Università degli Studi di Pisa

Corso di laurea specialistica in Matematica

Intertemporal utility maximization problems with state constraints: existence theorems and dynamic programming

Tesi di laurea specialistica

Francesco Bartaloni

Relatori

Prof. Paolo Acquistapace
Università di Pisa

Prof. Fausto Gozzi
LUISS Università Guido Carli

Controrelatore

Prof. Giuseppe Buttazzo
Università di Pisa

Anno accademico 2013-14

Contents

Part I. Introduction

Utility maximization problems represent a fundamental part of modern economic growth models, since the works by Ramsey (1929), Lucas (1954), Romer (1986), Barro and Sala-i-Martin (1995).

These models aim to formalize the dynamics of an economy throughout the quantitative description of the consumers' behaviour. Consumers are seen as homogeneous entities, as far as their operative decisions are concerned; hence the time series of their consuming choices, or consumption path, is represented by a single function, and they as a collective are named after *social planner*, or simply *agent*.

This definition suggests the idea that the analysis of these models finds its natural mathematical framework in the techniques and methods of Control Theory. This is precisely the case, if we look at the consumption of the agent as the control strategy, and if we assume that this function is involved in a suitable (dynamical) relation with other significant economic quantities, such as the average income.

Turning back to the illustration of this class of models, the social planner's purpose is to maximize the utility in function of the series of the consumption choices in a fixed time interval; this can be finite or more often (as far as economic growth literature is concerned) infinite.

From the application viewpoint, the target of the analysis is the study of the optimal – in relation to this utility functional – trajectories: regularity, monotonicity, asymptotic behaviour properties and similar are expected to be investigated. Hence good existence results are specially needed, as well as handy sufficient and necessary conditions for the optimum.

As outlined above, these problems are treated mathematically as optimal control problems; often external reasons such as the pursuit of more empirical description power imply the presence of additional control and state constraints, which we call "static" constraints since do not involve the derivative of the state variable.

It is worth noticing that the introduction of the static state constraints usually makes the problem quite harder, insomuch that it is considered extraneous to the usual setting of control theory. As an example, we see that the main properties of optimal trajectories are still not characterized in recent literature, at least in the case of non-concave production function.

Hence this kind of program is quite complex, especially in the above mentioned case – and has to be dealt with in many phases. With this dissertation we undertake the work providing an existence result and various necessary conditions related to the Hamilton-Jacobi-Bellman problem, on the basis of a draft by F. Gozzi and D. Fiaschi ([1]) and of some original results.

The main method which we rely on in order to find the proper necessary conditions for the value function (the supremum of the objective functional) and for the optimal control is the so-called Dynamic Programming (which sometimes also provides sufficient conditions). The structure of this method can be summarized in some main steps:

- letting the initial data vary, find a suitable functional equation for the value function: this will be called Bellman functional equation (BE);
- consider the infinitesimal version of BE, the Hamilton-Jacobi-Bellman equation (HJB), which is a non-linear first order PDE;
- solve (whenever possible) the HJB equation to find the value function.

Sometimes one can go further and try to prove that the present value of the optimal control strategy can be expressed as a function of the present value of the optimal state trajectory: this will be the so-called *closed loop (or feedback) relation* for the optimal control. If this is accomplished, then the control by its closed loop expression in the state equation, and try to solve the equation he has obtained - which is called Closed Loop Equation.

In our case we chose to give a direct existence proof for the optimal control, because of the very peculiar weakness of certain assumptions on the data we imposed to ourselves, with the result that couplings with the traditional literature such as Cesari, Zabczyk and Yong-Zhou were missing.

Of course it is not clear a priori (and not true in general, see on this Bardi and Capuzzo Dolcetta (1997)) that the value function solves the HJB equation. Two main problems arise: the value function is not necessarily differentiable, so it could not satisfy the HJB equation in the classical sense; moreover, the equation could have other solutions.

In general these problems are not easy. The dynamic programming approach consists in fact in dealing with such problems studying directly the HJB equation in relation to the weaker notion of *viscosity solution.* This new notion is a kind of nonsmooth solution to partial differential equations, whose key feature is to replace the conventional derivatives by the (set-valued) super- / sub-differentials while maintaining the uniqueness of the solution under very mild conditions. These make the theory a powerful tool in tackling optimal control problems. The viscosity solutions that we are going to discuss can be merely continuous (not necessarily differentiable).

This notion can be characterized both in terms of super- and sub-differentials and of test functions; in any case these auxiliary tools must match the necessary restrictions to the domain of the Hamiltonian function involved in the equation, at least for the solutions we are interested in verifying. Hence, naming "Hamiltonian problem" the question whether the value function is a viscosity solution to the proper HJB equation, we see that the well–posedness itself of the Hamiltonian problem is in general at risk. Fortunately, we are able to prove certain regularity properties of the value function ensuring that this is not the case. The peculiar fact is that this proof involves the existence of the optimal control - quite naturally, indeed.

1 Plan of work

It is straightforward that the contents of the dissertation have been organized in accordance with these observations.

For the sake of completeness, the dissertation begins with a section on finite-horizon minimization problems. There we treat a wide class of problems characterized by very general functional and dynamics. In this case the well-posedeness of the Hamiltonian problem reduce to the continuity of the value function, which is its turn not obvious. Then the advantages of the dynamic programming approach are deeply examined, proving that

(i) the value function solves the Bellman functional equation (the so-called Dynamic Programming Principle, of which we also provide a set-theoretic formulation);

(ii) the Hamilton-Jacobi-Bellman equation (HJB) is also solved by the same function, supposed that it is regular enough.

The limits of the methods are also faced up, as we give an example of an optimization problem generating a non differentiable value function and a HJB equation with no differentiable solution. Hence we introduce the notion of viscosity solution for a general class of PDE's, and prove that the value function is in our case a viscosity solution to the HJB equation introduced before. Finally, we give a uniqueness result for the viscosity solutions to HJB in the class of continuous functions satisfying a suitable boundary condition, which the value function belongs to.

As far as utility maximization in infinite-horizon framework is concerned, a few words have to be spent in order to pinpoint the specific problems one has to face up in the analysis of the Gozzi-Fiaschi model. Of course one wants to implement at least part of the techniques developed in the first part of the thesis; nevertheless many technical difficulties arise as an effect of the above mentioned generality of the hypothesis on the data, which are supposed to be the reason of the versatility and wide-range applicability of this model.

In particular the dynamics (where the state variable represents total endowment of the social planner or average capital of the representative dynasty) contains a convex-concave function representing production.

It is well known that the presence of non concavity in an optimization problem can lead to many difficulties in establishing the necessary and sufficient conditions for the optimum, as well as in examining the regularity properties of the value function.

Moreover, the presence of the static state constraint makes any admissibility proof much more complicated than usual.

As a third relevant (and unusual) feature, we require that the admissible controls are not more than locally integrable in the positive half-line: this is the maximal class if one wants the control strategy to be a function and the state equation to have solution. This is a weak regularity requirement which is of very little help; conversely it generates unexpected issues in various respects.

We can summarize the main criticalities entailed by these three traits as follows:

1. Certain questions appear that in other "bounded-control" models are not even present, such as the finiteness of the value function and the well-posedness of the Hamiltonian problem.

2. The problem of the existence of an optimal control strategy (for every fixed initial state) is unusually difficult. Specifically, it is a natural idea to make use of the traditional compactness results, such as the Dunford-Pettis criterion, in order to generate a convergent approximation procedure. As we commit ourselves to deal with merely (locally) integrable control functions, the application of such compactness results is not straightforward. Indeed, a very careful preliminary work

is needed in order to set up a proper approximation procedure, which has to be further refined so that we can find a limit function which is admissible in the sense that it satisfies the state static constraint, and so that the approximation works also as far as the objective functionals are concerned.

3. Additional work to the usual proof of the fact that the value function indeed solves HJB is needed; in fact we not only use the optimal control, but also, separately, a preliminary result which appears in the optimality construction: the fundamental Lemma 33.

4. The regularity property stated in Theorem 50.ii), which is necessary in order that the HJB problem is well-posed, not only requires – as we have seen – optimal controls. It can be proven by a standard argument under the hypothesis that the admissible controls are locally bounded; in our case it shows again to be useful to come back to the preliminary tools (Lemmas 33 and 34) in order to move around the obstacle and have the result proven with merely integrable control functions.

The contents of this second part are consequently arranged: first, the reader will come across an introductory paragraph which intends to clear up the genesis of the model and the economic motivations for the assumptions.

Then comes a section dedicated to the preliminary results that are crucial for the development of the theory. All of them are technical lemmas strictly connected to the nature of the problem they are going to be applied to (even if likely applicable to a wider class of problems), except Lemma 27, and Corollary 28 which states and proves the comparison principle for ordinary differential equations in a context-adapted form.

Afterwards, some basic properties of the value function are proven, such as its behaviour near the origin and near $+\infty$. These results require careful manipulations of the data and some standard results about ordinary differential equations, but do not require the existence of optimal control functions.

Next comes the pivotal section in which we prove the existence of an optimal control strategy for every initial state. Here we make wide use of the preliminary lemmas in association with a special diagonal procedure generating a weakly convergent sequence of control functions from a family of sequences which are not extracted neatly one from the other, as in the Ascoli-Arzelà's theorem.

Afterwards, we will be able to prove other important regularity properties of the value function, using optimal functions.

Eventually we give an application of the methods of Dynamic Programming to our model. As mentioned before, the proof of the admissibility of the value function as a viscosity solution is made more complicated by the use of the preliminary lemmas and of the optimal control function, but it allows to obtain the result independently of the regularity of the Hamiltonian function, which contributes to make this problem peculiar and hopefully a source of further motives of scientific interest.

Last but not least, I feel the moral duty and the pleasure to remind that I owe everything of the good that may harbour in this work to the inspiration, the support and the guidance of my masters, Paolo Acquistapace and Fausto Gozzi, to whom I express my sincere gratitude.

Part II. Finite-horizon optimization and viscosity solutions to Hamilton-Jacobi-Bellman equations

2 Basic definitions and continuity of the value function.

Let us now state the basic properties required for the functions describing the dynamics of the sistem, which we indicate by b, and the cost functional. In fact, the requirements for b are a little stronger than the hypotheses of Cauchy's existence and uniqueness result:

$$b : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}^n \text{ is uniformly continuous over the whole domain} \tag{1}$$

$$\text{There exists a real number } L \geq 0 \text{ such that}$$
$$\begin{cases} \|b(t,x,u) - b(t,y,u)\| \leq L\|x-y\| & \forall t \in [0,T],\ x,y \in \mathbb{R}^n,\ u \in U \\ \|b(t,0,u)\| \leq L & \forall t \in [0,T],\ u \in U \end{cases} \tag{2}$$

Note that condition (2) implies

$$\begin{aligned} &\|b(t,x,u)\| \leq \|b(t,x,u) - b(t,0,u)\| + \|b(t,0,u)\| \leq L(\|x\|+1) \\ &\forall t \in [0,T],\ x \in \mathbb{R}^n,\ u \in U \end{aligned} \tag{3}$$

We will use this fact together with global uniform continuity many times later.

Now let f and h be functions satisfying analogous assumptions.

$$f : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R} \text{ is uniformly continuous over the whole domain} \tag{4}$$

There exists a real number $L \geq 0$ such that

$$\begin{cases} |f(t,x,u) - f(t,y,u)| \leq L\|x-y\| & \forall t \in [0,T],\ x,y \in \mathbb{R}^n,\ u \in U \\ |f(t,0,u)| \leq L & \forall t \in [0,T],\ u \in U \end{cases} \tag{5}$$

Hence

$$|f(t,x,u)| \leq L(\|x\|+1) \tag{6}$$

$$\forall t \in [0,T],\ x \in \mathbb{R}^n,\ u \in U \tag{7}$$

Finally

$$h : \mathbb{R}^n \to \mathbb{R} \text{ is uniformly continuous} \tag{8}$$

There exists a real number $L \geq 0$ such that

$$\begin{cases} |h(x) - h(y))| \leq L\|x-y\| & \forall x,y \in \mathbb{R}^n \\ |h(0)| \leq L \end{cases} \tag{9}$$

First of all, we observe that the above conditions (1), (2) imply by Cauchy's theorem that every differential (controlled) system

$$\begin{cases} \dot{x}(t) = b(t,x(t),u(t)) & t \in (s,T] \\ x(s) = y \end{cases}$$

with $(s,y) \in [0,T] \times \mathbb{R}^n$, is solved by a unique function in $\mathcal{C}^1([0,T],\mathbb{R}^n)$ which we denote by $x(\cdot;s,y,u)$ or simply $x(\cdot)$ if there is no possible misunderstanding.

Definition 1. Let $(s,y) \in [0,T] \times \mathbb{R}^n$ and

$$\Lambda(s) := \{u(\cdot) : [s,T] \to U \ / \ u(\cdot) \text{ is measurable}\}.$$

The *cost functional* $J(\cdot; s, y) : \Lambda(s) \to \mathbb{R}$ is

$$J(u; s, y) = \int_s^T f(t, x(t; s, y, u), u(t))\mathrm{d}t + h(x(T; s, y, u)) \quad \forall u \in \Lambda(s)$$

(note that $J(u; T, y) = h(x(T; T, y, u)) = y$ does not depend on u).

The *value function* $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is

$$\begin{cases} V(s, y) = \inf\limits_{u \in \Lambda(s)} J(u; s, y) & \forall (s, y) \in [0, T) \times \mathbb{R}^n \\ V(T, y) = h(y) & \forall y \in \mathbb{R}^n \end{cases}$$

We now state and prove the basic result about the value function; but first we need to establish a very important property of measurable functions.

Lemma 2 (**Gronwall's inequality**). *Let* $f \in L^1([a, b], \mathbb{R})$ *satifying the integral inequality:*

$$f(t) \le g(t) + N \int_a^t f(s)\, ds \quad \textit{for a.e. } t \in [a, b]$$

where $g \in L^\infty([a, b], \mathbb{R})$ *and* $N \ge 0$. *Then*

$$f(t) \le g(t) + N e^{Nt} \int_a^t g(s)\, e^{-Ns} ds \quad \textit{for a.e. } t \in [a, b].$$

In particular, if g *is increasing, the last quantity is bounded above by*

$$g(t)\, e^{N(t-a)}$$

for every $t \in [a, b]$.

