a=1}^{a_\dagger}\gamma_{k,a}^{n}b_a\Big)\sin n\theta\right]\phi_k^n(r)e^{-((z_k^n)^2+\mu)t}.
\end{aligned} \tag{3.78}
$$

Now integrating with respect to $t$ we obtain the recurrence relation (3.59), where the index $m$ again appears:

$$
\begin{cases}
u_1^{m+1}(r,\theta) = S_0(U^m(r,\theta))\left[\sum_{k=1}^{\infty}\Big(\sum_{a=1}^{a_\dagger}\alpha_{k,a}^{0,m}b_a\Big)\dfrac{\phi_k^0(r)}{((z_k^0)^2+\mu)}\right. \\
\left.+\sum_{n,k=1}^{\infty}\left[\Big(\sum_{a=1}^{a_\dagger}\alpha_{k,a}^{n,m}b_a\Big)\cos n\theta + \Big(\sum_{a=1}^{a_\dagger}\gamma_{k,a}^{n,m}b_a\Big)\sin n\theta\right]\dfrac{\phi_k^n(r)}{((z_k^n)^2+\mu)}\right] \\
u_a^{m+1}(r,\theta) = \sigma_{a-1}\sum_{k=1}^{\infty}\eta_{k,m}^{0,a}\phi_k^0(r) \\
+\sigma_{a-1}\sum_{n,k=1}^{\infty}\phi_k^n(r)(\eta_{k,m}^{n,a}\cos n\theta + \xi_{k,m}^{n,a}\sin n\theta)
\end{cases} \tag{3.79}
$$

### Neumann boundary condition

In this case the problem is the following:

$$
\begin{cases}
\dfrac{\partial p}{\partial t}(t,r,\theta) = \dfrac{1}{r^2}\dfrac{\partial^2 p}{\partial\theta^2}(t,r,\theta) + \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial p}{\partial r}(t,r\theta)\right) - \mu p(t,r,\theta) \\
p(0,r,\theta) = \sum_{a=1}^{a_\dagger} b_a u_a^m(r,\theta) \\
p_r(t,1,\theta) = 0
\end{cases} \tag{3.80}
$$

We can proceed as above, using the *separation of variables.* The only difference is that the set of eigenvalues of (3.72) with Neumann boundary conditions is $\{\lambda_i^n\}_{i\in\mathbb{N}^+}$, with $\lambda_k^n = (\xi_k^n)^2$, where $\xi_k^n$ is the $k$-th zero of $J_n'$. On the other hand, the set of eigenfunction is $\{\psi_i^n\}_{i\in\mathbb{N}^+}$ where $\psi_k^n(r) = J_n(\xi_k^n r)$ (Dini functions, see Appendix B). Except for that, the computations are the same, i.e. we expand the initial condition in Dini series and calculate the solution of (3.80). Observe that the expansion in Dini series is valid, since the new system of eigenfunctions is again orthonormal and complete (see Appendix B). Therefore the solution of (3.80) is:

$$\begin{aligned} p(t,r,\theta) &= \sum_{k=1}^{\infty}(\sum_{a=1}^{a_\dagger}\delta_{k,a}^{0}b_a)\psi_k^0(r)e^{-((\xi_k^0)^2+\mu)t} \\ &+ \sum_{n,k=1}^{\infty}\Big[(\sum_{a=1}^{a_\dagger}\delta_{k,a}^{n}b_a)\cos n\theta + (\sum_{a=1}^{a_\dagger}\nu_{k,a}^{n}b_a)\sin n\theta\Big]\psi_k^n(r)e^{-((\xi_k^n)^2+\mu)t} \end{aligned} \tag{3.81}$$

where

$$\delta_{k,a}^0 = \frac{\displaystyle\int_0^1\int_0^{2\pi} u_a(r,\theta)\psi_k^0(r) r d\theta dr}{\displaystyle\int_0^1\int_0^{2\pi} (\psi_k^0(r))^2 r dr},$$

$$\delta_{k,a}^n = \frac{\displaystyle\int_0^1\int_0^{2\pi} u_a(r,\theta)\psi_k^n(r)\cos n\theta r d\theta dr}{\displaystyle\int_0^1\int_0^{2\pi} (\psi_k^n(r))^2 r dr},$$

$$\nu_{k,a}^n = \frac{\displaystyle\int_0^1\int_0^{2\pi} u_a(r,\theta)\psi_k^n(r)\sin n\theta r d\theta dr}{\displaystyle\int_0^1\int_0^{2\pi} (\psi_k^n(r))^2 r dr}.$$

Now substituting it in (3.59) and integrating with respect to $t$ we obtain:

$$\begin{cases} u_1^{m+1}(r,\theta) = S_0(U^m(r,\theta))\Big[\displaystyle\sum_{k=1}^{\infty}(\sum_{a=1}^{a_\dagger}\delta_{k,a}^{0,m}b_a)\frac{\psi_k^0(r)}{((\xi_k^0)^2+\mu)} \\ +\displaystyle\sum_{n,k=1}^{\infty}\Big[(\sum_{a=1}^{a_\dagger}\delta_{k,a}^{n,m}b_a)\cos n\theta + (\sum_{a=1}^{a_\dagger}\nu_{k,a}^{n,m}b_a)\sin n\theta\Big]\frac{\psi_k^n(r)}{((\xi_k^n)^2+\mu)}\Big] \\ u_a^{m+1}(r,\theta) = \sigma_{a-1}\displaystyle\sum_{k=1}^{\infty}\eta_{k,m}^{0,a}\phi_k^0(r) \\ +\sigma_{a-1}\displaystyle\sum_{n,k=1}^{\infty}\phi_k^n(r)(\eta_{k,m}^{n,a}\cos n\theta + \xi_{k,m}^{n,a}\sin n\theta) \end{cases} \tag{3.82}$$

### 3.3.3 Regularity and uniqueness of solutions

Here we deal with the regularity and uniquess of solutions of (3.60) and (3.65) with Dirichlet or Neumann boundary conditions.
Concerning the uniqueness we use the *energy method*. In particular we have the following result:

**Theorem 3.3.1.** *Let $A = \Omega\times]0,T[$, where $\Omega$ is an open bounded set of $\mathbb{R}^n$ with $\partial\Omega$ piecewise continuously differentiable. Then if $f \in C(A), \phi \in C(\Omega)$ and $\psi \in C(\partial\Omega \times [0,T])$, then the problem:*

$$\begin{cases} u_t - \Delta u = f & (x,t) \in A \\ u(x,0) = \phi(x) & x \in \Omega \\ \left(\alpha u(\cdot,t) + \beta\frac{\partial u(\cdot,t)}{\partial\nu}\right)\Big|_{\partial\Omega} = \psi & t \in [0,T], \end{cases} \tag{3.83}$$

*where $\alpha, \beta$ are non negative constants and not both zero, has at most one solution $u \in C(\bar{A}) \cap C^{2,1}(A)$ with $\frac{\partial u}{\partial\nu} \in C(\bar{A})$*

*Proof.* Let $u, v$ two solutions of the problem and put $w = u - v$, then $w$ satisfies (3.83) with $f, \phi, \psi = 0$, then multiplying the first equation for $w$ and integrating among $\Omega$ with fixed $t$ and applying the *Green formula*, we get

$$\begin{aligned} \frac{1}{2}\int_\Omega |w(x,t)|^2 dx &= \int_\Omega w_t u dx = \int_\Omega (f + \Delta w) w dx \\ &= \int_{\partial\Omega} \frac{\partial w}{\partial\nu} w d\sigma - \int_\Omega |Dw|^2 dx. \end{aligned}$$

Now we note that the last equation of (3.83) imply that in $\partial\Omega$

$$w\frac{\partial w}{\partial\nu} = -\frac{\beta}{\alpha}\left|\frac{\partial u}{\partial\nu}\right|^2 \leq 0 \quad \text{if } \alpha > 0, \qquad w\frac{\partial w}{\partial\nu} = -\frac{\alpha}{\beta}|w|^2 \leq 0 \quad \text{if } \beta > 0$$

then

$$\int_{\partial\Omega} w\frac{\partial w}{\partial\nu} d\sigma \leq 0.$$

This implies $w \equiv 0$ and the thesis is proved. □

The previous theorem can be applied to both (3.60) and (3.65) with Dirichlet or Neumann boundary conditions. Obviously the above argument implies also uniqueness for the discrete problem (3.59).
Now we deal with existence of solutions of (3.60): if this problem as a solution, it has the form (3.61) for Dirichlet boundary conditions, and (3.63) for Neumann boundary conditions. Now we observe that, if

$$\sum_{n=1}^{\infty} |\gamma_{n,a}| < \infty, \quad a = 1, \ldots, a_\dagger,$$

then, the series expansions of the initial conditions, converges uniformly to continuous functions. Moreover the series (3.61) is uniformly convergent with all

its derivatives, due to the presence of a negative exponential; hence (3.61) solves the differential equation and (3.62) solves the discrete problem. We can rephrase the previous argument for Neumann boundary condition: instead in this case we require that $\sum_{n=1}^{\infty} |\delta_{n,a}| < \infty$.
Concerning the two dimensional problems, we need more restrictive hypotheses: under Dirichlet boundary conditions, due to the uniform boundedness of the Bessel functions (see Appendix B), if

$$\sum_{k=0}^{\infty} |\alpha_{k,a}^{0}| < \infty, \quad \sum_{n,k=1}^{\infty} n^2[|\alpha_{k,a}^{n}| + |\gamma_{k,a}^{n}|] < \infty, \quad \text{for } a = 1, \ldots, a_{\dagger},$$

then the series $p$ with its first and second derivatives (with respect to $r$ and $\theta$) converges uniformly and also the derivative with respect to $t$ converges uniformly. Then (3.78) is a solution of (3.65), hence (3.79) solves the discrete problem. For Neumann boundary conditions, we impose the same conditions on $\delta_{k,a}^{n}$ and $\nu_{k,a}^{n}$. The above assumptions about the diffusion problem in dimension 1 or 2, are clearly sufficient in order to obtain a unique regular solution. For the sake of brevity we do not care to look for optimal conditions still guaranteeing this result.

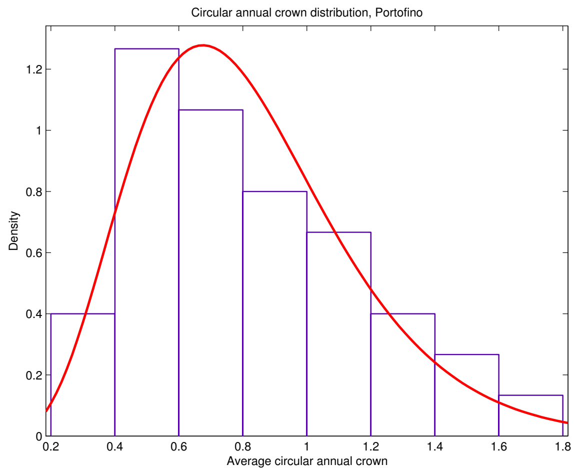
# Chapter 4

# Numerical computations

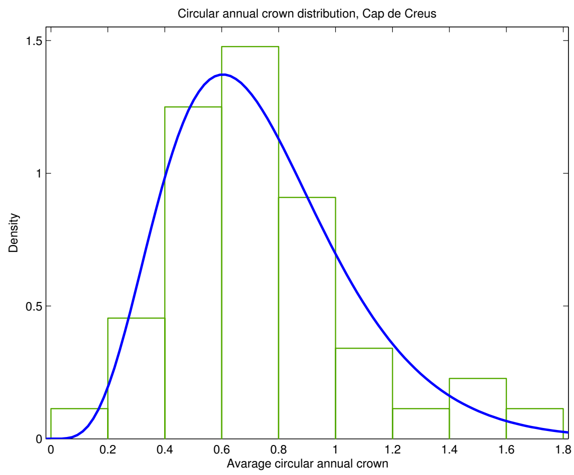
In this chapter we perform the computations concerning the two discrete models presented in chapter 3 ((3.3) and (3.8)), for both Portofino and Cap de Creus populations. In particular, we determine the associated parameters and functions from the data and analyze the stability or unstability, of their steady states.
The presented data (see chapter 2) allow us to divide Portofino population in 90 age classes and Cap de Creus population in 60 age classes, this age structure will be taken for both our discrete models. The considered cutoff function $S_0$ has the form (3.1). Moreover, due to the lack of experimental values, we used the same function of [SBI07] (scaled with respect to the basic reproduction number of the population). In the first section we present the estimation of the required parameters for both models. In the second section, assuming the current status of the populations as a steady state, we deal with the stability fo this equilibrium.

## 4.1 Determination of parameters

Here we present the estimations of parameters and functions within the two discrete models: we start with general ones and then we deal with survival and reprocuctive parameters.
To determine the function $g(\omega)$ within the model (3.8), we approximate the distribution of the growth rates (with respect to the area) of the colonies contained in the second level of data (see chapter 2). In both cases we obtain a fit with two *Gamma distribution functions* (i.e. with density of the form $g(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{a\alpha-1} e^{\frac{-x}{\beta}}$, with $\alpha$ and $\beta$ positive parameters), as seen in figure 4.1.




(a) Portofino distribution: $\alpha = 5.84, \beta = 0.1394$




(b) Cap de Creus distribution:$\alpha = 5.48, \beta = 0.1349$

Figure 4.1: Average growth rate distributions

Concerning the definitions of density of the population within a certain area (in particular the third equation of (3.3) and (3.8)), we set $\omega_i = \frac{1}{48}$, since the analyzed region have an area of $48\ \mathrm{dm}^2$ and $H_i(\omega) = \frac{i\omega}{48}$. We remark that this choice does not satisfy condition (3.11). This is not a problem since we have chosen $g(\omega) = ce^{-\frac{\omega}{b}}$, so that the integral (see chapter 3)

$$\int_0^{\infty} H_i(\omega)\alpha_i(\omega)d\omega = \int_0^{\infty} H_i(\omega)\Pi(\omega)g(\omega)d\omega$$

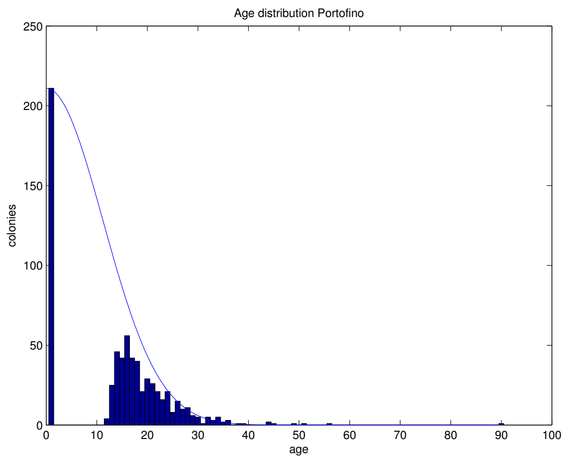
is finite.

### 4.1.1 Survival rates

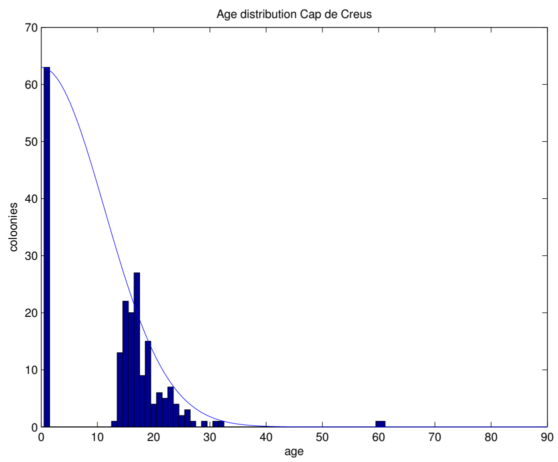
The survival rates $\sigma_i$ have been estimated through a fit of the age distributions in figure 2.4: due to the scarcity of informations, we have assumed for both models that the survival rates depend only on age (i.e. they are constant with respect to the variable $\omega$). We approximate the age distributions with *Rayleigh distribution functions*, i.e. with densities of the form

$$f(x) = \frac{x}{\beta^2} e^{-(\frac{x^2}{2\beta^2})},$$

where $\beta$ is a positive parameter.




(a) Portofino survival function: $\beta = 11.13$




(b) Cap de Creus survival function: $\beta = 11.25$

Figure 4.2: Fit of survival functions

From figure (4.2) we see that the lack of several age classes within the distributions determines an unsatisfactory accuracy in the estimation.
Next, assuming that the actual distributions are stationary, we use them in order to evaluate the survival rates from them, so that the estimation of these rates cannot be better than the previous ones.

### 4.1.2 Reproductive parameters

To estimate the reproductive parameters we use, as much as possible, the data presented in [Vie09]; but they were sometimes not sufficient, hence we take the values of the missing parameters from the study of Calafuria population, [SBI07].
The average number of planulae produced by a colony with age $i$ are estimated, in both two models, in the following way:

$$b_i = q \cdot F_i \cdot sr \cdot P_i, \quad i = 1, \dots, a_\dagger$$

where

- $sr$ is the sex ratio (that we suppose independent from the age);
- $F_i$ is the *fertility*, i.e. the percentage of fertile female colonies;
- $P_i$ is the number of polyps in each colony;
- $q$ is the *fecundity*, i.e. the planulae produced by each polyp (we assume that this parameter does not depend on age $i$, too).