Proof. Multiplying both sides of the intergal inequality we obtain, for almost every $t \in [a,b]$:

$$f(t)\, e^{-N(t-a)} - N e^{-N(t-a)} \int_a^t f(s)\,\mathrm{d}s \le g(t)\, e^{-N(t-a)}.$$

Observe that the left-hand side coincides with $\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-N(t-a)} \int_a^t f(s)\,\mathrm{d}s\right)$. Hence integrating both members of the latter inequality between a and any $t \in (a,b]$ such that the inequality holds, we obtain

$$e^{-N(t-a)} \int_a^t f(s)\,\mathrm{d}s \le \int_a^t g(s)\, e^{-N(s-a)} \mathrm{d}s.$$

Hence by hypothesis

$$f(t)\, e^{-N(t-a)} \le g(t)\, e^{-N(t-a)} + N \int_a^t g(s)\, e^{-N(s-a)} \mathrm{d}s$$

which implies

$$f(t) \le g(t) + N e^{Nt} \int_a^t g(s)\, e^{-Ns} \mathrm{d}s;$$

as this holds for almost every $t \in (a,b]$, we have the thesis. $\square$

Theorem 3. *The value function $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous. Precisely, for some $K > 0$*

$$|V(s,y) - V(\bar{s},\bar{y})| \le K\{\|y-\bar{y}\| + (1+\max\{\|y\|,\|\bar{y}\|\})|s-\bar{s}|\}$$
$$\forall (s,y),(\bar{s},\bar{y}) \in [0,T] \times \mathbb{R}^n$$

Proof. We split the proof in various inequalities.

First, let $(s,y) \in [0,T] \times \mathbb{R}^n,\ u \in \Lambda(s),\ t \in [s,T],\ x(t) := x(t;s,y,u)$.

Since $\dot{x}(r) = b(t,x(r),u(r))$ for every $r \in [s,t]$, integrating between s and t leads to $x(t) - y = \int_s^t b(t,x(r),u(r))\mathrm{d}r$, an equation between $n-$vectors.

This implies

$$\begin{aligned}\|x(t)\| &\leq \|y\| + \left\|\int_s^t b\left(r, x(r), u(r)\right) \mathrm{d}r\right\| \\ &\leq \|y\| + \int_s^t \|b\left(r, x(r), u(r)\right)\| \mathrm{d}r \\ &\leq^{(3)} \|y\| + \int_s^t L(\|x(r)\| + 1)\mathrm{d}r \\ &\leq \|y\| + LT + \int_s^t L\|x(r)\|\mathrm{d}r\end{aligned}$$

which *is* in the form of the antecedent of *ii*). So we deduce

$$\|x(t)\| \leq (\|y\| + LT)\exp(\int_s^t L\,\mathrm{d}r) \leq (\|y\| + LT)e^{LT}$$

If $LT \leq 1$ we set $K_0 := e^{LT}$; otherwise, $K_0 := LTe^{LT}$. In any case we obtain

$$\|x(t)\| \leq (\|y\| + 1)K_0$$

that is

$$\|x(t; s, y, u)\| \leq (\|y\| + 1)K_0 \quad \forall t \in [s, T] \tag{10}$$

Note that K_0 does not depend on s, y, u; so it does not depend on the solution $x(\cdot; s, y, u)$ either.
Now we prove another inequality; let
$(s, y), (\bar{s}, \bar{y}) \in [0, T] \times \mathbb{R}^n,\ u \in \Lambda(\min\{s, \bar{s}\})$
$x(t) := x(t; s, y, u),\ \bar{x}(t) := x(t; \bar{s}, \bar{y}, u),\ t \in [\max\{s, \bar{s}\}, T],$. From

$$x(t) = y + \int_s^t b\left(r, x(r), u(r)\right) \mathrm{d}r$$

and

$$\bar{x}(t) = \bar{y} + \int_{\bar{s}}^t b(r, \bar{x}(r), u(r))\mathrm{d}r$$

for all $t \in [\max\{s, \bar{s}\}, T]$, we get

$$\begin{aligned} x(t) - \bar{x}(t) &= y - \bar{y} + \int_s^t b(r, x(r), u(r))\mathrm{d}r - \int_{\bar{s}}^t b(r, x(r), u(r))\mathrm{d}r \\ &\quad + \int_{\bar{s}}^t b(r, x(r), u(r))\mathrm{d}r - \int_{\bar{s}}^t b(r, \bar{x}(r), u(r))\mathrm{d}r \end{aligned}$$

for all $t \in [\max\{s, \bar{s}\}, T]$.

Hence by (10),

$$\begin{aligned} \|x(t) - \bar{x}(t)\| &\leq \|y - \bar{y}\| + \int_{s\wedge\bar{s}}^{\bar{s}\vee s} \|b(r, x(r), u(r))\|\mathrm{d}r \\ &\quad + \int_{\bar{s}}^t \|b(r, x(r), u(r)) - b(r, \bar{x}(r), u(r))\|\mathrm{d}r \\ &\leq \|y - \bar{y}\| + \int_{s\wedge\bar{s}}^{\bar{s}\vee s} L(\|x(r)\| + 1)\mathrm{d}r + \int_{\bar{s}}^t L\|x(r) - \bar{x}(r)\|\mathrm{d}r \\ &\leq \|y - \bar{y}\| + |s - \bar{s}|L + |s - \bar{s}|LK_0(\|y\| + 1) \\ &\quad + \int_{\bar{s}}^t L\|x(r) - \bar{x}(r)\|\mathrm{d}r \end{aligned}$$

for all $t \in [\max\{s, \bar{s}\}, T]$.

Observe that $\|y-\bar{y}\|+|s-\bar{s}|L(1+K_0+K_0\|y\|)$ is independent of t, which gives us the possibility of applying Lemma 2 and obtain, for $t \in [\max\{s, \bar{s}\}, T]$,

$$\begin{aligned} \|x(t; s, y, u) - x(t; \bar{s}, \bar{y}, u)\| &\leq \{\|y - \bar{y}\| + |s - \bar{s}|L(1 + K_0 + K_0\|y\|)\}e^{(t-\bar{s})L} \\ &\leq \{\|y - \bar{y}\| + |s - \bar{s}|L(1 + K_0 + K_0 M(y, \bar{y})\}e^{TL} \end{aligned} \tag{11}$$

where we have set $M(y, \bar{y}) = \max\{\|y\|, \|\bar{y}\|\}$. Observe that this proves the continuous dependence of the orbit on the initial state, with fixed control.

Now we can give the proper proof of the continuity of the value function.

Let $(s, y), (\bar{s}, \bar{y}) \in [0, T) \times \mathbb{R}^n$. We will exhibit an upper bound of $|V(s, y) - V(\bar{s}, \bar{y})|$.

Let $\epsilon > 0$ and suppose without loss of generality that $s \geq \bar{s}$. Since $V(\bar{s}, \bar{y}) = \inf\{J(u; \bar{s}, \bar{y})/u \in \Lambda(\bar{s})\}$ we can find $u_\epsilon \in \Lambda(\bar{s})$ such that

$$V(\bar{s}, \bar{y}) + \epsilon > J(u_\epsilon; \bar{s}, \bar{y})$$

so that

$$\begin{aligned} V(s,y) - V(\bar{s},\bar{y}) &\leq J(u_\epsilon \restriction_{[s,T]}; s, y) - J(u_\epsilon; \bar{s}, \bar{y}) + \epsilon \\ &= \int_s^T f(t, x(t; s, y, u_\epsilon), u_\epsilon(t))\mathrm{d}t - \int_{\bar{s}}^T f(t, x(t; \bar{s}, \bar{y}, u_\epsilon), u_\epsilon(t))\mathrm{d}t \\ &\quad + h(x(T; s, y, u_\epsilon)) - h(x(T; \bar{s}, \bar{y}, u_\epsilon)) + \epsilon \end{aligned}$$

Hence, setting $x(t) = x(t; s, y, u_\epsilon)$ and $\bar{x}(t) = x(t; \bar{s}, \bar{y}, u_\epsilon)$,

$$\begin{aligned} V(s,y) - V(\bar{s},\bar{y}) &\leq \int_s^T f(t, x(t), u_\epsilon(t))\mathrm{d}t - \int_{\bar{s}}^T f(t, x(t), u_\epsilon(t))\mathrm{d}t \\ &\quad + \int_{\bar{s}}^T f(t, x(t), u_\epsilon(t))\mathrm{d}t - \int_{\bar{s}}^T f(t, \bar{x}(t), u_\epsilon(t))\mathrm{d}t \\ &\quad + h(x(T)) - h(\bar{x}(T)) + \epsilon \\ &= \int_s^{\bar{s}} f(t, x(t), u_\epsilon(t))\mathrm{d}t + \int_{\bar{s}}^T [f(t, x(t), u_\epsilon(t)) - f(t, \bar{x}(t), u_\epsilon(t))]\mathrm{d}t \\ &\quad + h(x(T)) - h(\bar{x}(T)) + \epsilon \end{aligned}$$

We get by (5),(7), (5) and (9):

$$\begin{aligned}
|V(s,y)-V(\bar{s},\bar{y})| &\leq \int_s^{\bar{s}} L(\|x(t)\|+1)\mathrm{d}t + \int_{\bar{s}}^{T} L(\|x(t)-\bar{x}(t)\|\mathrm{d}t \\
&\leq +L\|x(T)-\bar{x}(T)\|+\epsilon \\
&\leq |s-\bar{s}|L+|s-\bar{s}|LK_0(M(y,\bar{y})+1) \\
&\quad +TLe^{TL}\{\|y-\bar{y}\|+|s-\bar{s}|L(1+K_0+K_0M(y,\bar{y})\} \\
&\quad +Le^{TL}\{\|y-\bar{y}\|+|s-\bar{s}|L(1+K_0+K_0M(y,\bar{y})\}+\epsilon
\end{aligned}$$

Remind that $TLe^{TL} \leq K_0$ so setting $K_1 := Le^{TL}+K_0$ it follows that

$$\begin{aligned}
|V(s,y)-V(\bar{s},\bar{y})| &\leq \|y-\bar{y}\|K_1+\{1+K_0M(y,\bar{y})+2K_0+K_0^2+K_0^2M(y,\bar{y}) \\
&\quad +K_1+K_1K_0+K_1K_0M(y,\bar{y})\}|s-\bar{s}|L+\epsilon \\
&= \|y-\bar{y}\|K_1+\{LK_2+LK_3M(y,\bar{y})\}|s-\bar{s}|+\epsilon
\end{aligned}$$

where we put $K_2 = 1+2K_0+K_0^2+K_1+K_1K_0$ and
$K_3 = K_0+K_0^2+K_0K_1$. First, note that none of the K_i's depends on control u_ϵ - so they neither depend on ϵ:

$$\begin{aligned}
|V(s,y)-V(\bar{s},\bar{y})| &\leq \|y-\bar{y}\|K_1+\{LK_2+LK_3M(y,\bar{y})\}|s-\bar{s}| \\
&\leq K\{\|y-\bar{y}\|+(1+M(y,\bar{y}))|s-\bar{s}|\}
\end{aligned}$$

where $K := \max\{K_1, LK_2, LK_3\}$. Besides, the K_i's do not depend on $s, y, \bar{s}, \bar{y}$ hence also K does not. For this reason the above inequality proves that V is continuous on $[0,T)\times\mathbb{R}^n$.

The case $\bar{s} < s = T$ is similar. We take $u \in \Lambda(\bar{s})$ and $y, \bar{y} \in \mathbb{R}^n$. Then

$$\begin{aligned}
V(\bar{s},\bar{y})-V(T,y) &\leq J(u;\bar{s},\bar{y})-h(y) \\
&= \int_{\bar{s}}^{T} f(t,x(t;\bar{s},\bar{y},u),u(t))\mathrm{d}t+h(x(T;\bar{s},\bar{y},u))-h(x(T;T,y,u))
\end{aligned}$$

Hence

$$
\begin{aligned}
|V(\bar{s},\bar{y}) - V(T,y)| &\leq \int_{\bar{s}}^{T} L(\|x(t;\bar{s},\bar{y},u)\| + 1)\mathrm{d}t \\
&\quad + |h(x(T;\bar{s},\bar{y},u)) - h(x(T;T,y,u))| \\
&\leq |T-\bar{s}|L + |T-\bar{s}|LK_0(\|\bar{y}\| + 1) \\
&\quad + L\|x(T;\bar{s},\bar{y},u) - x(T;T,y,u)\| \\
&\leq |T-\bar{s}|L\{1 + K_0(\|\bar{y}\| + 1)\} + L\{\|y-\bar{y}\| \\
&\quad + |T-\bar{s}|L(1 + K_0 + K_0\|\bar{y}\|)\}e^{TL} \\
&\leq L\|y-\bar{y}\| + (K_0^\star + K_0^\star M(y,\bar{y}))|T-\bar{s}|L \\
&\leq K^\star\{\|y-\bar{y}\| + (1 + M(y,\bar{y}))|T-\bar{s}|\}
\end{aligned}
$$

where $K_0^\star > 0$ is a suitable number independent of $\bar{s}, \bar{y}, y, u$ and $K^\star := \max\{L, LK_0^\star\}$.

Finally,

$$
|V(T,y) - V(T,\bar{y})| = |h(y) - h(\bar{y})| \leq L\|y-\bar{y}\|
$$

So the continuity of V on the whole definition domain $[0,T] \times \mathbb{R}^n$ is proven by

$$
|V(s,y) - V(\bar{s},\bar{y})| \leq \tilde{K}\{\|y-\bar{y}\| + (1 + M(y,\bar{y}))|s-\bar{s}|\} \quad \forall (s,y), (\bar{s},\bar{y}) \in [0,T] \times \mathbb{R}^n \tag{12}
$$

for a suitable constant $\tilde{K}$. □

3 Dynamic Programming

We are now about to state a principle that is quite meaningful in the context of optimal control formulation of economic models like the one we will study in Chapter 3. Moreover, the principle has an algorithmic structure which makes it a useful tool in numerical applications.

In its purely set-theoretical version it says that, if one wants to find the infimum of a set which is the image of a function $f(x,y)$ of two variables, and the variables are subject to constraints $p(x,y)$ and $q(x)$, then one can

i) fix a $\bar{x}$ satisfying the constraint $q(\bar{x})$

ii) search for the least $f(\bar{x},y)$ when y varies subject to the constraint $p(\bar{x},y)$

iii) minimize the value obtained at ii) letting $\bar{x}$ vary in the set of the points satisfying $q(\bar{x})$.

Formally:

Proposition 4. *Let X be a set, $(D,<)$ an ordered set and $f : X \to D$. If $p(\cdot,\cdot)$ and $q(\cdot)$ are properties over the elements of X, then*

$$\inf\{f(x,y)\ /\ p(x,y)\ \textit{and}\ q(x)\} = \inf\{\inf\{f(x,y)/p(x,y)\}\ /\ q(x)\}$$

Proof. We show that the right member, say μ, is the infimum of the set at the left member, say A.

i)μ is a lower bound of A.

Let (x,y) such that $p(x,y)$ and $q(x)$; then $f(x,y) \geq \inf\{f(x,b)\ /\ p(x,b)\} \geq \inf\{\inf\{f(a,b)/p(a,b)\}\ /\ q(a)\} = \mu$

So being (x,y) generic, $\mu \leq \inf\{f(x,y)\ /\ p(x,y)$ and $q(x)\}$

ii) μ is the greatest lower bound of A.