For both models we set $sr = \frac{1}{2}$ and

$$F_i = \begin{cases} 1 & \text{if } i \geq 7 \\ 0 & \text{if } i < 7 \end{cases}, \quad F_i = \begin{cases} 1 & \text{if } i \geq 12 \\ 0 & \text{if } i < 12 \end{cases}$$

for Portofino and Cap de Creus respectively.
Then we set $q = 0.87$, as estimated in [SBI07] for Calafuria population.
The number of polyps for age is estimated in two different ways within the two models: in (3.3) we use the estimation done in [Vie09]:

$$\begin{aligned} P(i) &= 29.71 \cdot 1.09^i \quad \text{for Portofino;} \\ P(i) &= 24.97 \cdot 1.11^i \quad \text{for Cap de Creus.} \end{aligned}$$

On the other hand for (3.8) we use an estimation of the number of polyps with respect to the average diameter, expressed as a function of the circular annual crown $\omega$ and age $i$:

$$\begin{aligned} P(i, \omega) &= 72.69\sqrt{i\omega} - 118.51 \quad \text{for Portofino;} \\ P(i, \omega) &= 85.01\sqrt{i\omega} - 160.822 \quad \text{for Cap de Creus.} \end{aligned}$$

## 4.2 Stability of the equilibria

Here we investigate whether the actual equilibria are stable: we transform (3.8) in the form (3.21), then we linearize both systems obtaining two systems of the form (3.6). Then we compute the eigenvalues of the associated matrices and apply theorem 3.1.1. Afterwards, we modify the basic reproduction numbers of the two populations, to see whether this variations implies a destabilization of the equilibria.
We must remark that, in order to study the linearized systems, we needed to modify the obtained age distributions to fill the lack within several age classes. Then we evaluate the missing age multiplying the number of recruits for the evalated survival rates; this distortion of the input data return another reason, to consider this computations not so significant.

| | **Cap de Creus discrete model** | **Cap de Creus discrete model with crown distribution** | **Portofino discrete model** | **Portofino discrete model with crown distribution** |
|---|---|---|---|---|
| $R_0$ | unstable | stable | stable | stable |
| $\frac{1}{2}R_0$ | unstable | unstable | unstable | stable |
| $2R_0$ | unstable | stable | stable | stable |
| $\frac{1}{4}R_0$ | unstable | unstable | unstable | unstable |
| $4R_0$ | unstable | stable | stable | stable |

The systems concerning Portofino appear stable also with respect to a slight destabilization, on the other hand the population of Cap de Creus looks vulnerable and possibly unstable. Furthermore the two populations, analyzed with the crown distribution model, seem to have a better behaviour. However, the non-homogeneity and scarcity of the data forces us to consider the outcome of this computations as nothing more than a shady indication.

# Appendix A

# Laplace transform and Volterra integral equations

This appendix is devoted to present some results concerning the applications of Laplace transform theory to the study of Volterra equations. Indeed we may investigate the asymptotic behaviour of solutions of Volterra integral equations by analyzing the roots of certain equations involving the Laplace transform. This presentation is based on [Ian94], except for the proof of Paley-Wiener Theorem that has been reconstructed directly from Paley-Wiener [PW33].
In the first section we present two equations involving the Laplace transform of a function and prove two result concerning their roots. In the second section we state some basic results about linear integral equations (without proof), then we prove the Paley-Wiener Theorem [PW33] and finally we analyze a special nonlinear integral equation.
Our references are G. Doetsch's book [Doe74] for the Laplace transform, and the monography G. Gripenberg, S.O. Londen and O. Staffans [GLS90] for Volterra equations.
Let us define the Laplace transform and prove some basic properties of this map:

**Definition A.0.1.** *Let $f(\cdot) \in L^1_{loc}(\mathbb{R}_+, \mathbb{R})$ and $\lambda \in \mathbb{C}$. We say that $f(\cdot)$ is Laplace transformable at $\lambda \in \mathbb{C}$, if*

$$\hat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt \tag{A.1}$$

*converges as an improper integral. Moreover $f(\cdot)$ is said absolutely Laplace transformable at $\lambda$ if the integral in (A.1) is absolutely convergent.*

## A.1 Laplace transform equations

In this section we deal with some simple specific equations involving the Laplace transform of a function.
Let $K \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, with $K(t) \geq 0 \quad a.e.$, $\|K\|_1 > 0$, and $K(t) = 0$ for $t > T$ for some $T > 0$ (these assumptions for $K$ hold throughout in this section).
The first equation we analyze is:

$$\hat{K}(\lambda) = 1 \tag{A.2}$$

and we can state that:

**Proposition A.1.1.** *The equation (A.2) has a unique real root $\alpha^*$ which is simple. Furthermore $\alpha^* < 0$ if and only if $\int_0^\infty K(t)dt < 1$ and $\alpha^* \geq 0$ if and only if $\int_0^\infty K(t)dt \geq 1$. Any other root $\alpha$ of (A.2) is such that $\Re\alpha < \alpha^*$. Moreover in each strip $\sigma_1 < \Re\lambda < \sigma_2$ there is at most a finite number of roots.*

*Proof.* Considering the real function

$$x \to \hat{K}(x) = \int_0^\infty e^{-xt} K(t)dt, \quad x \in \mathbb{R}, \tag{A.3}$$

is easy to see that it is strictly decreasing and tends to $0$ as $x \to \infty$. Hence it has only one real root $\alpha^*$, with

$$\frac{d}{dx}\hat{K}(x)\bigg|_{x=\alpha^*} = -\int_0^\infty t e^{-\alpha^* t} K(t)dt < 0;$$

hence $\alpha^*$ is simple. Moreover the sign of $\alpha^*$ depends on the sign of $\hat{K}(0) - 1 = \int_0^\infty K(t)dt - 1$.
Let $\alpha$ be another solution of (A.2); then:

$$\begin{aligned}\int_0^\infty e^{-\alpha^* t} dt = 1 &= \int_0^\infty e^{-\alpha t} K(t)dt = \Re \int_0^\infty e^{-\alpha t} K(t)dt \\ &= \int_0^\infty e^{-\Re\alpha t} \cos(\Im\alpha t) K(t)dt < \int_0^\infty e^{-\Re\alpha t} K(t)dt\end{aligned}$$

consequently we get $\Re\alpha < \alpha^*$. Concerning the last part we can observe that, since $\hat{K}(\lambda) \to 0$ as $|\lambda| \to \infty$, then all roots in a strip of the form $\sigma_1 < \Re\lambda < \sigma_2$ must belong to some bounded subset. Hence they must be finitely many, since otherwise the entire function $\hat{K}(\lambda)$ would vanish identically. □

Now we consider another equation in the complex plane which is a generalization of the previous one:

$$\hat{K}(\lambda) + F(\lambda, \tau) = 1. \tag{A.4}$$

Here we assume that : $K$ satisfies (in addition to the previous hypotheses)

$$\int_0^\infty K(t)dt = 1. \tag{A.5}$$

On the other hand we suppose that $F(\lambda, \tau) : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ is continuosly differentiable and verifies:

$$F(\lambda, 0) = 0 \qquad \forall \lambda \in \mathbb{C}, \tag{A.6}$$

$$\frac{\partial F}{\partial \tau}(0,0) > 0, \tag{A.7}$$

$$\exists M, \beta > 0 : |F(\lambda, \tau)| < M|\tau| \text{ for } \Re\lambda \geq -\beta \text{ and } \tau \text{ small enough.} \tag{A.8}$$

Under these particolar assumptions, we have:

**Theorem A.1.1.** *There exists $\delta > 0$ such that, if $\tau \in [0, \delta]$, then (A.4) has a real positive root. If otherwise $\tau \in [-\delta, 0]$, all roots have negative real part.*

*Proof.* From the previous proposition, (A.5) implies that the equation

$$\hat{K}(\lambda) = 1$$

has the real root $\lambda_0 = 0$, which is unique in the whole half plane $\Re\lambda \geq \alpha$, where $\alpha \in (-\beta, 0)$. With sufficiently small $\beta$ set

$$m = \inf_{y \in \mathbb{R}} |1 - \hat{K}(\alpha + iy)| > 0$$

and take $L > 0$ such that

$$\frac{1}{2} < |1 - \hat{K}(\lambda)| \quad \text{for } |\lambda| < L, \Re\lambda \geq \alpha$$

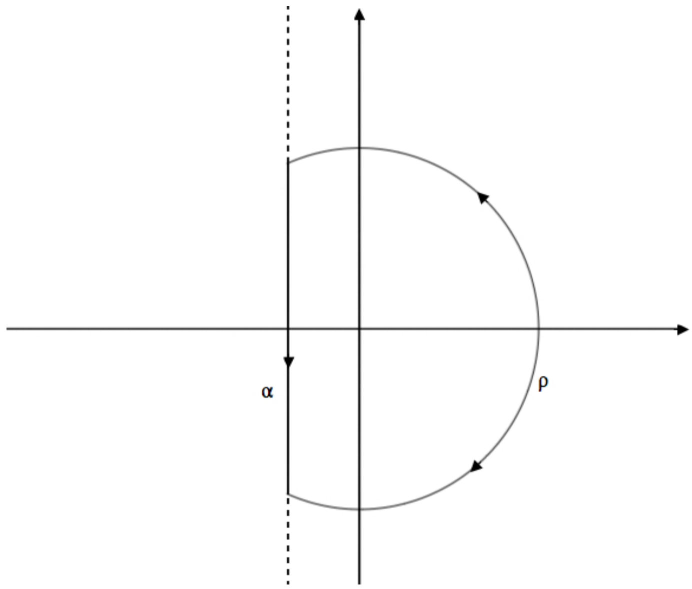



Figure A.1

Now, if $\tau$ satisfies (A.8) and is such that:

$$|\tau| < \frac{\left(m \wedge \frac{1}{2}\right)}{M},$$

we obtain that, on the contour of any domain $\Sigma_\rho$ (as shown in figure A.1) with $\rho > L$, the following inequalities hold:

$$|F(\lambda, \tau)| < \left(m \wedge \frac{1}{2}\right) < |1 - \hat{K}(\lambda)|.$$

Now for Rouché theorem, equation (A.4) has only one root in the interior of $\Sigma_\rho$ and no roots outside; hence it has only one root in the whole half plane $\Re\lambda \geq \alpha$. Concerning the location of this root, let $\lambda(\tau)$ be the differentiable path originating from $\lambda(0) = 0$, such that $\lambda(\tau)$ is a root of (A.4). We differentiate with respect to $\tau$ equation (A.4) and, by (A.6), we get

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=0} = \frac{\frac{\partial F}{\partial \tau}(0,0)}{\int_0^\infty tK(t)dt}.$$

In particular, by (A.7), we have $\frac{d}{d\tau}\Re\lambda(\tau) > 0$ in a neighbourhood of $0$. Hence we see that the path goes to the right of the imaginary axis as $\tau$ increases from $0$, nd goes to the left if $\tau$ decreases. □

## A.2 Volterra linear equations and a special nonlinear case

In this section we present some results from the theory of Volterra integral equations. A reference for the first results (presented without proof) is [GLS90]; on the other hand our proof of Paley Wiener Theorem is different from that provided in this book and is based on what stated in [PW33].

### A.2.1 Introduction to the linear case

We consider the linear Volterra system of equations

$$u(t) = \int_0^t K(t-s)u(s)ds + f(t) \tag{A.9}$$

where $u(t), f(t)$ are $n$-vectors and $K(t)$ is a $n \times n$ matrix. In order to treat these equations we need some assumptions: we suppose that

$$K \in L^1([0,\infty);\mathcal{L}(\mathbb{R}^n)), \tag{A.10}$$

$$f \in L^1([0,\infty);\mathbb{R}^n). \tag{A.11}$$

Now we present a result that proves the existence of a function $R(t)$, said *resolvent*, which helps to analyze equation (A.9).

**Theorem A.2.1.** *Let $K$ satisfy (A.10); then there exists a unique $R \in L^1_{loc}([0,\infty);\mathcal{L}(\mathbb{R}^n))$ such that*

$$R(t) = -K(t) + \int_0^t K(t-s)R(s)ds, \tag{A.12}$$

$$R(t) = -K(t) + \int_0^t R(t-s)K(s)ds. \tag{A.13}$$

*Moreover for all $f \in L^1([0,\infty);\mathbb{R}^n)$*

$$u(t) = f(t) - \int_0^t R(t-s)f(s)ds \tag{A.14}$$

*is the unique solution of (A.9).*

This result is a useful tool to represent the solutions of (A.9). Furthermore, as we will see later, the stability for solutions of (A.9) is determined by some properties of resolvent function.
Now, let $u(t) \equiv 0$ be the *trivial* solution of (A.9) with $f(t) = 0$. We define a concept of stability for this solution for continuous and bounded inputs (i.e. $f \in C_B([0,\infty);\mathbb{R}^n)$).

**Definition A.2.1.** *The trivial solution of (A.9) is stable if for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $\|f\|_\infty < \delta$ *then* $\|u\|_\infty < \epsilon$.
*Moreover it is asymptotically stable if it is stable and* $\lim_{t\to\infty} f(t) = 0$ *implies* $\lim_{t\to\infty} u(t) = 0$.

With this definition we can present the result announced before:

**Theorem A.2.2.** *Let* $f \in C_B([0,\infty); \mathbb{R}^n)$*; then the trivial solution of (A.9) is asymptotically stable if and only if* $R$ *satisfy:*

$$R \in L^1([0,\infty); \mathcal{L}(\mathbb{R}^n)). \tag{A.15}$$

From this theorem we can develop some results concerning the stability of solutions of Volterra linear and nonlinear equations, using Laplace transform, as we will see in the next section.

### A.2.2 The Paley-Wiener theorem

We have just seen that a necessary and sufficient condition for stability of the trivial solution of (A.9) is that the resolvent is absolutely integrable. Now we state an equivalent condition involving the Laplace transform of the kernel $K$: the Paley-Wiener theorem.

**Theorem A.2.3** (Paley-Wiener). *Suppose that* $K$ *satisfies (A.10); then the resolvent* $R(\cdot)$ *satisfies* $R(\cdot) \in L^1([0,\infty); \mathcal{L}(\mathbb{R}^n))$ *if and only if*

$$\det(I - \hat{K}(\lambda)) \neq 0 \qquad \text{for } \Re\lambda \geq 0. \tag{A.16}$$

*Proof.* We prove the result in the scalar case since the general one is analogous. First suppose that (A.15) holds; then $R(\cdot)$ is absolutely Laplace transformable for $\Re\lambda \geq 0$ and from (A.12) we get $\hat{K}(\lambda)\hat{R}(\lambda) - \hat{K}(\lambda) - \hat{R}(\lambda) = 0$, i.e.

$$(1 - \hat{K}(\lambda))(1 - \hat{R}(\lambda)) = 1 \quad \text{for } \Re\lambda \geq 0.$$

So (A.16) is satisfied.
Concerning the sufficiency, we start with proving that if (A.16) holds, then $R(\cdot)$ is absolutely Laplace transformable for $\Re\lambda$ sufficiently large. Let $\lambda_0 > 0$ be such that

$$\int_0^\infty e^{-\lambda_0 t}|K(t)|dt = a < 1$$

then from (A.12) we obtain, for $\Re\lambda \geq \lambda_0$:

$$\begin{aligned} \int_0^T e^{-\lambda t}|R(t)|dt &\leq a + \int_0^T e^{-\lambda t}\int_0^t |K(t-s)||R(s)|dsdt \\ &= a + \int_0^T e^{-\lambda s}|R(s)|ds \int_s^T e^{-\lambda(t-s)}|K(t-s)|dt \end{aligned}$$

and hence

$$\int_0^T e^{-\lambda s}|R(s)|ds \leq \frac{a}{1-a},$$

which for $T \to \infty$ implies the absolute Laplace transformability of $R(\cdot)$ for $\Re\lambda \geq \lambda_0$. Moreover $\hat{R}(\lambda)$ satisfies, by (A.12),

$$\hat{R}(\lambda) = \frac{\hat{K}(\lambda)}{\hat{K}(\lambda)-1} \quad \text{for } \Re\lambda \geq \lambda_0. \tag{A.17}$$

Now for convenience of notation we set $f(w) = \hat{K}(w)$, and

$$g(w) = \frac{f(w)}{f(w)-1}; \tag{A.18}$$

we want to prove that $g$ is the Laplace transform of a function belonging to $L^1(\mathbb{R})$.
Consider for $A \in \mathbb{R}^+$ the function:

$$\phi_A(u) = \begin{cases} 1 & \text{if } |u| < A \\ 2 - \frac{|u|}{A} & \text{if } A \leq |u| < 2A \\ 0 & \text{if } |u| \geq 2A. \end{cases} \tag{A.19}$$

Then the two functions:

$$g_1(u) = \phi_A(u)g(iu), \tag{A.20}$$

$$g_2(u) = [1-\phi_A(u)]g(iu), \tag{A.21}$$

satisfy $g(iu) = g_1(u)+g_2(u)$. We are going to prove that, if $A$ is sufficiently large, then $g_1$ and $g_2$ are Fourier transforms of functions in $L^1$. Applying the backward Fourier transform to $\phi_A$ we get:

$$\begin{aligned} h_A(x) &= \frac{1}{2\pi}\int_0^\infty e^{ixu}\phi_A(u)du \\ &= \frac{1}{2\pi}\int_{-2A}^{A} e^{ixu}\phi_A(u)du + \frac{1}{2\pi}\int_{-A}^{A} e^{ixu}du + \frac{1}{2\pi}\int_{A}^{2A} e^{ixu}\phi_A(u)du \\ &= \frac{1}{2\pi}\frac{e^{-iAx}-e^{-2iAx}}{x^2A} - \frac{1}{2\pi}\frac{e^{2iAx}-e^{-iAx}}{x^2A} \\ &= \frac{1}{\pi A}\left[\frac{\cos(Ax)-\cos(2Ax)}{x^2}\right]. \end{aligned}$$