If $\epsilon > 0$, there is $\bar{x}$ such that $q(\bar{x})$ and $\mu+\epsilon = \mu+\frac{\epsilon}{2}+\frac{\epsilon}{2} > \inf\{f(\bar{x},y)/p(\bar{x},y)\}+\frac{\epsilon}{2} > f(\bar{x},\bar{y})$ for some $\bar{y}$ such that $p(\bar{x},\bar{y})$. Thus $\mu+\epsilon > f(\bar{x},\bar{y})$ such that $p(\bar{x},\bar{y})$ and $q(\bar{x})$. $\square$

Now we can go through the version of the principle which is related to the identification of the value function in differential controlled sistems.

Theorem 5 (**Bellman's dynamic programming principle**). *The value function $V : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following functional equation:*

$$\forall (s,y) \in [0,T) \times \mathbb{R}^n : \forall \bar{s} \in [s,T] :$$
$$V(s,y) = \inf \left\{ \int_s^{\bar{s}} f\left(t, x(t;s,y,u), u(t)\right) dt + V\left(\bar{s}, x(\bar{s};s,y,u)\right) \ / \ u \in \Lambda(s) \right\} \tag{13}$$

Proof. Fix $(s,y) \in [0,T) \times \mathbb{R}^n$ and $\bar{s} \in [s,T]$.
Remembering that $V(s,y) = \inf\{J(u;s,y) \ / \ u \in \Lambda(s)\}$, and setting

$$\mu(s,\bar{s},y) := \inf\{\int_s^{\bar{s}} f(t,x(t;s,y,u),u(t))\mathrm{d}t + V(\bar{s},x(\bar{s};s,y,u)) \ / \ u \in \Lambda(s)\}, \tag{14}$$

we show that:

$$\forall u \in \Lambda(s) : \ \mu(s,\bar{s},y) \le J(u;s,y) \tag{15}$$

Let $u \in \Lambda(s)$, $\bar{u} := u \restriction_{[\bar{s},T]}$, $\bar{y} := x(\bar{s};s,y,u)$; then

$$\begin{aligned}
\mu(s,\bar{s},y) &\le \int_s^{\bar{s}} f(t,x(t;s,y,u),u(t))\mathrm{d}t + V(\bar{s},\bar{y}) \\
&\le \int_s^{\bar{s}} f(t,x(t;s,y,u),u(t))\mathrm{d}t + J(\bar{u};\bar{s},\bar{y}) \\
&= \int_s^{\bar{s}} f(t,x(t;s,y,u),u(t))\mathrm{d}t + \int_{\bar{s}}^{T} f(t,x(t;\bar{s},\bar{y},\bar{u}),u(t))\mathrm{d}t + h(x(T;\bar{s},\bar{y},\bar{u})) \\
&= J(u;s,y)
\end{aligned}$$

where the last equality holds because $x(\cdot;s,y,u) = x(\cdot;\bar{s},\bar{y},\bar{u})$ over $[\bar{s},T]$ for the uniqueness of the orbit.

In the second place, we prove that

$$\forall \epsilon > 0 : \ \exists u_\epsilon \in \Lambda(s) : \ \mu(s,\bar{s},y) + \epsilon > J(u_\epsilon;s,y) \tag{16}$$

Let $\epsilon > 0$ and $v_\epsilon \in \Lambda(s)$ such that

$$\mu(s, \bar{s}, y) + \frac{\epsilon}{2} > \int_s^{\bar{s}} f(t, x(t; s, y, v_\epsilon), v_\epsilon(t))\mathrm{d}t + V(\bar{s}, x(\bar{s}; s, y, v_\epsilon))$$

Then we can find $z_\epsilon \in \Lambda(\bar{s})$ such that, taking $\bar{y} := x(\bar{s}; s, y, v_\epsilon)$

$$\mu(s, \bar{s}, y) + \epsilon > \int_s^{\bar{s}} f(t, x(t; s, y, v_\epsilon), v_\epsilon(t))\mathrm{d}t + J(z_\epsilon; \bar{s}, \bar{y})$$

Now define $u_\epsilon : [s, T] \to U$ as

$$u_\epsilon(t) := \begin{cases} v_\epsilon(t) & \text{if } t \in [s, \bar{s}] \\ z_\epsilon(t) & \text{if } t \in (\bar{s}, T] \end{cases}$$

so that $u_\epsilon \in \Lambda(s)$.

Now observe that the orbit $x(\cdot; s, y, u_\epsilon)$ does reach the state $(\bar{s}, \bar{y})$ because

$$x(\bar{s}; s, y, u_\epsilon) = \lim_{t \uparrow \bar{s}} x(t; s, y, u_\epsilon) = \lim_{t \uparrow \bar{s}} x(t; s, y, v_\epsilon) = x(\bar{s}; s, y, v_\epsilon) = \bar{y}$$

Hence

$$x(t; s, y, u_\epsilon) = \begin{cases} x(t; s, y, v_\epsilon) & t \in [s, \bar{s}] \\ x(t; \bar{s}, \bar{y}, z_\epsilon) & t \in [\bar{s}, T] \end{cases}$$

and the above inequality turns into

$$\mu(s, \bar{s}, y) + \epsilon > J(u_\epsilon; s, y)$$

□

Definition 6. For $(s, y) \in [0, T] \times \mathbb{R}^n$, we say that the control $u \in \Lambda(s)$ is *optimal respect to a state* (s, y) if

$$V(s, y) = J(u; s, y)$$

As a consequence of the above theorem, we have a quite predictible result: a control which is optimal respect to a state is optimal respect to every "successive" state.

Corollary 7. *If* $(s, y) \in [0, T] \times \mathbb{R}^n$ *and* $u^\star \in \Lambda(s)$ *with* $V(s, y) = J(u^\star; s, y)$, *then for every* $\bar{s} \in [s, T]$

$$V(\bar{s}, x(\bar{s}; s, y, u^\star)) = J(u^\star \restriction_{[\bar{s},T]}; \bar{s}, x(\bar{s}; s, y, u^\star))$$

Proof. Let $s, y, u^\star$ as in the hypothesis and $\bar{y} := x(\bar{s}; s, y, u^\star)$. Then, since $V(s, y) \leq \mu(s, \bar{s}, y)$ for the above theorem, we have

$$\begin{aligned} V(s, y) &\leq \int_s^{\bar{s}} f(t, x(t; s, y, u^\star), u^\star(t))\mathrm{d}t + V(\bar{s}, \bar{y}) \\ &\leq \int_s^{\bar{s}} f(t, x(t; s, y, u^\star), u^\star(t))\mathrm{d}t + J(u^\star \restriction_{[\bar{s},T]}; \bar{s}, \bar{y}) \\ &= J(u^\star; s, y) = V(s, y) \end{aligned}$$

which implies the thesis. □

4 The Hamilton-Jacobi-Bellman equation

Now let us extract some other consequences of the value function being a solution of Bellman equation.

Suppose as usual $(s, y) \in [0, T) \times \mathbb{R}^n$ and $\bar{s} \in [s, T]$, and let $\mathbf{u} \in U$ (the space of the control values). Let $x(\cdot)$ be the orbit for (s, y) controlled by the constant control $\mathbf{u}$, that is $x(t) := x(t; s, y, \mathbf{u})$. From $V(s, y) \leq \mu(s, \bar{s}, y)$ it follows that

$$\begin{aligned} V(s, x(s)) &\leq \int_s^{\bar{s}} f(t, x(t), \mathbf{u})\mathrm{d}t + V(\bar{s}, x(\bar{s})) \\ &\Longleftrightarrow \\ \frac{V(s, x(s)) - V(\bar{s}, x(\bar{s}))}{\bar{s} - s} &\leq \frac{1}{\bar{s} - s} \int_s^{\bar{s}} f(t, x(t), \mathbf{u})\mathrm{d}t \end{aligned} \tag{17}$$

Now if V is differentiable, being $x(\cdot) \in \mathcal{C}^1([0,T],\mathbb{R}^n)$ we can take the limit for $\bar{s} \downarrow s$ of the left hand side of the above inequality (being sure of its existence), which is

$$\begin{aligned}
\lim_{\bar{s}\downarrow s} \frac{V(s,x(s)) - V(\bar{s},x(\bar{s}))}{\bar{s}-s} &= -\frac{\mathrm{d}}{\mathrm{d}s}V(s,x(s)) \\
&= -\left\langle DV(s,x(s))\,,\,(1,\dot{x}(s))\right\rangle \\
&= -V_t(s,x(s)) - \left\langle V_x(s,x(s))\,,\,b(s,x(s),\mathbf{u})\right\rangle
\end{aligned}$$

where $DV(\tau,z) = (V_t(\tau,z),V_x(\tau,z)) \in \mathbb{R}\times\mathbb{R}^n$ is the gradient of $V:[0,T]\times\mathbb{R}^n \to \mathbb{R}$ at the point $(\tau,z)\in(0,T)\times\mathbb{R}^n$.
So taking the limit in (17) leads to

$$-V_t(s,y) - \left\langle V_x(s,y)\,,\,b(s,y,\mathbf{u})\right\rangle - f(s,y,\mathbf{u}) \leq 0$$

and being $\mathbf{u}$ generic and independent from (s,y)

$$\sup_{\mathbf{u}\in U}\left\{\left\langle -V_x(s,y)\,,\,b(s,y,\mathbf{u})\right\rangle - f(s,y,\mathbf{u})\right\} \leq V_t(s,y) \tag{18}$$

Now the idea is to use the other "side" of Bellman's Dynamic Programming Equation to show that the reverse inequality also holds.

First of all let us establish a useful property of *sequences of orbits* having the same initial point:

Lemma 8. *For every $(s,y)\in[0,T]\times\mathbb{R}^n$ and every sequence $(u_\epsilon)_{\epsilon>0}\subset\Lambda(s)$ we have:*

$$\forall\epsilon>0 : \forall t\in[s,s+\epsilon] : \|x(t;s,y,u_\epsilon) - y\| \leq \epsilon L\left(\|y\|+1\right)e^{(t-s)L}.$$

Proof. Let $\epsilon>0$ and $t\in[s,s+\epsilon]$. Integrating both sides of

$$\frac{\mathrm{d}x(r;s,y,u_\epsilon)}{\mathrm{d}r} = b\left(r,x\left(r;s,y,u_\epsilon\right)u_\epsilon\left(r\right)\right)$$

for $r \in [s,t]$, adding and removing $\int_s^t b\left(r, y, u_\epsilon(r)\right) \mathrm{d}r$ to the obtained identity, and then passing to the norms, we get by (2) and(3):

$$\begin{aligned} \|x(t;s,y,u_\epsilon) - y\| &\leq \int_s^t \|b\left(r, x(r;s,y,u_\epsilon), u_\epsilon(r)\right) - b\left(r, y, u_\epsilon(r)\right)\| \mathrm{d}r + \\ &\quad + \int_s^t \|b\left(r, y, u_\epsilon(r)\right)\| \mathrm{d}r \\ &\leq \int_s^t L \|x(r;s,y,u_\epsilon) - y\| \mathrm{d}r + \epsilon L \left(\|y\| + 1\right) \end{aligned}$$

By Gronwall's inequality

$$\|x(t;s,y,u_\epsilon) - y\| \leq \epsilon L \left(\|y\| + 1\right) e^{(t-s)L}$$

□

Now observe that for $\bar{s} = s + \epsilon$ (and for a fixed $\epsilon > 0$) Bellman's Principle becomes

$$V(s,y) = \inf \left\{ \int_s^{s+\epsilon} f\left(t, x(t;s,y,u), u(t)\right) \mathrm{d}t + V\left(s+\epsilon, x(s+\epsilon;s,y,u)\right) \ / \ u \in \Lambda(s) \right\}$$

Hence there exists $u_\epsilon \in \Lambda(s)$ such that

$$V(s,y) + \epsilon^2 \geq \int_s^{s+\epsilon} f\left(t, x(t;s,y,u_\epsilon), u_\epsilon(t)\right) \mathrm{d}t + V\left(s+\epsilon, x(s+\epsilon;s,y,u_\epsilon)\right)$$

which implies, setting $x_\epsilon(\cdot) := x(\cdot;s,y,u_\epsilon)$

$$\int_s^{s+\epsilon} \left[V_t\left(t, x_\epsilon(t)\right) + \left\langle V_x\left(t, x_\epsilon(t)\right), b\left(t, x_\epsilon(t), u_\epsilon(t)\right)\right\rangle + f\left(t, x_\epsilon(t), u_\epsilon(t)\right)\right] \mathrm{d}t \leq \epsilon^2$$

Now assume:

Claim 9. The function:

$$(t, y) \to \sup_{\mathbf{u} \in U} \{\langle -V_x(t, y), b(t, y, \mathbf{u})\rangle - f(t, y, \mathbf{u}))\}$$

is continuous over $[0, T] \times \mathbb{R}^n$.

□

In particular, the function:

$$G_\epsilon(t) = -V_t(t, x_\epsilon(t)) + \sup_{\mathbf{u} \in U} \{\langle -V_x(t, x_\epsilon(t)), b(t, x_\epsilon(t), \mathbf{u})\rangle - f(t, x_\epsilon(t), \mathbf{u}))\}$$

is measurable over $[s, s+\epsilon]$. Hence we have:

$$\begin{aligned}
-\epsilon^2 &\leq \int_s^{s+\epsilon} [-V_t(t, x_\epsilon(t)) + \langle -V_x(t, x_\epsilon(t)), b(t, x_\epsilon(t), u_\epsilon(t))\rangle - f(t, x_\epsilon(t), u_\epsilon(t))] \, \mathrm{d}t \\
&\leq \int_s^{s+\epsilon} \left[-V_t(t, x_\epsilon(t)) + \sup_{\mathbf{u} \in U} \{\langle -V_x(t, x_\epsilon(t)), b(t, x_\epsilon(t), \mathbf{u})\rangle - f(t, x_\epsilon(t), \mathbf{u}))\}\right] \mathrm{d}t
\end{aligned}$$

Consequently:

$$\begin{aligned}
-\epsilon &\leq \frac{1}{\epsilon} \int_s^{s+\epsilon} \left[-V_t(t, x_\epsilon(t)) + \sup_{\mathbf{u} \in U} \{\langle -V_x(t, x_\epsilon(t)), b(t, x_\epsilon(t), \mathbf{u})\rangle - f(t, x_\epsilon(t), \mathbf{u}))\}\right] \mathrm{d}t \\
&\leq -V_t(\tau_\epsilon, x_\epsilon(\tau_\epsilon)) + \sup_{\mathbf{u} \in U} \{\langle -V_x(\tau_\epsilon, x_\epsilon(\tau_\epsilon)), b(\tau_\epsilon, x_\epsilon(\tau_\epsilon), \mathbf{u})\rangle - f(\tau_\epsilon, x_\epsilon(\tau_\epsilon), \mathbf{u}))\}
\end{aligned}$$

where $\tau_\epsilon \in [s, s+\epsilon]$ is one maximum point of the function G_ϵ over $[s, s+\epsilon]$. Now by Lemma 8 we have:

$$\|x_\epsilon(\tau_\epsilon) - y\| \leq \epsilon L (\|y\| + 1) e^{\epsilon L}$$

so that $(\tau_\epsilon, x_\epsilon(\tau_\epsilon)) \to (s, y)$ as $\epsilon \to 0$. By Claim 9 and the continuity of V_t we obtain, for $\epsilon \to 0$:

$$0 \leq -V_t(s, y) + \sup_{\mathbf{u} \in U} \{\langle -V_x(s, y), b(s, y, \mathbf{u})\rangle - f(s, y, \mathbf{u})\}. \tag{19}$$

Observe that this is precisely the reverse inequality of (18).

To complete the argument, we have to prove the former claim.

Proof of Claim 9. Let us call G the function whose continuity over $[0,T]\times\mathbb{R}^n$ is to be proven, and let $(t_0, y_0) \in [0,T] \times \mathbb{R}^n,\ \epsilon > 0$.
First of all observe that, for the properties of the supremum:

$$|G(t,y) - G(t_0,y_0)| \leq$$
$$\sup_{\mathbf{u}\in U} |\langle -V_x(t,y), b(t,y,\mathbf{u})\rangle - f(t,y,\mathbf{u})) + \langle V_x(t_0,y_0), b(t_0,y_0,\mathbf{u})\rangle + f(t_0,y_0,\mathbf{u}))|$$

for every $(t,y) \in [0,T] \times \mathbb{R}^n$. Hence, in order to prove the continuity of G in (t_0,y_0), it is sufficient to show that there exists a $\delta > 0$, depending only on ϵ and (t_0,y_0), such that if $\sqrt{|t-t_0|^2 + \|y-y_0\|^2} < \delta$ and $\mathbf{u} \in U$, then the absolute value on the right hand side of the latter inequality is less than a linear function of ϵ.
By the uniform continuity of b and f we can find a $\eta_1 > 0$ such that:

$$\sqrt{|r_1 - r_2|^2 + \|y_1 - y_2\|^2} < \eta_1 \implies \begin{cases} |f(r_1,y_1,\mathbf{u}) - f(r_2,y_2,\mathbf{u})| \leq \epsilon \\ \|b(r_1,y_1,\mathbf{u}) - b(r_2,y_2,\mathbf{u})\| \leq \epsilon \end{cases}$$

for every $(r_1,y_1,\mathbf{u}), (r_2,y_2,\mathbf{u}) \in [0,T] \times \mathbb{R}^n \times U$.
By the continuity of V_x there exists a $\eta_2 > 0$ such that:

$$\sqrt{|t - t_0|^2 + \|y - y_0\|^2} < \eta_2 \implies \begin{cases} \|V_x(t_0,y_0) - V_x(t,y)\| \leq \epsilon \\ \|V_x(t,y)\| \leq (1 + \|V_x(t_0,y_0)\|) \end{cases}$$

for every$(t,y), (t_0,y_0) \in [0,T] \times \mathbb{R}^n$.
Now observe that, for $(t,\mathbf{u}) \in [0,T] \times U$:

$$\langle V_x(t_0,y_0), b(t_0,y_0,\mathbf{u})\rangle - \langle V_x(t,y), b(t,y,\mathbf{u})\rangle =$$
$$\langle V_x(t_0,y_0), b(t_0,y_0,\mathbf{u})\rangle - \langle V_x(t,y), b(t,y,\mathbf{u})\rangle \pm \langle V_x(t,y), b(t_0,y_0,\mathbf{u})\rangle =$$
$$\langle V_x(t_0,y_0) - V_x(t,y), b(t_0,y_0,\mathbf{u})\rangle + \langle V_x(t,y), b(t_0,y_0,\mathbf{u}) - b(t,y,\mathbf{u})\rangle$$

Hence, for $\sqrt{|t-t_0|^2+\|y-y_0\|^2}<\delta:=\min\{\eta_1,\eta_2\}$ and for a generic $\mathbf{u}\in U$, using Schwarz's inequality we find:

$$\begin{aligned}
& \left|\langle -V_x(t,y), b(t,y,\mathbf{u})\rangle - f(t,y,\mathbf{u})) + \langle V_x(t_0,y_0), b(t_0,y_0,\mathbf{u})\rangle + f(t_0,y_0,\mathbf{u}))\right| \\
\leq\ & \left|\langle V_x(t_0,y_0), b(t_0,y_0,\mathbf{u})\rangle - \langle V_x(t,y), b(t,y,\mathbf{u})\rangle\right| + \left|f(t_0,y_0,\mathbf{u})) - f(t,y,\mathbf{u}))\right| \\
\leq\ & \|V_x(t_0,y_0)-V_x(t,y)\|\,\|b(t_0,y_0,\mathbf{u})\| + \|V_x(t,y)\|\,\|b(t_0,y_0,\mathbf{u})-b(t,y,\mathbf{u})\| + \epsilon \\
\leq\ & \epsilon L(1+\|y_0\|) + \epsilon(1+\|V_x(t_0,y_0)\|) + \epsilon
\end{aligned}$$

As δ and the last upper bound depend only on ϵ and (t_0,y_0), we have proven that

$$\lim_{(t,y)\to(t_0,y_0)} G(t,y) = G(t_0,y_0)$$

□

Hence combining (18) with (19) we see that we have given a proof of the following important result:

Theorem 10. *Suppose $V\in\mathcal{C}^1([0,T)\times\mathbb{R}^n,\mathbb{R})$. In this case V is a solution of*

$$-\mathrm{v}_t(s,y)+\sup\{\langle -\mathrm{v}_x(s,y)\,,\,b(s,y,\mathbf{u})\rangle - f(s,y,\mathbf{u})\,/\,\mathbf{u}\in U\}=0 \qquad \forall (s,y)\in[0,T)\times\mathbb{R}^n \tag{20}$$

in the unknown $\mathrm{v}:[0,T)\times\mathbb{R}^n\to\mathbb{R}$.

The function

$$H(s,y,p):=\sup_{\mathbf{u}\in U}\{\langle -p, b(s,y,\mathbf{u})\rangle - f(s,y,\mathbf{u})\} \quad \forall (s,y,p)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$$

is called *Hamiltonian* and the above equation is usually referred to as the *Hamilton-Jacobi-Bellman equation* (or **HJB**).

The basic property of the Hamiltonian function is the following continuity result:

Theorem 11. *The Hamiltonian function* $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *satisfies:*
i)

$$|H(t,x,p) - H(s,y,q)| \leq$$
$$L(1+\|y\|)\|p-q\| + \sup_{\mathbf{u}\in U}\{\|p\|\|b(s,y,\mathbf{u}) - b(t,x,\mathbf{u})\| + |f(s,y,\mathbf{u}) - f(t,x,\mathbf{u})|\}$$

ii) $H \in \mathcal{C}^0([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$
iii) $|H(t,x,p) - H(t,y,p)| \leq L(1+\|p\|)\|x-y\|$
iv) $|H(t,x,p) - H(t,x,q)| \leq L(1+\|x\|)\|p-q\|$
for every $(t,x,p), (s,y,q) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$

Proof. For i), we follow the proof of Claim 9. In the first place:

$$|H(t,x,p) - H(s,y,q)| \leq$$
$$\sup_{\mathbf{u}\in U}|\langle q, b(s,y,\mathbf{u})\rangle - \langle p, b(t,x,\mathbf{u})\rangle + f(s,y,\mathbf{u}) - f(t,x,\mathbf{u})|$$

In the second place, for every $\mathbf{u} \in U$:

$$\begin{aligned} & |\langle q, b(s,y,\mathbf{u})\rangle - \langle p, b(t,x,\mathbf{u})\rangle + f(s,y,\mathbf{u}) - f(t,x,\mathbf{u})| \\ \leq\ & |\langle (p-q), b(s,y,\mathbf{u})\rangle| + |\langle p, [b(s,y,\mathbf{u}) - b(t,x,\mathbf{u})]\rangle| + |f(s,y,\mathbf{u}) - f(t,x,\mathbf{u})| \\ \leq\ & L\|p-q\|(1+\|y\|) + \|p\|\|b(s,y,\mathbf{u}) - b(t,x,\mathbf{u})\| + |f(s,y,\mathbf{u}) - f(t,x,\mathbf{u})|. \end{aligned}$$

For ii), it is sufficient to observe that the last quantity is less than

$$L\|p-q\|(1+\|y\|) + (1+\|q\|)\epsilon + \epsilon$$

for any $\epsilon > 0$ and for (t,x,p) sufficiently near to (s,y,q), thanks to the uniform continuity of functions b, f. iii) and iv) follow easily from i). $\square$

Remark 12. The above theorem implies that for every function $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R}^n, \mathbb{R})$, the function

$$(t,y) \to \sup_{\mathbf{u} \in U} \{\langle -\varphi_x(t,y), b(t,y,\mathbf{u})\rangle - f(t,y,\mathbf{u})\}$$

is continuous.

The value function need not be differentiable in general, nor must HJB have a unique solution. So solving HJB could lead to a function which is different from the value function.

In the following we give an example of an optimal control problem (that is, of a particular controlled differential system) which defines a non-differentiable value function and a HJB equation with no differentiable solutions.

5 A non differentiable value function

Define a controlled one-dimensional sistem by setting

i) $U := [-1,1]$ as the control space,

ii) $[0,1]$ as the time horizon,

iii) $\forall s \in [0,1] : \Lambda(s) := \{u : [s,1] \to U \,/\, u \text{ is measurable}\}$ as the set of feasible controls starting at time s.

Define a set of initial value problems

$$\begin{cases} \dot{x}(t) = u(t)x(t) & t \in [s,1] \\ x(s) = y \end{cases}$$

for $(s,y) \in [0,1] \times \mathbb{R}$

with associated cost functional given by

$$J(u; s, y) := x(1; s, y, u) \quad \forall u \in \Lambda(s)$$

(in this case $f \equiv 0$ and $h = Id$). We show that:

1. The value function associated to this problem is not differentiable - even though it is continuous (as a consequence of Thoerem 3) and admits both partial derivatives in its definition domain. So it is not possible to obtain the value function as a solution of HJB because the proof of Thorem 10 explicitly requires the differentiability of V.

2. The HJB equation associated to this problem does not admit *any* differentiable solution.

For point 1, begin by observing that $x(\cdot; s, y, u) = y \exp\left(\int_s^{\cdot} u\right)$, $J(u; s, y) = y \exp\left(\int_s^1 u\right)$ which implies

$$V(s, y) = \inf_{u \in \Lambda(s)} \{J(u; s, y)\} = \begin{cases} ye^{s-1} & \text{if } y \geq 0 \\ ye^{1-s} & \text{if } y < 0 \end{cases} \quad \forall (s, y) \in [0, 1] \times \mathbb{R}.$$

Hence $V : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and admits both partial derivatives in its domain; nonetheless $V \notin \mathcal{C}^1([0, 1] \times \mathbb{R}, \mathbb{R})$ since

$$V_x(s, y) = \begin{cases} e^{s-1} & \text{if } y \geq 0 \\ e^{1-s} & \text{if } y < 0 \end{cases} \quad \forall (s, y) \in (0, 1) \times \mathbb{R}$$

is not continuous at any point of the segment $(0, 1) \times \{0\}$.

As far as point 2 is concerned, observe in the first place that for $(s, y, \mathbf{u}, p) \in [0,1] \times \mathbb{R} \times U \times \mathbb{R}$, $\langle p\,,\, b(s, y, \mathbf{u})\rangle - f(s, y, \mathbf{u}) = py\mathbf{u}$; hence the hamiltonian is

$$H(s, y, p) = \sup_{\mathbf{u} \in U} \{py\mathbf{u}\} = \begin{cases} py & \text{if } py \geq 0 \\ -py & \text{if } py < 0 \end{cases} = |py|$$

and HJB takes the form

$$\begin{cases} -\mathrm{v}_t(s, y) + |\mathrm{v}_x(s, y)y| = 0 & \forall (s, y) \in [0, 1) \times \mathbb{R} \\ \mathrm{v}(1, y) = y & \forall y \in \mathbb{R}. \end{cases}$$

Suppose now that v is a solution of HJB in $\mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R})$. In particular v_x is continuous over the line $\{1\} \times \mathbb{R}$, and because $\mathrm{v}_x(1, y) = 1$ for every $y \in \mathbb{R}$, there is a neighbourhood (for instance a strip) of $\{1\} \times \mathbb{R}$ (say $\mathcal{N}$) where $\mathrm{v}_x \geq 0$. This allows us to drop down the absolute value in HJB and obtain that v satisfies the following equations:

$$\begin{cases} -\mathrm{v}_t(s, y) + \mathrm{v}_x(s, y)y = 0 & \forall (s, y) \in \mathcal{N} \cap \{y \geq 0\} \\ \mathrm{v}(1, y) = y & \forall y \geq 0 \end{cases}$$

$$\begin{cases} -\mathrm{v}_t(s, y) - \mathrm{v}_x(s, y)y = 0 & \forall (s, y) \in \mathcal{N} \cap \{y < 0\} \\ \mathrm{v}(1, y) = y & \forall y < 0. \end{cases}$$

These are “standard” PDEs solved by the change of variable $z = xe^t$.

$$\frac{\mathrm{dv}(t, ze^{-t})}{\mathrm{d}t} = \mathrm{v}_t(t, ze^{-t}) - \mathrm{v}_x(t, ze^{-t})ze^{-t} = 0$$

if $(t, ze^{-t}) \in \mathcal{N} \cap \{y \geq 0\}$. Hence for (t, z) such that $(t, ze^{-t}) \in \mathcal{N} \cap \{y \geq 0\}$ the function $(t, z) \to \mathrm{v}(t, ze^{-t})$ does not depend on t so

$$\mathrm{v}(t, ze^{-t}) = \mathrm{v}(1, ze^{-1}) = ze^{-1}.$$

Substituting the expression for z we obtain

$$\mathrm{v}(s,y) = ye^{s-1} \quad \forall (s,y) \in \mathcal{N} \cap \{y \geq 0\}\,.$$

In a very similar way we deduce that

$$\mathrm{v}(s,y) = ye^{1-s} \quad \forall (s,y) \in \mathcal{N} \cap \{y < 0\}$$

which means that v and V coincide over $\mathcal{N}$. But we saw at point 1 that V_x was discontinuous in the segment $(0,1) \times \{0\}$ and particulary at the points of its nonempty intersection with $\mathcal{N}$. So we have found a non empty subset of $[0,1] \times \mathbb{R}$ in which v_x is not continuous, against the hypothesis that $\mathrm{v} \in \mathcal{C}^1([0,1] \times \mathbb{R}, \mathbb{R})$.