Hence $h_A$ belongs to $L^1([0,\infty))$. Moreover extending $K$ as 0 in $\mathbb{R}^-$, we have that $f(iu) = \mathscr{F}(K)(u)$. On the other hand, let us define

$$\eta(u) = \begin{cases} \frac{1}{f(iu)-1} & \text{if } |u| \leq 2A \\ \eta_1(u) & \text{if } 2A < |u|, \end{cases}$$

where $\eta_1(u)$ is a $C_0^\infty$ function such that $\eta_1(u) = 1$ for $|u| \leq 2A$. Then we have $\eta \in C_0^\infty \subseteq \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space. As the Fourier transform is bijective on $\mathcal{S}(\mathbb{R})$, there is $\theta \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ such $\mathscr{F}(\theta)(u) = \eta(u)$ for every $u \in \mathbb{R}$. In particular $g_1(u) = \phi_A(u) f(iu)\eta(u) = \mathscr{F}(h_A * K * \theta)$ is the Fourier transform of a convolution product of $L^1$ functions, which belongs to $L^1$.
Concerning $g_2$, we have:

$$g_2(u) = [1-\phi_A(u)]g(iu) = [1-\phi_A(u)]\frac{f(iu)[1-\phi_{\frac{A}{2}}(u)]}{f(iu)[1-\phi_{\frac{A}{2}}(u)]-1};$$

formally we also have

$$\begin{aligned} g_2(u) &= -[1-\phi_A(u)]f(iu)[1-\phi_{\frac{A}{2}}(u)]\textstyle\sum_{n=0}^{\infty} f(iu)^n[1-\phi_{\frac{A}{2}}(u)]^n \\ &= -[1-\phi_A(u)]f(iu)\textstyle\sum_{n=0}^{\infty} f(iu)^n[1-\phi_{\frac{A}{2}}(u)]^n \end{aligned} \tag{A.22}$$

Hence if we prove that $\|\mathscr{F}^{-1}[f(iu)(1-\phi_{\frac{A}{2}}(u))]\|_1 < 1$, for large $A$, then the series on the right-hand side of (A.22) converges and so $\mathscr{F}^{-1}(g_2)$ is a series of functions belonging to $L^1(\mathbb{R})$. More precisely we remark that

$$\frac{2}{\pi A}\int_{-\infty}^{\infty} \frac{\cos(\frac{A}{2}\eta) - \cos(A\eta)}{\eta^2} d\eta = \int_{-\infty}^{\infty} h_{\frac{A}{2}}(\eta) d\eta = 1;$$

then, since $\mathscr{F}(h_{\frac{A}{2}}) = \phi_{\frac{A}{2}}$, we have

$$\|\mathscr{F}^{-1}[f(iu)(1-\phi_{\frac{A}{2}}(u))]\|_1 = \|K - K * h_{\frac{A}{2}}\|_1. \tag{A.23}$$

We consider now a sequence $\{K_m\}_{m\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R})$ such that

$$K_m \to K \text{ in } L^1(\mathbb{R}).$$

For fixed $m$, take $\epsilon \in ]0,1[$ and choose $\delta > 0$ such that

$$|K_m(x) - K_m(x+h)| \leq \epsilon \quad \forall x \in \mathbb{R}, \forall |h| \leq 2\delta,$$

and select $A$ large, in order that $\int_\delta^\infty |h_{\frac{A}{2}}(\eta)|d\eta < \epsilon$. Now let $M > 0$ such that $suppK_m \subseteq [-\frac{M}{2}, \frac{M}{2}]$, and $\max|K_m| \leq M$. Then

$$\begin{aligned}
\|K_m - K_m * \nu_{\frac{A}{2}}\|_1 &\leq \int_{-M}^{M}\int_{-\infty}^{\infty} |K(\xi) - K(\xi-\eta)||h_{\frac{A}{2}}(\eta)|d\eta d\xi \\
&\leq \int_{-M}^{M}\int_{\delta}^{\infty} |K(\xi) - K(\xi-\eta)| \cdot |h_{\frac{A}{2}}(\eta)|d\eta d\xi \\
&\quad + \int_{-M}^{M}\int_{-\delta}^{\delta} |K(\xi) - K(\xi-\eta)| \cdot |h_{\frac{A}{2}}(\eta)|d\eta d\xi \\
&\quad + \int_{-M}^{M}\int_{-\infty}^{-\delta} |K(\xi) - K(\xi-\eta)| \cdot |h_{\frac{A}{2}}(\eta)|d\eta d\xi \\
&\leq 4M^2\epsilon + 2M\epsilon^2 + 4M^2\epsilon.
\end{aligned}$$

Finally, choose $m$ such that $\|K - K_m\|_1 < \epsilon$. Then, by (A.23)

$$\begin{aligned}
\|K - K * h_{\frac{A}{2}}\|_1 &\leq \|K - K_m\|_1 + \|K_m - K_m * h_{\frac{A}{2}}\|_1 + \|K_m * h_{\frac{A}{2}} - K_m * h_{\frac{A}{2}}\|_1 \\
&\leq 2\epsilon + \|K_m - K_m * h_{\frac{A}{2}}\|_1 \leq c\epsilon.
\end{aligned}$$

We have so proved that $\|\mathscr{F}^{-1}(f(iu)(1-\phi_{\frac{A}{2}}(u)))\|_1$ is small provided $A$ is sufficiently large. Hence $g_2(u) = \mathscr{F}^{-1}(\gamma)(u)$, where

$$\gamma = \sum_{n=0}^{\infty}(K - K * h_A) * \underbrace{(K - K * h_{\frac{A}{2}}) * \ldots * (K - K * h_{\frac{A}{2}})}_{n} \in L^1(\mathbb{R}.)$$

This is equivalent to

$$\int_{-\infty}^{0} \gamma(\xi)e^{-iu\xi}d\xi = -\int_{0}^{\infty} \gamma(\xi)e^{-iu\xi}d\xi + g(iu).$$

Now for $\Re w \geq 0$ we have $f(w) \to 0$ as $|w| \to \infty$; hence by (A.16) there exists $c > 0$ such that $|f(w) - 1| \geq c$ for every $w$ in the positive half-plane. Thus

$$-\int_{0}^{\infty} \gamma(\xi)e^{-w\xi}d\xi + g(w) \tag{A.24}$$

is bounded and analytic for $\Re w \geq 0$. Similarly

$$\int_{-\infty}^{0} \gamma(\xi)e^{-w\xi}d\xi \tag{A.25}$$

is bounded and analytic in the left half-plane (including the imaginary axis). Furthermore the two functions coincide on the imaginary axis: then they are parts of the same analytic and bounded function on the whole complex plane. Hence by Liouville theorem, this function is constant. Now since

$$\lim_{w\to\infty}\int_{-\infty}^{0}\gamma(\xi)e^{-w\xi}d\xi = 0$$

this function is identically $0$. In particular (A.24) implies that:

$$g(w) = \int_{0}^{\infty}\gamma(\xi)e^{-w\xi}d\xi \tag{A.26}$$

Hence $g(w)$ is the Laplace transform of a function in $L^1(\mathbb{R})$; from (A.18) and (A.17) we have $g(w) = \hat{R}(w)$ for $\Re w \geq \lambda_0$, so that by uniqueness of the Laplace transform, we get $\gamma(w) = R(\omega)$, and the thesis is proved. □

### A.2.3 A nonlinear perturbation for the Volterra equation

In this section we consider a perturbation of the system of linear integral equations (A.9) of this form:

$$u(t) = \int_{0}^{t} K(t-s)u(s) + \mathcal{P}[u(\cdot), c(\cdot)](t) \tag{A.27}$$

where $K$ is linear and satisfies (A.10). The nonlinear part

$$\mathcal{P} : C_0([0,\infty);\mathbb{R}^n) \times L^1([a,b];\mathbb{R}^m) \to C_0([0,\infty);\mathbb{R}^n),$$

is supposed to satisfy the following conditions:

$$\mathcal{P}[0,0] = 0; \tag{A.28}$$

there exists a constant $M > 0$, such that:

$$\|\mathcal{P}[0,c(\cdot)]\|_\infty \leq M|c|_1 \quad \forall c \in L^1([a,b];\mathbb{R}^m); \tag{A.29}$$

there exists a function $S \mapsto L(S)$ with $\lim_{S\to 0} L(S) = 0$, such that for $\|u\|_\infty, \|\tilde{u}\|_\infty, |c|_1 < S$ :

$$\|\mathcal{P}[u(\cdot), c(\cdot)] - \mathcal{P}[\tilde{u}(\cdot), c(\cdot)]\|_\infty \leq L(S)\|u - \tilde{u}\|_\infty. \tag{A.30}$$

Note that in (A.27) the term $c$ is supposed to be assigned as a sort of initial condition.