6 Viscosity solutions to a Hamilton-Jacobi equation

The previous section shows that the hypotheses of Theorem 10 are too strong for practical purposes. Hence we introduce a notion which is weaker than the notion of regular solution to a PDE, and which implies the uniqueness of the solution under very mild conditions, as we will see.

Definition 13. Let Ω an open subset of $\mathbb{R}^n$ and $F \in \mathcal{C}^0\left(\Omega \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}\right)$, and consider the equation

$$F\left(x, \mathrm{v}\left(x\right), D\mathrm{v}\left(x\right)\right) = 0 \quad \forall x \in \Omega \tag{21}$$

in the unknown $\mathrm{v} : \Omega \to \mathbb{R}$. Then a function $u \in \mathcal{C}^0\left(\Omega, \mathbb{R}\right)$ is a *viscosity subsolution* [*supersolution*] to (21) if, and only if:

for every $\varphi \in \mathcal{C}^1\left(\Omega, \mathbb{R}\right)$ and for every local maximum [minimum] point $x_0 \in \Omega$

of $u-\varphi$:

$$F\left(x_0, u\left(x_0\right), D\varphi\left(x_0\right)\right) \leq$$
$$[\geq \quad 0]$$

Remark 14. It follows from the definition that $u \in \mathcal{C}^0\left(\Omega, \mathbb{R}\right)$ is a viscosity solution of (21) if and only if $-u$ is a viscosity solution of

$$-F\left(x, -\mathrm{v}\left(x\right), -D\mathrm{v}\left(x\right)\right) = 0 \quad \forall x \in \Omega$$

The test for the viscosity sub- / super-solution property can be weakend according to the following reflections, which are developed for the notion of viscosity *sub*-solution for simplicity.

Remark 15. Fix $u \in \mathcal{C}^0\left(\Omega, \mathbb{R}\right)$ and let $\varphi \in \mathcal{C}^1\left(\Omega, \mathbb{R}\right)$, $x_0 \in \Omega$ satisfying

$$x_0 \text{ is a local maximum of } u-\varphi \text{ in } \Omega \tag{22}$$

$$F\left(x_0, u\left(x_0\right), D\varphi\left(x_0\right)\right) \leq 0 \tag{23}$$

Hence if we define:

i) $\varphi_1 := \varphi - \varphi\left(x_0\right) + u\left(x_0\right)$

ii) $\varphi_2\left(y\right) := \varphi\left(y\right) + \left\|y - x_0\right\|^2$

iii) $\varphi_3\left(y\right) := \varphi\left(y\right) - \varphi\left(x_0\right) + u\left(x_0\right) + \left\|y - x_0\right\|^2$

then we have

- x_0 is a local maximum of $u-\varphi_1$ in Ω, $\varphi_1\left(x_0\right) = u\left(x_0\right)$ and $D\varphi_1\left(x_0\right) = D\varphi\left(x_0\right)$;

- x_0 is a *strict* local maximum of $u-\varphi_2$ in Ω and $D\varphi_2\left(x_0\right) = D\varphi\left(x_0\right)$;

- x_0 is a *strict* local maximum of $u-\varphi_2$ in Ω, $\varphi_3\left(x_0\right) = u\left(x_0\right)$ and $D\varphi_3\left(x_0\right) = D\varphi\left(x_0\right)$;

For istance to see the that this last property holds, it is sufficient to observe that by (22), for y near x_0 and $y \neq x_0$:

$$\begin{aligned} u(y) - \varphi_3(y) &= u(y) - \varphi(y) + \varphi(x_0) - u(x_0) - \|y - x_0\|^2 \\ &\leq -\|y - x_0\|^2 < 0 = u(x_0) - \varphi_3(x_0) \end{aligned}$$

Remark 16. Let $u \in \mathcal{C}^0(\Omega, \mathbb{R})$ and define:

$$\begin{aligned} \mathcal{A}_u^{0,-} &:= \left\{(x_0, \varphi) \in \mathcal{C}^1(\Omega, \mathbb{R}) \times \Omega / (22) \text{ holds}\right\} \\ \mathcal{B}_u^- &:= \left\{(x_0, \varphi) \in \mathcal{C}^1(\Omega, \mathbb{R}) \times \Omega / (23) \text{ holds}\right\}, \end{aligned}$$

then u is a viscosity subsolution to (21) if and only if $\mathcal{A}_u^{0,-} \subseteq \mathcal{B}_u^-$.

Now define

$$\begin{aligned} \mathcal{A}_u^{1,-} &:= \left\{(x_0, \varphi) \in \mathcal{C}^1(\Omega, \mathbb{R}) \times \Omega / x_0 \text{ is a strict local maximum of } u - \varphi \text{ in } \Omega\right\} \\ \mathcal{A}_u^{2,-} &:= \left\{(x_0, \varphi) \in \mathcal{C}^1(\Omega, \mathbb{R}) \times \Omega / (22) \text{ holds and } \varphi(x_0) = u(x_0)\right\} \\ \mathcal{A}_u^{3,-} &:= \mathcal{A}_u^{1,-} \bigcap \mathcal{A}_u^{2,-}. \end{aligned}$$

Hence Remark 15 shows that $\mathcal{A}_u^{i,-} \subseteq \mathcal{B}_u^-$ implies $\mathcal{A}_u^{0,-} \subseteq \mathcal{B}_u^-$, for $i = 1, 2, 3$. This means that for every $i = 1, 2, 3$, u is a viscosity subsolution to (21) if and only if $\mathcal{A}_u^{i,-} \subseteq \mathcal{B}_u^-$.

So one can choose the test-set $\mathcal{A}_u^{i,-}$ in the most suitable way, in relation to the context.

Moreover we can define $\mathcal{A}_u^{i,+}$ and $\mathcal{B}_u^+$, in the obvious way, and obtain that, for every $i = 0, 1, 2, 3$, u is a viscosity supersolution to (21) if and only if $\mathcal{A}_u^{i,+} \subseteq \mathcal{B}_u^+$.

Of course the notion of viscosity solution wolud be useless if it was not an extension of the classical notion.

Proposition 17. *Let $u \in \mathcal{C}^1(\Omega, \mathbb{R})$. Then u is a (classical) solution of* (21) *if and only if u is a viscosity solution of* (21).

Proof. ($\Longrightarrow$) Let u be a classical solution of (21) and $(\varphi, x_0) \in \mathcal{A}_u^{0,-}$. Then $Du(x_0) = D\varphi(x_0)$. In particular $F(x_0, u(x_0), D\varphi(x_0)) \leq 0$, that is to say $(\varphi, x_0) \in \mathcal{B}_u^-$. The same way we see that $\mathcal{A}_u^{0,+} \subseteq \mathcal{B}_u^+$.

($\Longleftarrow$) Let u be a viscosity solution of (21). Hence, for any $x \in \Omega$, $(u, x) \in \mathcal{A}_u^{0,-} \bigcap \mathcal{A}_u^{0,+}$ which implies by viscosity $(u, x) \in \mathcal{B}_u^- \bigcap \mathcal{B}_u^+$ - and this means $F(x, u(x), Du(x)) = 0$. As x is generic, u is a classical solution of (21). $\square$

As an example, consider the HJ equation

$$-|\mathrm{v}'(x)| + 1 = 0 \quad \forall x \in (-1, 1) \tag{24}$$

(generated by the function $F(x, r, p) := -|p| + 1$, $(x, r, p) \in (-1, 1) \times \mathbb{R} \times \mathbb{R}$).

It is clear that the function $u(x) = |x|$ is a regular solution of (24) in $\Omega' := (-1, 0) \cup (0, 1)$; hence by Proposition 17 u is also a viscosity solution in Ω'.

We show that u is a viscosity solution of (24) in $\Omega := (-\epsilon, \epsilon)$.

As far as the subsolution property is cencerned, let $(\varphi, x_0) \in \mathcal{A}_u^{1,-}$, $x_0 \in \Omega$. If $x_0 \neq 0$ then there exists $u'(x_0) = \varphi'(x_0)$ and so $-|\varphi'(x_0)| + 1 = -|u'(x_0)| + 1 = 0$, which implies $(\varphi, x_0) \in \mathcal{B}_u^-$.

If 0 is a strict local maximum of $u - \varphi$ then for x near 0 and $x \neq 0$:

$$|x| - \varphi(x) < -\varphi(0) \iff 1 < \frac{\varphi(x) - \varphi(0)}{|x|}$$

which implies $\left|\varphi'_+(0)\right| \neq \left|\varphi'_-(0)\right|$, a contradiction. Hence $(\varphi, x_0) \in \mathcal{A}_u^{1,-}$ implies $x_0 \neq 0$, and so $(\varphi, x_0) \in \mathcal{B}_u^-$.

For the supersolution property, consider $(\varphi, x_0) \in \mathcal{A}_u^{1,+}$. If $x_0 \neq 0$ then

$(\varphi, x_0) \in \mathcal{B}_u^+$; if $x_0 = 0$ then for x sufficiently near to 0 and $x \neq 0$ we have:

$$|x| - \varphi(x) > -\varphi(0) \iff 1 > \frac{\varphi(x) - \varphi(0)}{|x|},$$

hence $\varphi'(0) \in [-1, 1]$ which obviously implies $(\varphi, 0) \in \mathcal{B}_u^+$.

This proves that u is a viscosity solution of (24) in $(-1, 1)$. Now observe that, according to Remark 14, $-u$ is a viscosity solution of

$$|\mathrm{v}'(x)| - 1 = 0 \quad \forall x \in (-1, 1). \tag{25}$$

This simple unidimensional example shows the complexity of the behaviour of viscosity solutions, as it is possible to prove that u is *not* a viscosity supersolution of the above equation.

Indeed, the function $\varphi(x) = x^2$ is such that $(\varphi, 0) \in \mathcal{A}_u^{1,+}$ but $|\varphi'(0)| - 1 < 0$, which means $(\varphi, 0) \notin \mathcal{B}_u^+$.

Similarly, considering the test function $\varphi(x) = -x^2$, we see that $-u$ is not a subsolution of (24).

More in general, we have, for any $K \in \mathbb{R}$:

i) $|\cdot| + K$ is a viscosity solution of (24), while it is not a viscosity supersolution of (25)

ii) $-|\cdot| + K$ is a viscosity solution of (25), while it is not a viscosity subsolution of (24).

Hence, as far as the preservation of the viscosity solutions is concerned, multiplying both sides of an equation by the same negative quantity is not allowed.

7 The value function as a viscosity solution of HJB

Turning back to the study of finite horizon optimization problems, we see that equation (20) can be written in the Hamilton-Jacobi form:

$$F\left(t, x, \mathrm{v}(t,x), D\mathrm{v}(t,x)\right) = 0 \quad \forall\,(t,x) \in (0,T) \times \mathbb{R}^n$$

setting, for every $(t,x,r,p) \in (0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{n+1}$:

$$F\left(t,x,r,p\right) = -e_1 \bullet p + \sup_{\mathbf{u}\in U}\left\{-p \bullet \begin{pmatrix} 0 & \dots & 0 \\ & \mathrm{I}_n & \end{pmatrix} \bullet b\left(t,x,\mathbf{u}\right) - f\left(t,x,\mathbf{u}\right)\right\}.$$

Moreover, we can profitably focus on a notion that is even slightly weaker then the notion of the latter section.

Definition 18. A function $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is called a *non -regular viscosity subsolution* [*supersolution*] of (HJB) -(20) if, ond only if:
for every $\varphi \in \mathcal{C}^1\left([0,T) \times \mathbb{R}^n, \mathbb{R}\right)$ and for every local maximum [minimum] point $(t_0, x_0) \in [0,T) \times \mathbb{R}^n$ of $v - \varphi$:

$$\begin{aligned} -\varphi_t\left(t_0,x_0\right) + \sup_{\mathbf{u}\in U}\left\{\left\langle -\varphi_x\left(t_0,x_0\right), b\left(t_0,x_0,\mathbf{u}\right)\right\rangle - f\left(t_0,x_0,\mathbf{u}\right)\right\} &= \\ -\varphi_t\left(t_0,x_0\right) + H\left(t_0,x_0,\varphi_x\left(t_0,x_0\right)\right) &\leq 0 \\ &[\geq 0] \end{aligned}$$

If v is also continuous then we say that v is a *viscosity subsolution* [*supersolution*] of (HJB).
Moreover, v is called a *viscosity solution* of (HJB) if it is both a viscosity supersolution and a viscosity subsolution of (HJB).

The next step is to show that every (continuous) solution to *Bellman's Dynamic Programming Equation* is also a solution to (HJB) in the viscosity sense.

Lemma 19. i) *Every solution* $v : [0,T) \times \mathbb{R}^n :\to \mathbb{R}$ *to the Bellman inequality:*

$$\forall (s,y) \in [0,T) \times \mathbb{R}^n : \forall \bar{s} \in [s,T] :$$
$$v(s,y) \leq \inf_{u \in \Lambda(s)} \left\{ \int_s^{\bar{s}} f\left(t, x(t;s,y,u), u(t)\right) dt + v\left(\bar{s}, x(\bar{s};s,y,u)\right) \right\}$$

is also a non-regular viscosity subsolution of (HJB).

ii) *Every solution* $v : [0,T) \times \mathbb{R}^n :\to \mathbb{R}$ *to the Bellman inequality:*

$$\forall (s,y) \in [0,T) \times \mathbb{R}^n : \forall \bar{s} \in [s,T] :$$
$$v(s,y) \geq \inf_{u \in \Lambda(s)} \left\{ \int_s^{\bar{s}} f\left(t, x(t;s,y,u), u(t)\right) dt + v\left(\bar{s}, x(\bar{s};s,y,u)\right) \right\}$$

is also a non-regular viscosity supersolution of (HJB).

Proof. i) Let $v : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ satisfy the hypothesis, $\varphi \in \mathcal{C}^1\left([0,T) \times \mathbb{R}^n, \mathbb{R}\right)$ and $(s,y) \in [0,T) \times \mathbb{R}^n$ a local maximum point of $v - \varphi$, so that:

$$v\left(s,y\right) - v \geq \varphi\left(s,y\right) - \varphi$$

in a neighbourhood of (s,y). Then pick $\mathbf{u} \in U$, set $x(\cdot) = x(\cdot;s,y,\mathbf{u})$ and observe that by assumption:

$$v\left(s,y\right) - v\left(\bar{s}, x\left(\bar{s}\right)\right) \leq \int_s^{\bar{s}} f\left(t, x\left(t\right), \mathbf{u}\right) \mathrm{d}t$$

for every $\bar{s} \in [s,T]$. Hence, for $\bar{s}$ sufficiently near to s we have:

$$\varphi\left(s,y\right) - \varphi\left(\bar{s}, x\left(\bar{s}\right)\right) \leq \int_s^{\bar{s}} f\left(t, x\left(t\right), \mathbf{u}\right) \mathrm{d}t$$

which implies, following the first part of the proof of Theorem (10):

$$-\varphi_t\left(s,y\right) - \left\langle \varphi_x\left(s,y\right), b\left(s,y,\mathbf{u}\right)\right\rangle - f\left(s,y,\mathbf{u}\right) \leq 0$$

As the argument is independent of $\mathbf{u}$ we can take the sup with respect to $\mathbf{u}$ and reach the thesis.

ii) Let $v : [0,T) \times \mathbb{R}^n \to \mathbb{R}$ as in the assumption. Hence:

$$\begin{aligned}&\forall\epsilon > 0 : \exists\, u_\epsilon \in \Lambda(s) :\\ &v(s,y) + \epsilon^2 \geq \int_s^{s+\epsilon} f(t, x(t; s, y, u_\epsilon), u_\epsilon(t))\,\mathrm{d}t + v(s+\epsilon, x(s+\epsilon; s, y, u_\epsilon))\end{aligned}$$

Let $x_\epsilon := x(\cdot; s, y, u_\epsilon)$ for every $\epsilon > 0$ and (s,y), φ such that φ is continuously differentiable in $[0,T) \times \mathbb{R}^n$ and:

$$v - \varphi \geq v(s,y) - \varphi(s,y)$$

over $(s-\delta, s+\delta) \times B_n(y,R)$ for some $\delta, R > 0$.

Now let $\hat{\epsilon} > 0$ such that $\hat{\epsilon} L (\|y\| + 1) e^{\hat{\epsilon}L} \leq R$. Hence, remembering Lemma (8), for every $\epsilon < \min\{\delta, \hat{\epsilon}\}$ we have:

$$\begin{aligned}&\varphi(s+\epsilon, x_\epsilon(s+\epsilon)) - \varphi(s,y) + \int_s^{s+\epsilon} f(t, x_\epsilon(t), u_\epsilon(t))\,\mathrm{d}t \leq\\ &v(s+\epsilon, x_\epsilon(s+\epsilon)) - v(s,y) + \int_s^{s+\epsilon} f(t, x_\epsilon(t), u_\epsilon(t))\,\mathrm{d}t \leq \epsilon^2\end{aligned}$$

Hence, since φ is differentiable near (s,y), for every $\epsilon < \min\{\delta, \hat{\epsilon}\}$:

$$\begin{aligned}-\epsilon &\leq \frac{1}{\epsilon}\int_s^{s+\epsilon}\Big\{-\varphi_t(t, x_\epsilon(t)) - \langle\varphi_x(t, x_\epsilon(t)), b(t, x_\epsilon(t), u_\epsilon(t))\rangle - f(t, x_\epsilon(t), u_\epsilon(t))\Big\}\mathrm{d}t\\ &\leq \frac{1}{\epsilon}\int_s^{s+\epsilon}\Big\{-\varphi_t(t, x_\epsilon(t)) + \sup_{\mathbf{u}\in U}\left[\langle-\varphi_x(t, x_\epsilon(t)), b(t, x_\epsilon(t), \mathbf{u})\rangle - f(t, x_\epsilon(t), \mathbf{u})\right]\Big\}\,\mathrm{d}t\\ &\leq -\varphi_t(\tau_\epsilon, x_\epsilon(\tau_\epsilon)) + \sup_{\mathbf{u}\in U}\left[\langle-\varphi_x(\tau_\epsilon, x_\epsilon(\tau_\epsilon)), b(\tau_\epsilon, x_\epsilon(\tau_\epsilon), \mathbf{u})\rangle - f(\tau_\epsilon, x_\epsilon(\tau_\epsilon), \mathbf{u})\right]\end{aligned}$$

where the second inequality is justified by Remark (12), and where τ_ϵ is the maximum point over the interval $[s, s+\epsilon]$.

By Remark (12) and Lemma (8) we also deduce that the right hand member of the last inequality tends to

$$-\varphi_t(s,y) + \sup_{\mathbf{u}\in U}\left[\langle -\varphi_x(s,y), b(s,y,\mathbf{u})\rangle - f(s,y,\mathbf{u})\right]$$

as $\epsilon \to 0$, which implies the thesis. $\square$

Corollary 20. *The value function is a viscosity solution to* (HJB) *satisfying*

$$\mathrm{v}(T,x) = h(x) \quad \forall x \in \mathbb{R}^n$$

Proof. By definition (1), Theorems (3) and (5), and Lemma (19). $\square$

8 Uniqueness of the viscosity solution of HJB

Now we go through the core of this chapter, a very important result which "compares" a viscosity subsolution to a viscosity supersolution. The result is what one would expect, by the proof is by no means trivial.

Theorem 21 (**Comparison principle for viscosity solutions of HJB**)**.**

Let $v, \hat{v} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ *be a viscosity subsolution of* (HJB) *and a viscosity supersolution of* (HJB) *respectively, satisfying:*

$$v(T,x) = \hat{v}(T,x) \ \ \forall x \in \mathbb{R}^n$$

Then $v \leq \hat{v}$ *over* $[0,T] \times \mathbb{R}^n$.

Proof. Suppose by contradiction that the thesis is not true. Then by continuity of v and $\hat{v}$ there exist a set $\mathcal{N}$ interior to $[0,T] \times \mathbb{R}^n$ and some $\bar{\gamma} > 0$ such that:

$$\sup_{\mathcal{N}} \{v - \hat{v}\} \geq \bar{\gamma}.$$

We may assume that $\mathcal{N}$ is a triangle of the form:

$$\mathcal{N} := \{(t,x) \in (T-T_0,T) \times \mathbb{R}^n / \, \|x\| < L_0 \, [t-(T-T_0)]\}$$

where $0 < T_0 < T$ and $L_0 > 0$.

Then, by Theorem (11):

$$\begin{aligned} |H(t,x,p) - H(t,x,q)| &\leq L(1+\|x\|)\,\|p-q\| \\ &\leq L(1+L_0T_0)\,\|p-q\| \leq M_0\,\|p-q\| \end{aligned} \tag{26}$$

for $p,q \in \mathbb{R}^n$, $(t,x) \in \overline{\mathcal{N}}$ and $M_0 := \max\{L_0, L(1+L_0T_0)\}$.

Preliminary remark. We know that every function which is continuous over $[0,T]\times\mathbb{R}^n$ is uniformly continuous over the compact subset $\overline{\mathcal{N}} \subset [0,T]\times\mathbb{R}^n$. Hence:

$$\forall \epsilon > 0 : \exists \delta > 0 : \sup_{\substack{|t-s|+\|x-y\|\leq\delta \\ (t,x,s,y)\in\overline{\mathcal{N}\times\mathcal{N}}}} \{|v(t,x)-v(s,y)| + |\hat{v}(t,x)-\hat{v}(s,y)|\} \leq \epsilon.$$

Setting

$$\forall r \geq 0 : \, \eta(r) := \frac{1}{2} \sup_{\substack{|t-s|+\|x-y\|\leq r \\ (t,x,s,y)\in\overline{\mathcal{N}\times\mathcal{N}}}} \{|v(t,x)-v(s,y)| + |\hat{v}(t,x)-\hat{v}(s,y)|\},$$

we observe that η is positive defined and increasing, so that the above condition implies $\lim\limits_{r\to 0^+} \eta(r) = 0$.

Moreover, since $\mathcal{N}$ is bounded, we can take $R > 0$ such that $\eta(R) = \eta(R')$ for all $R' \geq R$ and by monotonicity $\eta_0 := \eta(R) = \max\limits_{[0,\infty)} \eta$.

Now let $\delta, \epsilon, K > 0$ and $\zeta_\delta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that:

$$\begin{cases} K > \max\limits_{(t,x,s,y)\in\overline{\mathcal{N}\times\mathcal{N}}} \{v(t,x) - \hat{v}(s,y)\} \\ \zeta_\delta(r) = 0 & \text{if } r \leq -\delta \\ \zeta'_\delta(r) \leq 0 & \text{if } r \in (-\delta, 0) \\ \zeta_\delta(r) = -K & \text{if } r \geq 0 \end{cases}$$

With those elements (that is, for any fixed $\delta, \epsilon > 0$ as K depends on the problem's data) and for $\alpha, \beta, \gamma > 0$ define, over $\overline{\mathcal{N}\times\mathcal{N}}$, the family of auxiliary functions:

$$\begin{aligned} \Phi_{\alpha\beta\gamma\delta\epsilon}(t,x,s,y) := {} & v(t,x) - \hat{v}(s,y) - \frac{\|x-y\|^2}{\alpha} - \frac{|t-s|^2}{\beta} \\ & +\zeta_\delta\left[\sqrt{\|x\|^2+\epsilon^2} - L_0(t-T+T_0)\right] \\ & +\zeta_\delta\left[\sqrt{\|y\|^2+\epsilon^2} - L_0(s-T+T_0)\right] + \gamma(t+s) - 2\gamma T \end{aligned}$$

each of which approximates from below the function $\Phi(t,x,s,y) := v(t,x) - \hat{v}(s,y)$.

Now let $(t^*, x^*, s^*, y^*) \in \overline{\mathcal{N}\times\mathcal{N}}$ such that $\Phi_{\alpha\beta\gamma\delta\epsilon}$ attains its maximum value over $\overline{\mathcal{N}\times\mathcal{N}}$ in (t^*, x^*, s^*, y^*). So this point depends on the constants $\alpha, \beta, \gamma, \delta, \epsilon$. We split the proof, which is quite technical, in various steps.

Step 1. For every $\alpha, \beta, \gamma, \delta, \epsilon > 0$, the following inequalities hold:

$$\begin{cases} \|x^*-y^*\| \leq \sqrt{\alpha\eta_0} \\ |t^*-s^*| \leq \sqrt{\beta\eta_0} \\ \dfrac{\|x^*-y^*\|^2}{\alpha} + \dfrac{|t^*-s^*|^2}{\beta} \leq \eta\left(\sqrt{\alpha\eta_0} + \sqrt{\beta\eta_0}\right) \end{cases}$$

In the first place observe that (t^*, x^*, t^*, x^*), $(s^*, y^*, s^*, y^*) \in \overline{\mathcal{N}\times\mathcal{N}}$ because

$\overline{\mathcal{N} \times \mathcal{N}} = \overline{\mathcal{N}} \times \overline{\mathcal{N}}$; then by maximality

$$\Phi_{\alpha\beta\gamma\delta\epsilon}\left(t^*, x^*, t^*, x^*\right) + \Phi_{\alpha\beta\gamma\delta\epsilon}\left(s^*, y^*, s^*, y^*\right) \leq 2\Phi_{\alpha\beta\gamma\delta\epsilon}\left(t^*, x^*, s^*, y^*\right),$$

that is to say:

$$\begin{aligned}
&v\left(t^*, x^*\right) - \hat{v}\left(t^*, x^*\right) + 2\zeta_\delta\left[\sqrt{\left\|x^*\right\|^2 + \epsilon^2} - L_0\left(t^* - T + T_0\right)\right] + 2\gamma t^* - 2\gamma T + \\
&v\left(s^*, y^*\right) - \hat{v}\left(s^*, y^*\right) + 2\zeta_\delta\left[\sqrt{\left\|y^*\right\|^2 + \epsilon^2} - L_0\left(s^* - T + T_0\right)\right] + 2\gamma s^* - 2\gamma T \leq \\
&2v\left(t^*, x^*\right) - 2\hat{v}\left(s^*, y^*\right) - \frac{2\left\|x^* - y^*\right\|^2}{\alpha} - \frac{2\left|t^* - s^*\right|^2}{\beta} + \\
&2\zeta_\delta\left[\sqrt{\left\|x^*\right\|^2 + \epsilon^2} - L_0\left(t^* - T + T_0\right)\right] + \\
&2\zeta_\delta\left[\sqrt{\left\|y^*\right\|^2 + \epsilon^2} - L_0\left(s^* - T + T_0\right)\right] + 2\gamma\left(t^* + s^*\right) - 4\gamma T.
\end{aligned}$$

This gives, by definition of η:

$$\begin{aligned}
\frac{2\left\|x^* - y^*\right\|^2}{\alpha} + \frac{2\left|t^* - s^*\right|^2}{\beta} &\leq v\left(t^*, x^*\right) - v\left(s^*, y^*\right) + \hat{v}\left(t^*, x^*\right) - \hat{v}\left(s^*, y^*\right) \\
&\leq 2\eta\left(\left|t^* - s^*\right| + \left\|x^* - y^*\right\|\right) \leq 2\eta_0
\end{aligned}$$

and the assertions follow easily, using the monotonicity of η for the third one.

Step 2. There exist $\mathfrak{c}, \mathfrak{d}, \mathfrak{e} > 0$ such that, for every $\alpha, \beta > 0$,

$$\Phi_{\alpha\beta\mathfrak{c}\mathfrak{d}\mathfrak{e}}\left(t^*, x^*, s^*, y^*\right) \geq \frac{\bar{\gamma}}{2} > 0.$$

Let us choose $(\underline{t}, \underline{x}) \in \mathcal{N}$ and $\mathfrak{c} > 0$ such that:

$$\begin{cases}
\sup\limits_{\mathcal{N}}\left\{v - \hat{v}\right\} - \dfrac{\bar{\gamma}}{4} \leq v\left(\underline{t}, \underline{x}\right) - \hat{v}\left(\underline{t}, \underline{x}\right) \\
2\mathfrak{c}T_0 \leq \dfrac{\bar{\gamma}}{4}
\end{cases}$$

Hence, remembering that $\bar{\gamma} \leq \sup_{\mathcal{N}} \{v - \hat{v}\}$:

$$\begin{aligned}
\frac{\bar{\gamma}}{2} = \bar{\gamma} - \frac{\bar{\gamma}}{4} - \frac{\bar{\gamma}}{4} &\leq v\left(\underline{t}, \underline{x}\right) - \hat{v}\left(\underline{t}, \underline{x}\right) - \frac{\bar{\gamma}}{4} \\
&\leq v\left(\underline{t}, \underline{x}\right) - \hat{v}\left(\underline{t}, \underline{x}\right) - 2\mathfrak{c}T_0 \\
&< v\left(\underline{t}, \underline{x}\right) - \hat{v}\left(\underline{t}, \underline{x}\right) + 2\mathfrak{c}\left(\underline{t} - T\right)
\end{aligned}$$

where the last inequality holds by definition of $\mathcal{N}$. Observe that, since $|\underline{x}| < L_0\left(\underline{t} - T + T_0\right)$, there exist $\mathfrak{e} > 0$ and $0 < \mathfrak{d} < L_0\left(\underline{t} - T + T_0\right)$ such that:

$$\sqrt{|\underline{x}|^2 + \mathfrak{e}^2} - L_0\left(\underline{t} - T + T_0\right) \leq -\mathfrak{d}$$

which implies

$$\zeta_{\mathfrak{d}}\left[\sqrt{|\underline{x}|^2 + \mathfrak{e}^2} - L_0\left(\underline{t} - T + T_0\right)\right] = 0.$$

Hence for every $\alpha, \beta, \gamma > 0$:

$$\Phi_{\alpha\beta\gamma\mathfrak{d}\mathfrak{e}}\left(\underline{t}, \underline{x}, \underline{t}, \underline{x}\right) = v\left(\underline{t}, \underline{x}\right) - \hat{v}\left(\underline{t}, \underline{x}\right) + 2\gamma\left(\underline{t} - T\right)$$

which clearly proves the claim taking $\gamma = \mathfrak{c}$. Observe that the point $\left(\underline{t}, \underline{x}\right) \in \mathcal{N}$ depends only on the initial data and $\bar{\gamma}$, and so does $\mathfrak{c}$. Moreover, the numbers $\mathfrak{d}$ and $\mathfrak{e}$ depend only on $\left(\underline{t}, \underline{x}\right)$. Hence the sub-family of auxiliary functions $\{\Phi_{\alpha\beta\mathfrak{c}\mathfrak{d}\mathfrak{e}}/\alpha, \beta > 0\}$ is uniquely determined by the initial data and $\bar{\gamma}$.