Writing the resolvent equation associated to (A.27) we obtain:

$$u(t) = \mathcal{P}[u(\cdot), c(\cdot)](t) - \int_{0}^{t} R(t-s)\mathcal{P}[u(\cdot), c(\cdot)](s)ds \tag{A.31}$$

The following result allows us to apply the Paley-Wiener theorem to the system (A.27).

**Theorem A.2.4.** *Let* $K, R \in L^1([0,\infty);\mathcal{L}(\mathbb{R}^n))$, *and let* $\mathcal{P}$ *satisfy the assumptions (A.28)-(A.29). Then for any* $\epsilon > 0$ *there exists* $\delta$ *such that for any* $c \in L^1([a,b];\mathbb{R}^m)$ *with* $|c|_1 \leq \delta$, *the equation (A.27) has a unique solution* $u \in C_0([0,+\infty);\mathbb{R}^n)$ *such that* $\|u\|_\infty < \epsilon$.

*Proof.* Let $\epsilon > 0$; we take $\eta < \epsilon$ such that

$$L(\eta) < \frac{1}{2(1+M)(1+\|R\|_1)}, \tag{A.32}$$

where $L$ is the function of (A.30), $M$ is the constant of (A.29) and $\|R\|_1$ is the $L^1$ norm of $R$. Then we set

$$\delta = L(\eta)\eta$$

and consider

$$\mathcal{K} = \{u \in C_0([0,+\infty);\mathbb{R}^n) | \|u\|_\infty \leq \eta\}$$

which is a closed subset of $C_0([0,+\infty);\mathbb{R}^n)$. Now for any $c \in L^1([a,b];\mathbb{R}^m)$, with $|c|_1 \leq \delta$ we define the mapping $\mathcal{T}$ as

$$(\mathcal{T}u)(t) = \mathcal{P}[u(\cdot),c(\cdot)](t) - \int_0^t R(t-s)\mathcal{P}[u(\cdot),c(\cdot)](s)ds \quad \forall u \in \mathcal{K}. \tag{A.33}$$

Certainly $\mathcal{T}$ maps $\mathcal{K}$ into $C_0([0,+\infty);\mathbb{R}^n)$, but for $u \in \mathcal{K}$, using (A.32), we have

$$\begin{aligned} \|(\mathcal{T}u)(t)\|_\infty &\leq \|(\mathcal{T}u) - (\mathcal{T}0) + (\mathcal{T}0)\|_\infty \leq (1+|R|_1)(L(\eta)\|u\|_\infty + K|c|_1) \\ &\leq (1+\|R\|_1)(1+K)L(\eta)\eta < \eta \end{aligned}$$

so that $\mathcal{T}$ maps $\mathcal{K}$ into itself. Furthermore using (A.32) again, we have for $u, \tilde{u} \in \mathcal{K}$

$$\|(\mathcal{T}u) - (\mathcal{T}\tilde{u})\|_\infty \leq (1+\|R\|_1)L(\eta)\|u-\tilde{u}\|_\infty < \frac{1}{2}\|u-\tilde{u}\|_\infty.$$

Hence $\mathcal{T}$ is a contraction and has a unique fixed point in $\mathcal{K}$. This proves the theorem. □

# Appendix B

# Bessel functions

This appendix is devoted to some definitions and properties concerning Bessel functions. They are usually well known, but it is useful to collect here some of them, in order to justify easily some results and observations through the thesis. We just give few proofs, and we refer instead to two classical books: we suggest F. Bowman [Bow58] for an organic introduction to this argument and G.N. Watson [Wat44] for a complete treatment.
We restrict ourselves to Bessel functions of positive integer order, although there are more generally Bessel functions of arbitrarily real order. The results and properties presented here can be generalized to that case.

## B.1 Definitions and properties

In this section we define the Bessel functions of first and second kind and state some classic properties.
We introduce the Bessel functions of the first kind as power series:

**Definition B.1.1.** *The Bessel function of the first kind and zero order is defined by*

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}. \tag{B.1}$$

*Similarly, the Bessel function of the first kind and order* $n \in \mathbb{N}$*, is defined by:*

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n}. \tag{B.2}$$

It is obvious that for every $n \in \mathbb{N}$ the above series converges for all values of $x \in \mathbb{C}$, so that $J_n$ is an entire holomorphic function.

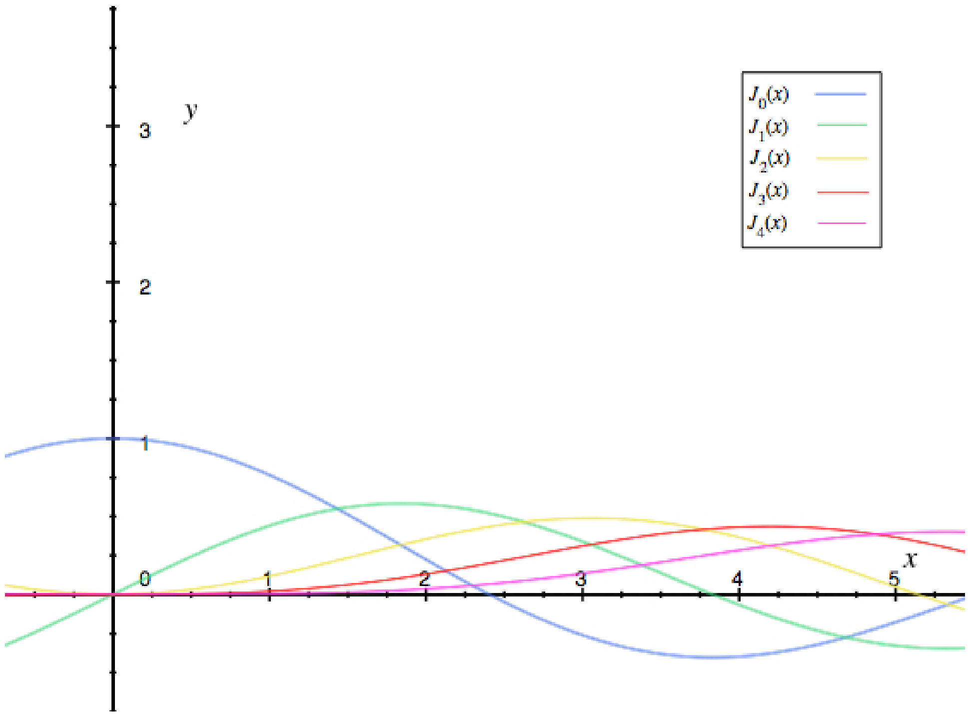



Figure B.1: Bessel function of first kind and order $i = 0, \ldots, 4$

Now we deduce the *Bessel differential equations*: from the above definition, differentiating (B.1) we obtain that

$$J_0'(x) = -J_1(x); \tag{B.3}$$

therefore multiplying $J_1(x)$ for $x$ and differentiating we get

$$\frac{d}{dx}[xJ_1(x)] = xJ_0x, \tag{B.4}$$

and substituting (B.3) in (B.4) we have that

$$\frac{d}{dx}\left[x\frac{d}{dx}J_0(x)\right] + xJ_0(x) = 0.$$

The above equation, fullfilled by $J_0(x)$, is called *Bessel equation of zero order*:

$$\frac{d}{dx}[xy'] + xy = 0; \tag{B.5}$$

it can be written also in the form

$$xy''J_0(x) + y' + xy = 0. \tag{B.6}$$

In the same way we obtain equations of order $n$: indeed

$$\begin{aligned} x\frac{d}{dx}\left[x\frac{d}{dx}J_n(x)\right] &= \sum_{m=0}^{\infty}(-1)^m\frac{(2m+n)^2}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n} \\ &= \sum_{m=0}^{\infty}(-1)^m\frac{n^2+4m(m+n)}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n} \\ &= (n^2-x^2)J_n(x), \end{aligned}$$

hence we have the *Bessel equation of order* $n$ :

$$x^2y'' + xy' + (x^2-n^2)y = 0, \tag{B.7}$$

which can be written also in the form:

$$y'' + \frac{1}{x}y' + (1-\frac{n^2}{x^2})y = 0. \tag{B.8}$$

Now we introduce Bessel functions of the second kind: as above we start with order zero and then we proceed with different orders. Consider (B.6): since it is a second order differential equation, it admits another solution which is independent of $J_0(x)$. Let $u$ be such solution and let $v = J_0(x)$; we have

$$\begin{aligned} xu'' + u' + xu &= 0 \\ xv'' + v' + xv &= 0. \end{aligned}$$

Hence multiplying respectively the two equations by $v$ and $u$ and subtracting, we get

$$x(u''v - v''u) + u'v - v'u = 0$$

which can be rewritten as

$$\frac{d}{dx}[x(u'v-v'u)] = 0.$$

Therefore

$$x(u'v - v'u) = B$$

where $B$ is a constant; then dividing by $xv^2$ we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v-uv'}{v^2} = \frac{B}{xv^2},$$

so that, by integration,

$$\frac{u}{v} = A + B\int\frac{dx}{xv^2}$$

where $A$ is another constant. Then we obtain

$$u(x) = AJ_0(x) + BJ_0(x)\int_1^x \frac{dt}{tJ_0^2(t)},$$

and we define the *Bessel function of the second kind and zero order* as follows:

$$Y_0(x) = J_0(x)\int_1^x \frac{dt}{tJ_0^2(t)}. \tag{B.9}$$

Note that inserting (B.1) in (B.9), we see that $Y_0(x)$ has a singularity in $x = 0$. Analogously, from (B.7) we define the Bessel functions of the second kind and order $n$, that have the same property.
The Bessel functions of the first kind $J_n(x)$ can be also defined as the coefficients of $t^n$, in the Laurent expansion of the function $e^{\frac{x}{2}(t-\frac{1}{t})}$ : indeed, writing this function as a Cauchy product of the Maclaurin expansions of $e^{\frac{x}{2}t}$ and $e^{-\frac{x}{2t}}$ we get

$$\begin{aligned}
e^{\frac{x}{2}(t-\frac{1}{t})} &= e^{\frac{x}{2}t}\cdot e^{-\frac{x}{2t}} = \left[\sum_{n=0}^{\infty}\frac{x^n t^n}{2^n n!}\right]\left[\sum_{p=0}^{\infty}\frac{(-1)^p x^p}{2^p t^p p!}\right] \\
&= \sum_{n=0}^{\infty}\left[\sum_{p=0}^{n}\frac{x^n t^{n-2p}(-1)^p}{2^n (n-p)!p!}\right] = \sum_{p=0}^{\infty}\left[\sum_{n=p}^{\infty}\frac{x^n t^{n-2p}(-1)^p}{2^n (n-p)!p!}\right] \\
&= \sum_{p=0}^{\infty}\left[\sum_{m=-p}^{\infty}\left(\frac{x}{2}\right)^{m+2p}\frac{t^m(-1)^p}{(m+p)!p!}\right] = \\
&= \sum_{p=0}^{\infty}\left[\sum_{m=0}^{\infty}\left(\frac{x}{2}\right)^{m+2p}\frac{t^m(-1)^p}{(m+p)!p!}\right] + \sum_{p=1}^{\infty}\left[\sum_{m=-p}^{-1}\left(\frac{x}{2}\right)^{m+2p}\frac{t^m(-1)^p}{(m+p)!p!}\right] \\
&= \sum_{m=0}^{\infty}J_m(x)t^m + \sum_{m=1}^{\infty}\left[\sum_{p=m}^{\infty}\left(\frac{x}{2}\right)^{2p-m}\frac{t^{-m}(-1)^p}{(p-m)!p!}\right] \\
&= \sum_{m=0}^{\infty}J_m(x)t^m + \sum_{m=1}^{\infty}(-1)^m J_m(x)t^{-m}.
\end{aligned}$$

Now setting $t = e^{i\theta}$, with $\theta \in [0, 2\pi]$, we have:

$$\begin{aligned} e^{ix\sin\theta} &= \sum_{m=0}^{\infty} J_m(x)e^{im\theta} + \sum_{m=1}^{\infty}(-1)^m J_m(x)e^{-im\theta} \\ &= \sum_{m=0}^{\infty} J_m(x)(\cos m\theta + i\sin m\theta) + \sum_{m=1}^{\infty}(-1)^m J_m(x)(\cos m\theta - i\sin m\theta) \\ &= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta) + i\sum_{n=0}^{\infty} J_{2n+1}\sin((2n+1)\theta). \end{aligned}$$

On the other hand

$$e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta).$$

Now, for the uniqueness of the Fourier expansion series, we obtain that the Bessel function of the first kind are the Fourier coefficient of the function $\cos(x\sin\theta) + i\sin(x\sin\theta)$. Hence for every $n \in \mathbb{N}$ :

$$J_{2n}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos(2n\theta)d\theta, \tag{B.10}$$

$$J_{2n+1}(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin((2n+1)\theta)d\theta. \tag{B.11}$$

Adding the two previous identities we get that:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta, \tag{B.12}$$

which is known as the *Bessel integral* for $J_n(x)$. In particular the Bessel functions of the first kind are uniformly bounded.

## B.2 Orthogonality and completeness

In this section we prove the orthogonality properties of Bessel functions of the first kind.
We begin with some recurrence formulae, involving functions of consecutive orders and their derivatives:

**Proposition B.2.1.** *Let $n \geq 1$. Then the Bessel functions of the first kind satisfy:*

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \tag{B.13}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \tag{B.14}$$

$$J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x) \tag{B.15}$$

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x) \tag{B.16}$$

*Proof.* To prove (B.13) we only have to write (B.2) for three consecutive orders:

$$\begin{aligned}
\frac{2n}{x} J_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m n}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n-1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n-1} - \sum_{m=0}^{\infty} \frac{(-1)^m m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n-1}
\end{aligned}$$

and we see that the first term is equal to $J_{n-1}(x)$ and the second to $-J_{n+1}(x)$. The proof of (B.14) is analogous:

$$\begin{aligned}
2J_n'(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n-1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n-1} + \sum_{m=0}^{\infty} \frac{(-1)^m m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n-1},
\end{aligned}$$

that is our thesis.
Summing (B.13) with (B.14) we obtain (B.15), and subtracting (B.14) from (B.13) we get (B.16). □

Now we observe that if $\alpha \neq 0$ is a real number, by (B.8) $J_n(\alpha x)$ satisfies

$$u'' + \frac{1}{x} u' + \left(\alpha^2 - \frac{n^2}{x^2}\right) u = 0. \tag{B.17}$$

As a consequence we have the following result.

**Proposition B.2.2.** *Let $n \in \mathbb{N}$, then the equation*

$$J_n(x) = 0 \tag{B.18}$$

*has infinitely many simple roots different from $0$. Moreover the solutions of (B.18) are not solutions of $J_{n+1}(x) = 0$.*

*Proof.* We prove that for $k > 1$ and sufficiently large $a > 0$, the equation $J_n(kx) = 0$ has at least one root between $a$ and $a + \pi$; this will imply the first part of the statement. The function $y(x) = J_n(kx)$ satisfies (B.17) with $\alpha = k$; if $u(x) = y(x)\sqrt{kx}$, then $u$ satisfies

$$\frac{d^2u}{dx^2} = -\left(k^2 - \frac{n^2 - \frac{1}{4}}{x^2}\right)u. \tag{B.19}$$

Now let $v = \sin(x - a)$; then

$$\frac{d^2v}{dx^2} = -v.$$

Hence multiplying this equation by $u$ and (B.19) by $v$ and subtracting, we get:

$$\frac{d}{dx}\left[u\frac{dv}{dx} - v\frac{du}{dx}\right] = \left(k^2 - 1 - \frac{n^2 - \frac{1}{4}}{x^2}\right)uv.$$

Now integrating between $a$ and $a + \pi$, we get

$$-u(a+\pi) - u(a) = \int_a^{a+\pi} \left(k^2 - 1 - \frac{n^2 - \frac{1}{4}}{x^2}\right)uvdx.$$

Within the integral on the right-hand side, $u$ is continuous, $v$ is positive and the remaining part of the integrand is positive for $a$ large enough. Hence, for the mean value theorem, there exists $\xi \in (a, a + \pi)$ such that

$$-u(a+\pi) - u(a) = u(\xi)\int_a^{a+\pi} \left(k^2 - 1 - \frac{n^2 - \frac{1}{4}}{x^2}\right)vdx.$$

Now the integral in the right-hand side is positive for $a$ sufficiently large; therefore $u(a), u(\xi), u(a+\pi)$ cannot all have the same sign, so the equation $u(x) = 0$ has at least one solution in the interval $(a, a+\pi)$. Hence the equation $J_n(x) = 0$ has at least one solution between $a$ and $a + k\pi$ for $k > 1$ and $a$ sufficiently large. Concerning the simplicity of the roots, suppose that $\alpha$ satisfies $J_n(\alpha) = J_n'(\alpha) = 0$; by (B.7) it must be $J_n''(\alpha) = 0$. Differentiating (B.7) we see that all derivatives of $J_n$ vanish in $\alpha$; hence $J_n \equiv 0$, a contradiction.
Finally if $J_n(\alpha) = J_{n+1}(\alpha) = 0$, by (B.16) we get $J_n'(\alpha) = 0$, which contradicts the previous statement. □

Now we can prove the orthogonality result for Bessel functions of the first kind:

**Theorem B.2.1.** *Let $n \in \mathbb{N}$ and $\alpha, \beta$ two different roots of $J_n(x) = 0$. Then:*

$$\int_0^1 J_n(\alpha x)J_n(\beta x)xdx = 0. \tag{B.20}$$

*Moreover*

$$\int_0^1 J_n^2(\alpha x)dx = \frac{J_{n+1}^2(\alpha)}{2}. \tag{B.21}$$

*Proof.* Let $u(x) = J_n(\alpha x)$ and $v(x) = J_n(\beta x)$; they satisfy an equation of the form (B.17). The same argument used in the previous section to define the Bessel functions of the second kind now yields

$$\frac{d}{dx}[x(uv' - v'u)] = (\beta^2 - \alpha^2)xuv.$$

Integrating in $[0, 1]$ we obtain

$$(\beta^2 - \alpha^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \alpha J_n'(\alpha)J_n(\beta) - \beta J_n'(\beta)J_n(\alpha) = 0. \tag{B.22}$$

that imply our thesis.
Concerning the second part, we multiply (B.17) by $2x^2u'$, where $u(x) = J_n(\alpha x)$, obtaining:

$$2xu'\frac{d}{dx}[xu'] + 2(\alpha^2x^2 - n^2)u'u = 0$$

which is equivalent to

$$\frac{d}{dx}[x^2(u')^2 + (\alpha^2x^2 - n^2)u^2] = 2\alpha^2xu^2.$$

Integrating the last equation and substituting $u = J_n(\alpha x)$ we obtain that

$$\int_0^1 xJ_n^2(\alpha x)dx = \frac{1}{2}\left[x^2(J_n'(\alpha x))^2 + \left(x^2 - \frac{n^2}{\alpha^2}\right)J_n^2(\alpha x)\right]_0^1, \tag{B.23}$$

and using (B.16) we get (B.21). □

The previous result can be generalized:

**Theorem B.2.2.** *Let $n \in \mathbb{N}$ and $\gamma, \delta$ two different roots of $xJ_n'(x) + HJ_n(x) = 0$, where $H$ is a real constant; then:*

$$\int_0^1 J_n(\delta x)J_n(\gamma x)xdx = 0. \tag{B.24}$$

*Moreover*

$$\int_0^1 J_n^2(\delta x)dx = \frac{(\delta^2 - n^2)J_n^2(\delta) + \delta^2(J_n'(\delta))^2}{2\delta^2}. \tag{B.25}$$

*Proof.* To prove the first part we proceed as in the previous theorem, obtaining (B.22), the first statement follows.
Also for the second part, we proceed as above and get (B.23), from which we deduce (B.25). □

Denote by $\{\lambda_m\}_{m\geq 0}$ the positive roots of the equation $J_n(x) = 0$, arranged in ascending order of magnitude, and let $f$ be a function defined in $(0,1)$. Using Theorem B.2.1 we want to investigate whether $f$ can be expanded in a series of the form $\sum_{m=0}^{\infty} a_m J_n(\lambda_m x)$, where $a_m = \frac{2}{J^2_{n+1}(\lambda_m)} \int_0^1 f(t) J_n(\lambda_m t) t dt$. When this expansion exists, it is called *Fourier-Bessel series* of $f$ of order $n$. We just give the final part of the proof; for the whole argument we refer to [Wat44],Ch.XVIII.

**Theorem B.2.3.** *Let $n \in \mathbb{N}$ and*

$$q_m(x) = \frac{2}{[J_{n+1}(\lambda_m)]} J_n(\lambda_m x)$$

*where $\{\lambda_m\}_{m\geq 0}$ are the roots of equation $J_n(x) = 0$ in ascending order. Then the system $\{q_m\}_{m\geq 0}$ is orthonormal and complete within the space $L^2([0,1]; xdx)$.*

*Proof.* The orthonormality of the system is proved in Theorem B.2.1. Concerning the completeness we refer to [Wat44],Ch.XVIII, §18.24, where it is shown that, for every $f \in L^1([0,1]; \sqrt{t}dt) \cap BV([0,1])$, the series $\sum_{m=0}^{\infty} a_m J_n(\lambda_m x)$ (called *Fourier-Bessel* series) is convergent with sum $\frac{1}{2}[f(x^+) + f(x^-)]$, where

$$a_m = \frac{2}{J^2_{n+1}(\lambda_m)} \int_0^1 f(t) J_n(\lambda_m t) t dt = \langle f, q_m \rangle_{L^2([0,1];tdt)}.$$

Now consider the subspace $B = span\{q_m\}_{m\in\mathbb{N}} \subseteq L^2([0,1]; tdt)$. By the above convergence result, we have $C^1([0,1]) \subseteq BV([0,1]) \cap L^1([0,1]; \sqrt{t}dt) \subseteq B$. As $C^1([0,1])$ is obviously dense in $L^2([0,1]; tdt)$, we obtain that $B$ is dense in $L^2([0,1]; tdt)$, and the result follows. □

An expansion similar to the previous one, but based upon the roots of the equation

$$x^{-n}[xJ'_n(x) + HJ_n(x)] = 0, \tag{B.26}$$

(where $H$ is a constant) is called the *Dini expansion* of $f$ of order $n$. We distinguish three cases, depending on the values of the constant $H$:

(i) If $H > -n$ the Dini expansion has the same form as the Fourier-Bessel one, i.e. $f(x) = \sum_{m=1}^{\infty} b_m J_n(\mu_m x)$, where $\{\mu_m\}_{m\geq 1}$ are the positive roots of (B.26) and

$$b_m = \frac{2\mu_m^2}{(\mu_m^2 - n^2)J_n^2(\mu_m) + \mu_m(J'_n(\lambda_m x))^2} \int_0^1 t f(t) J_n(\mu_m t) dt.$$

(ii) If $H = -n$, equation (B.26) becomes $x^{-n}[xJ_n'(x) - nJ_n(x)] = 0$, which by (B.16) is equivalent to

$$x^{-n+1}J_{n+1}(x) = 0$$

that has $x = 0$ as a double root. Hence we adjoin an initial term to the Dini expansion, obtaining the following form: $f(x) = b_0x^n + \sum_{m=1}^{\infty} b_m J_n(\mu_m x)$, which may be regarded as an expansion based upon the non-negative roots (including zero) of the equation $J_{n+1}(x) = 0$. The constant $b_0$ may be found in the same way

$$b_0 = \frac{\int_0^1 x^{n+1} f(x)dx}{\int_0^1 x^{2n+1}dx}.$$

The constants $b_m$ are given, analogously to the case of Fourier-Bessel series, by $b_m = \frac{2}{J_n^2(\mu_m)} \int_0^1 tf(t)J_n(\mu_m t)dt$.

(iii) If $H < -n$, equation (B.26) has two purely imaginary roots $\pm i\mu_0$ Hence the Dini expansion begins with a term depending on them and has the form $f(x) = b_0 I_n(\mu_0 x) + \sum_{m=1}^{\infty} b_m J_n(\mu_m x)$, where $I_n(x)$ is the *modified Bessel function of order* $n$ (see Bowman [Bow58], Ch. VI, §85). The coefficients $b_m$ (for $m \neq 0$) are determined as in the first case and $b_0 = \frac{2}{I_n^2(\mu_0) + I_{n-1}(\mu_0)I_{n+1}(\mu_0)} \int_0^1 tf(t)I_n(t\mu_0)dt$.

Again, we give here only the final part of the proof of completeness; the whole argument can be found in [Wat44], Ch. XVIII, where it is proved that for every $f \in L^1([0,1]; \sqrt{t}dt) \cap BV([0,1])$, the series $\sum_{m=1}^{\infty} b_m J_n(\mu_m x)$ (called *Fourier-Dini* series) is convergent with sum $\frac{1}{2}[f(x^+) + f(x^-)]$, where

$$b_m = \frac{2\mu_m^2}{(\mu_m^2 - n^2)J_n^2(\mu_m) + \mu_m(J_n'(\lambda_m x))^2} \int_0^1 tf(t)J_n(\mu_m t)dt = \langle f, p_m \rangle_{L^2([0,1];tdt)}.$$

As a consequence, we have the following theorem:

**Theorem B.2.4.** *Let* $n \in \mathbb{N}$ *and*

$$p_m(x) = \frac{2\mu_m^2}{(\mu_m^2 - n^2)J_n^2(\mu_m) + \mu_m(J_n'(\lambda_m x))^2} J_n(\mu_m x)$$

*where* $\{\mu_m\}_{m\geq 1}$ *are the roots of equation (B.26)* $xJ_n'(x) + HJ_n(x) = 0$ *in ascending order (for* $H > -n$*). Hence the system* $\{p_m\}_{m\geq 1}$ *is orthonormal and complete within the space* $L^2([0,1]; xdx)$*.*

*Proof.* The orthonormality was proved in Theorem B.2.2. About the completeness, using the convergence result quoted before, we may proceed as in the previous proof. □

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