Step 3. There exist $r_0 > 0$ such that for every $0 < \alpha, \beta \leq r_0$, the maximum point (t^*, x^*, s^*, y^*) of $\Phi_{\alpha\beta\mathfrak{c}\mathfrak{d}\mathfrak{e}}$ over $\overline{\mathcal{N} \times \mathcal{N}}$ belongs to $\mathcal{N} \times \mathcal{N}$.

Suppose that the assertion is false. Then there would exist two positive and infinitesimal sequences $\left(\alpha_m\right)_{m\in\mathbb{N}}, \left(\beta_m\right)_{m\in\mathbb{N}}$ such that, the corresponding maximum points of the corresponding auxiliary functions lie in the boundary of $\mathcal{N} \times \mathcal{N}$.

That is to say, there is a sequence $(t_m^*, x_m^*, s_m^*, y_m^*)_{n\in\mathbb{N}} \subset \partial\left(\mathcal{N}\times\mathcal{N}\right)$ such that

$$\Phi_{\alpha_m\beta_m\mathfrak{c}\mathfrak{d}\mathfrak{e}}\left(t_m^*, x_m^*, s_m^*, y_m^*\right) \geq \frac{\bar{\gamma}}{2}\ \forall m\in\mathbb{N}$$

But this condition implies that:

$$\begin{cases} |x_m^*| < L_0\left(t_m^* - T + T_0\right) \\ |y_m^*| < L_0\left(s_m^* - T + T_0\right) \end{cases} \quad \forall m\in\mathbb{N} \tag{27}$$

because on the contrary we would have for instance:

$$\sqrt{\left\|x_{\bar{m}}^*\right\|^2 + \mathfrak{e}^2} - L_0\left(t_{\bar{m}}^* - T + T_0\right) > |x_{\bar{m}}^*| - L_0\left(t_{\bar{m}}^* - T + T_0\right) \geq 0$$

and so $\zeta_{\mathfrak{d}}\left[\sqrt{\left\|x_{\bar{m}}^*\right\|^2 + \mathfrak{e}^2} - L_0\left(t_{\bar{m}}^* - T + T_0\right)\right] = -K$ by definition. Hence we would obtain:

$$\begin{aligned} 0 < \frac{\bar{\gamma}}{2} \leq \Phi_{\alpha_{\bar{m}}\beta_{\bar{m}}\mathfrak{c}\mathfrak{d}\mathfrak{e}}\left(t_{\bar{m}}^*, x_{\bar{m}}^*, s_{\bar{m}}^*, y_{\bar{m}}^*\right) &\leq v\left(t_{\bar{m}}^*, x_{\bar{m}}^*\right) - \hat{v}\left(s_{\bar{m}}^*, y_{\bar{m}}^*\right) - K + \\ &\quad +\mathfrak{c}\left(t_{\bar{m}}^* + s_{\bar{m}}^*\right) - 2\mathfrak{c}T \\ &< K - K + \mathfrak{c}\left(t_{\bar{m}}^* + s_{\bar{m}}^*\right) - 2\mathfrak{c}T \leq 0, \end{aligned}$$

a contradiction. So (27) holds. As the sequence lies in $\partial\left(\mathcal{N}\times\mathcal{N}\right)$, this implies

$$\forall m\in\mathbb{N} : t_m^* = T \ \vee\ s_m^* = T.$$

Observe that we can assume that $(t_m^*, x_m^*, s_m^*, y_m^*)_{m\in\mathbb{N}}$ converges (as it lies in a compact set), so that the last assertion, combined with *Step 1*, becomes:

$$\begin{cases} \lim\limits_{m\to\mathbb{N}} t_m^* = \lim\limits_{m\to\mathbb{N}} s_m^* = T \\ \lim\limits_{m\to\mathbb{N}} x_m^* = \lim\limits_{m\to\mathbb{N}} y_m^* = z \end{cases}$$

for some $z \in \mathbb{R}^n$ such that $(T, z, T, z) \in \partial(\mathcal{N} \times \mathcal{N})$. Hence for every $m \in \mathbb{N}$:

$$\begin{aligned} 0 < \frac{\bar{\gamma}}{2} \leq \Phi_{\alpha_m \beta_m \mathfrak{c}\mathfrak{d}\mathfrak{e}}(t_m^*, x_m^*, s_m^*, y_m^*) &\leq v(t_m^*, x_m^*) - \hat{v}(s_m^*, y_m^*) + \mathfrak{c}(t_m^* + s_m^*) - 2\mathfrak{c}T \\ &\to v(T, z) - \hat{v}(T, z) + 2\mathfrak{c}T - 2\mathfrak{c}T = 0 \end{aligned}$$

as $m \to \infty$, which is a contradiction. So there cannot exist positive infinitesimal sequences $(\alpha_m)_{m\in\mathbb{N}}, (\beta_m)_{m\in\mathbb{N}}$ with $(t_m^*, x_m^*, s_m^*, y_m^*)_{n\in\mathbb{N}} \subset \partial(\mathcal{N} \times \mathcal{N})$, and the claim is proved.

As a consequence we have:

Step 4. Call for every $\alpha, \beta > 0 : \left(t_{\alpha\beta}^*, x_{\alpha\beta}^*, s_{\alpha\beta}^*, y_{\alpha\beta}^*\right)$ the maximum point of $\Phi_{\alpha\beta\mathfrak{c}\mathfrak{d}\mathfrak{e}}$ over $\overline{\mathcal{N} \times \mathcal{N}}$. Then for every $0 < \alpha, \beta \leq r_0$:

$$\begin{aligned} 2\mathfrak{c} &\leq L_0 \left\{\zeta_{\mathfrak{d}}'(X_{\alpha\beta\mathfrak{e}}) + \zeta_{\mathfrak{d}}'(Y_{\alpha\beta\mathfrak{e}})\right\} - H\left(t_{\alpha\beta}^*,\, x_{\alpha\beta}^*,\, \frac{2}{\alpha}\left(y_{\alpha\beta}^* - x_{\alpha\beta}^*\right) + \zeta_{\mathfrak{d}}'(X_{\alpha\beta\mathfrak{e}}) \frac{x_{\alpha\beta}^*}{\left(x_{\alpha\beta}^*\right)_{\mathfrak{e}}}\right) \\ &\quad + H\left(s_{\alpha\beta}^*,\, y_{\alpha\beta}^*,\, \frac{2}{\alpha}\left(y_{\alpha\beta}^* - x_{\alpha\beta}^*\right) - \zeta_{\mathfrak{d}}'(Y_{\alpha\beta\mathfrak{e}}) \frac{y_{\alpha\beta}^*}{\left(y_{\alpha\beta}^*\right)_{\mathfrak{e}}}\right), \end{aligned}$$

where we have set $(z)_{\mathfrak{e}} := \sqrt{\|z\|^2 + \mathfrak{e}^2}$, $X_{\alpha\beta\mathfrak{e}} := \left(x_{\alpha\beta}^*\right)_{\mathfrak{e}} - L_0\left(t_{\alpha\beta}^* - T + T_0\right)$ and $Y_{\alpha\beta\mathfrak{e}} := \left(y_{\alpha\beta}^*\right)_{\mathfrak{e}} - L_0\left(s_{\alpha\beta}^* - T + T_0\right)$.

Set $0 < \alpha, \beta \leq r_0$; by *Step* 3 $\left(t_{\alpha\beta}^*, x_{\alpha\beta}^*, s_{\alpha\beta}^*, y_{\alpha\beta}^*\right) \in \mathcal{N} \times \mathcal{N}$ and so $\left(t_{\alpha\beta}^*, x_{\alpha\beta}^*\right) \in \mathcal{N}$ is a local maximum point for the (t, x) function:

$$\begin{aligned} \Phi_{\alpha\beta\mathfrak{c}\mathfrak{d}\mathfrak{e}}\left(t, x, s_{\alpha\beta}^*, y_{\alpha\beta}^*\right) :&= v(t, x) - \Big\{\hat{v}\left(s_{\alpha\beta}^*, y_{\alpha\beta}^*\right) + \frac{\left\|x - y_{\alpha\beta}^*\right\|^2}{\alpha} + \frac{\left|t - s_{\alpha\beta}^*\right|^2}{\beta} \\ &\quad - \zeta_{\mathfrak{d}}\left[(x)_{\mathfrak{e}} - L_0(t - T + T_0)\right] \\ &\quad - \zeta_{\mathfrak{d}}(Y_{\alpha\beta\mathfrak{e}}) - \mathfrak{c}\left(t + s_{\alpha\beta}^*\right) + 2\mathfrak{c}T\Big\}. \end{aligned}$$

Then being v a viscosity subsolution of (HJB), we have:

$$\begin{aligned} & \frac{2\left(s^*_{\alpha\beta}-t^*_{\alpha\beta}\right)}{\beta} - L_0\zeta'_{\mathfrak{d}}\left(X_{\alpha\beta\mathfrak{e}}\right) + c \\ +H & \left(t^*_{\alpha\beta}, x^*_{\alpha\beta}, \frac{2\left(y^*_{\alpha\beta}-x^*_{\alpha\beta}\right)}{\alpha} + \zeta'_{\mathfrak{d}}\left(X_{\alpha\beta\mathfrak{e}}\right)\frac{x^*_{\alpha\beta}}{\left(x^*_{\alpha\beta}\right)_{\mathfrak{e}}}\right) \leq 0 \end{aligned} \tag{28}$$

At the same time, $\left(s^*_{\alpha\beta}, y^*_{\alpha\beta}\right) \in \mathcal{N}$ is a local minimum point for the (s,y) function:

$$\begin{aligned} -\Phi_{\alpha\beta\mathfrak{c}\mathfrak{d}\mathfrak{e}}\left(t^*_{\alpha\beta}, x^*_{\alpha\beta}, s, y\right) : \ = \ & \hat{v}(s,y) - \Big\{ v\left(t^*_{\alpha\beta}, x^*_{\alpha\beta}\right) - \frac{\left\|y-x^*_{\alpha\beta}\right\|^2}{\alpha} - \frac{\left|s-t^*_{\alpha\beta}\right|^2}{\beta} \\ & +\zeta_{\mathfrak{d}}\left[(y)_{\mathfrak{e}} - L_0\left(s-T+T_0\right)\right] \\ & +\zeta_{\mathfrak{d}}\left(X_{\alpha\beta\mathfrak{e}}\right) + \mathfrak{c}\left(t^*_{\alpha\beta}+s\right) - 2\mathfrak{c}T\Big\}. \end{aligned}$$

Because $\hat{v}$ is a viscosity supersolution of (HJB) we get:

$$\begin{aligned} & \frac{2\left(s^*_{\alpha\beta}-t^*_{\alpha\beta}\right)}{\beta} + L_0\zeta'_{\mathfrak{d}}\left(Y_{\alpha\beta\mathfrak{e}}\right) - \mathfrak{c} \\ +H & \left(s^*_{\alpha\beta}, y^*_{\alpha\beta}, \frac{2\left(y^*_{\alpha\beta}-x^*_{\alpha\beta}\right)}{\alpha} - \zeta'_{\mathfrak{d}}\left(Y_{\alpha\beta\mathfrak{e}}\right)\frac{y^*_{\alpha\beta}}{\left(y^*_{\alpha\beta}\right)_{\mathfrak{e}}}\right) \geq 0 \end{aligned} \tag{29}$$

Combining (28) e (29) together, the claim is proved.

Step 5. Now let $0 < \alpha \leq r_0$, $\beta_m \downarrow 0$. We can assume that the sequence $\left(t^*_{\alpha\beta_m}, x^*_{\alpha\beta_m}, s^*_{\alpha\beta_m}, y^*_{\alpha\beta_m}\right)_{m\in\mathbb{N}} \subset \overline{\mathcal{N}\times\mathcal{N}}$ converges - else, we take a subsequence. Although the point $\left(t^*_{\alpha\beta_m}, x^*_{\alpha\beta_m}, s^*_{\alpha\beta_m}, y^*_{\alpha\beta_m}\right)$ is defined as the maximum point of the function $\Phi_{\alpha\beta_m\mathfrak{c}\mathfrak{d}\mathfrak{e}}$ over $\overline{\mathcal{N}\times\mathcal{N}}$ and by *Step 3* it belongs to $\mathcal{N}\times\mathcal{N}$, we can forget those two informations and rely only on the fact that *Step 4* holds for our α and for $\beta = \beta_m$ (for m sufficiently big).

Let $(t_\alpha, x_\alpha, s_\alpha, y_\alpha) := \lim_{m\to\infty}\left(t^*_{\alpha\beta_m}, x^*_{\alpha\beta_m}, s^*_{\alpha\beta_m}, y^*_{\alpha\beta_m}\right)$.

By *Step 1* we have $\left|t^*_{\alpha\beta_m} - s^*_{\alpha\beta_m}\right| \leq \sqrt{\beta_m \eta_0}$ for every $m \in \mathbb{N}$, and so

$$t_\alpha = s_\alpha.$$

Now set observe that both $(X_{\alpha\beta_m\mathfrak{e}})_{m\in\mathbb{N}}$ and $(Y_{\alpha\beta_m\mathfrak{e}})_{m\in\mathbb{N}}$ converge to a real number, which we call respectively $X_{\alpha\mathfrak{e}}, Y_{\alpha\mathfrak{e}}$. Set, for simplicity of notation: $\forall m \in \mathbb{N} : \left(t^*_{\alpha\beta_m}, x^*_{\alpha\beta_m}, s^*_{\alpha\beta_m}, y^*_{\alpha\beta_m}\right) =: (t^*_m, x^*_m, s^*_m, y^*_m)$.
Now observing that $(t_\alpha, x_\alpha) \in \overline{\mathcal{N}}$, by (26), *Step 4*, Theorem (11) and the continuity of $\zeta'_{\mathfrak{d}}$, we have:

$$\begin{aligned}
2\mathfrak{c} \quad \leq \quad & L_0 \left\{\zeta'_{\mathfrak{d}}\left(X_{\alpha\beta_m\mathfrak{e}}\right) + \zeta'_{\mathfrak{d}}\left(Y_{\alpha\beta_m\mathfrak{e}}\right)\right\} - H\left(t^*_m,\, x^*_m,\, \frac{2}{\alpha}\left(y^*_m - x^*_m\right) + \zeta'_{\mathfrak{d}}\left(X_{\alpha\beta_m\mathfrak{e}}\right)\frac{x^*_m}{\left(x^*_m\right)_{\mathfrak{e}}}\right) \\
& + H\left(s^*_m,\, y^*_m,\, \frac{2}{\alpha}\left(y^*_m - x^*_m\right) - \zeta'_{\mathfrak{d}}\left(Y_{\alpha\beta_m\mathfrak{e}}\right)\frac{y^*_m}{\left(y^*_m\right)_{\mathfrak{e}}}\right) \\
\to \quad & L_0 \left\{\zeta'_{\mathfrak{d}}\left(X_{\alpha\mathfrak{e}}\right) + \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\right\} - H\left(t_\alpha,\, x_\alpha,\, \frac{2}{\alpha}\left(y_\alpha - x_\alpha\right) + \zeta'_{\mathfrak{d}}\left(X_{\alpha\mathfrak{e}}\right)\frac{x_\alpha}{\left(x_\alpha\right)_{\mathfrak{e}}}\right) \\
& + H\left(t_\alpha,\, y_\alpha,\, \frac{2}{\alpha}\left(y_\alpha - x_\alpha\right) - \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\frac{y_\alpha}{\left(y_\alpha\right)_{\mathfrak{e}}}\right) \\
= \quad & L_0 \left\{\zeta'_{\mathfrak{d}}\left(X_{\alpha\mathfrak{e}}\right) + \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\right\} - H\left(t_\alpha,\, x_\alpha,\, \frac{2}{\alpha}\left(y_\alpha - x_\alpha\right) + \zeta'_{\mathfrak{d}}\left(X_{\alpha\mathfrak{e}}\right)\frac{x_\alpha}{\left(x_\alpha\right)_{\mathfrak{e}}}\right) \\
& + H\left(t_\alpha,\, y_\alpha,\, \frac{2}{\alpha}\left(y_\alpha - x_\alpha\right) - \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\frac{y_\alpha}{\left(y_\alpha\right)_{\mathfrak{e}}}\right) \pm \\
& H\left(t_\alpha,\, x_\alpha,\, \frac{2}{\alpha}\left(y_\alpha - x_\alpha\right) - \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\frac{y_\alpha}{\left(y_\alpha\right)_{\mathfrak{e}}}\right) \\
\leq \quad & M_0 \left\{\zeta'_{\mathfrak{d}}\left(X_{\alpha\mathfrak{e}}\right) + \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\right\} + M_0 \left\|\zeta'_{\mathfrak{d}}\left(X_{\alpha\mathfrak{e}}\right)\frac{x_\alpha}{\left(x_\alpha\right)_{\mathfrak{e}}} + \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\frac{y_\alpha}{\left(y_\alpha\right)_{\mathfrak{e}}}\right\| \\
& + L\left\{1 + \left\|\frac{2}{\alpha}\left(y_\alpha - x_\alpha\right) - \zeta'_{\mathfrak{d}}\left(Y_{\alpha\mathfrak{e}}\right)\frac{y_\alpha}{\left(y_\alpha\right)_{\mathfrak{e}}}\right\|\right\}\left\|x_\alpha - y_\alpha\right\|
\end{aligned}$$

$$\begin{aligned}
&\le M_0 \left\{\zeta_{\mathfrak{d}}'(X_{\alpha\mathfrak{e}}) + \zeta_{\mathfrak{d}}'(Y_{\alpha\mathfrak{e}})\right\} + M_0 \left\{|\zeta_{\mathfrak{d}}'(X_{\alpha\mathfrak{e}})| + |\zeta_{\mathfrak{d}}'(Y_{\alpha\mathfrak{e}})|\right\} \\
&\quad + L\left\{1 + |\zeta_{\mathfrak{d}}'(Y_{\alpha\mathfrak{e}})|\right\} \|x_\alpha - y_\alpha\| + \frac{2L}{\alpha}\|x_\alpha - y_\alpha\|^2 \\
&= L\left\{1 + |\zeta_{\mathfrak{d}}'(Y_{\alpha\mathfrak{e}})|\right\} \|x_\alpha - y_\alpha\| + \frac{2L}{\alpha}\|x_\alpha - y_\alpha\|^2
\end{aligned}$$

where the last equality hold because $\zeta_{\mathfrak{d}}' \le 0$ over $\mathbb{R}$. Because this inequality holds for every α sufficiently small, we can take the limit for $\alpha \to 0$. Observe that we can suppose that the sequence $(s_\alpha, y_\alpha)_{\alpha\in\mathbb{R}} \subseteq \overline{\mathcal{N}}$ converges, so that $Y_{\alpha\mathfrak{e}} = (y_\alpha)_{\mathfrak{e}} - L_0(s_\alpha - T + T_0)$ tends to a certain number Y as $\alpha \to 0$.
By *Step 1* we have $\|x_\alpha - y_\alpha\| \le \sqrt{\alpha\eta_0}$. Moreover, for every $m \in \mathbb{N}$,

$$\frac{\|x_m^* - y_m^*\|^2}{\alpha} \le \frac{\|x_m^* - y_m^*\|^2}{\alpha} + \frac{|t_m^* - s_m^*|^2}{\beta_m} \le \eta\left(\sqrt{\alpha\eta_0} + \sqrt{\beta_m\eta_0}\right)$$

which gives for $m \to \infty$, and by the continuity of η:

$$\frac{\|x_\alpha - y_\alpha\|^2}{\alpha} \le \eta\left(\sqrt{\alpha\eta_0}\right)$$

Hence:

$$2\mathfrak{c} \le L\left\{1 + |\zeta_{\mathfrak{d}}'(Y_{\alpha\mathfrak{e}})|\right\}\|x_\alpha - y_\alpha\| + \frac{2L}{\alpha}\|x_\alpha - y_\alpha\|^2 \to 0$$

as $\alpha \to 0$, which is a contraddiction because $\mathfrak{c}$ has been choosen a strictly positive number independent of α. □

Corollary 22. *The value function is the only viscosity solution to* (HJB) *satisfying*

$$\mathrm{v}(T, x) = h(x) \quad \forall x \in \mathbb{R}^n$$

Proof. By Corollary 20 and Theorem 21. □

Part III. An infinite-horizon economic-growth model with non-concave technology and static state constraint

9 The model

9.1 Qualitative description

We assume the existence of a representative dynasty in which all members share the same endowments and consume the same amount of a certain good. Our goal is to describe the dynamics of the capital accumulated by each member of the dynasty in an infinite-horizon period and to maximize, if possible, its intertemporal utility (considered as a function of the quantity of good c that has been consumed). Clearly, consuming is seen as the agent's control strategy, and the set of consumption functions (over time) will be a superset of the set of the admissible control strategies.

First, we need a notion of instantaneous utility, depending on the consumptions, in order to define the inter-temporal utility functional. We will assume that instantaneous utility, which we denote by u, is a strictly increasing and strictly concave function of the consumption path, and that it is twice continuously differentiable. Moreover, we will assume the usual Inada's conditions, that is to say:

$$\lim_{c \to 0^+} u'(c) = +\infty, \ \lim_{c \to +\infty} u'(c) = 0.$$

We will also use the following assumptions over u:

$$u(0) = 0, \ \lim_{c \to +\infty} u(c) = +\infty.$$

With this material, we can define the inter-temporal utility functional, which, as usual, must include a (exponential) discount factor expressing time preference for consumption:

$$U\left(c\left(\cdot\right)\right) := \int_{0}^{+\infty} e^{-\hat{\rho}t} e^{nt} u\left(c\left(t\right)\right) \mathrm{d}t \tag{30}$$

where $\hat{\rho} \in \mathbb{R}$ is the rate of time preference and $n \in \mathbb{R}$ is the growth rate of population. The number of members of the dynasty at time zero is normalized to 1.

9.2 Production function and constraints

We consider the production or output, denoted by F, as a function of the average capital of the representative dynasty, which we denote by k. First, we assume the usual hypothesis of monotonicity, regularity and unboundedness about the production, that is to say: F is strictly increasing and continuously differentiable from $\mathbb{R}$ to $\mathbb{R}$, and

$$F\left(0\right) = 0, \lim_{k \to +\infty} F\left(k\right) = +\infty$$

where we may assume $F\left(x\right) < 0$ for every $x \in \left(-\infty, 0\right)$, as the assumption that F is defined over $\left(-\infty, 0\right)$ is merely technical, as we will see in the first paragraph; this way we distinguish the “admissible” values of the production function from the ones which are not.

Next, we make some specific requirements. As we want to deal with a non-monotonic marginal product of capital, we assume that, in $[0, +\infty)$, F is first strictly concave, then strictly convex and then again strictly concave up to $+\infty$. This means that in the first phase of capital accumulation, the production shows decreasing returns to scale, which become increasing from a certain level of *pro capite* capital $\underline{k}$. Then, when *pro capite* endowment exceed a threshold $\overline{k} > \underline{k}$, decreasing returns to scale characterize the production anew.

Moreover, we ask that the marginal product in $+\infty$ is strictly positive, so that we can deal with endogenous growth. Observe that this limit surely exists, as F' is (strictly) decreasing in a neighbourhood of $+\infty$. Of course the assumption is equivalent to the fact that the average product of capital tends to a strictly positive quantity for large values of the average stock of capital. Moreover, requiring that the marginal product has a strictly positive lower bound is necessary to ensure a positive long-run growth rate.

As far as the agent's behaviour is concerned, the following constraints must be satisfied, for every time $t \geq 0$:

$$
\begin{aligned}
&k(t) \geq 0 \\
&c(t) \geq 0 \\
&i(t) + c(t) \leq F(k(t)) \\
&\dot{k}(t) = i(t)
\end{aligned}
$$

where $i(t)$ is the per capita investment at time t. Observe that the first assumption is needed in order to make the agent's optimal strategy possibly different from the case of monotonic marginal product. In fact if condition $\forall t \geq 0 : k(t) \geq 0$ was not present, then heuristically the convex range of production function would be not relevant to establish the long-run behaviour of economy, since every agent would have the possibility to get an amount of resources such that he can fully exploit the increasing return; therefore only the form of production function for large k would be relevant.

Another heuristic remark turns out to be crucial: the monotonicity of u respect to c implies that, if c is an optimal consumption path, then the production is completely allocated between investment and consumption, that is to say $i(t) + c(t) = F(k(t))$ for every $t \geq 0$. This remark, combined with the last of the above conditions implies that the dynamics of capital allocation, for an initial endowment $k_0 \geq 0$, is described by the following Cauchy's problem:

$$\begin{cases} \dot{k}(t) = F(k(t)) - c(t) & \text{for } t \geq 0 \\ k(0) = k_0 \end{cases} \tag{31}$$

Considering the first two constraints, the agent's target can be expressed the following way: given an initial endowment of capital $k_0 \geq 0$, maximize the functional in (30), when $c(\cdot)$ varies among measurable functions which are everywhere positive in $[0, +\infty)$ and such that the unique solution to problem (31) is also everywhere positive in $[0, +\infty)$; this is what is usually called a *state constraint.*

A few reflections are still necessary in order to begin the analytic work. First, we will consider only the case when the time discount rate $\hat{\rho}$ and the population growth rate n satisfy

$$\hat{\rho} - n > 0,$$

which is the most interesting from the economic point of view. Second, we weaken the requirement that c is measurable and positive in $[0, +\infty)$ (in order that c is admissible) to the requirement that c is locally integrable and almost everywhere positive in $[0, +\infty)$.

Finally, we need another assumption about instantaneous utility u so that the functional in (30) is finite. To identify the best hypothesis, we temporarily restrict our attention to the particular but significant case in which u is a concave power function and F is linear; namely:

$$u(c) = c^{1-\sigma}, \quad c \geq 0$$
$$F(k) = Lk, \quad k \geq 0$$

for some $\sigma \in (0, 1)$ and $L > 0$ (of course in this case F does not satisfy all of the previous assumptions). Using Gronwall's Lemma, it is easy to verify that for any admissible control c (starting from an initial state k_0) and for every time $t \geq 0$, $\int_0^t c(s)\,\mathrm{d}s \leq k_0 e^{Lt}$. Hence, setting $\rho = \hat{\rho} - n$:

$$\begin{aligned} U(c(\cdot)) &= \lim_{T\to+\infty} \int_0^T e^{-\rho t} u(c(t))\,\mathrm{d}t \\ &= \lim_{T\to+\infty} e^{-\rho T} \int_0^T u(c(s))\,\mathrm{d}s + \lim_{T\to+\infty} \rho \int_0^T e^{-\rho t} \int_0^t u(c(s))\,\mathrm{d}s\mathrm{d}t. \end{aligned}$$

Hence using Jensen inequality, we reduce the problem of the convergence of $U(c(\cdot))$ to the problem of the convergence of

$$\int_1^{+\infty} t e^{-\rho t} e^{L(1-\sigma)t}\mathrm{d}t$$

which is equivalent to the condition $L(1-\sigma) < \rho$. Perturbing this clause by the addition of a positive quantity ϵ_0 we get $(L+\epsilon_0)(1-\sigma) < \rho - \epsilon_0$ which is in its turn equivalent to the requirement that the function $e^{\epsilon_0 t} e^{-\rho t} \left(e^{(L+\epsilon_0)t}\right)^{1-\sigma} = e^{\epsilon_0 t} e^{-\rho t} u\left(e^{(L+\epsilon_0)t}\right)$ tends to 0 as $t \to +\infty$.

Turning back to the general case, we are suggested to assume precisely the same condition, taking care of defining the constant L as $\lim_{k\to+\infty} F'(k)$ (which has already been assumed to be strictly positive).

9.3 Quantitative description

Hence the mathematical frame of the economic problem can be defined precisely as follows:

Definition 23. For every $k_0 \geq 0$ and for every $c \in \mathcal{L}^1_{loc}([0,+\infty),\mathbb{R})$:
$k(\cdot;k_0,c)$ is the only solution to the Cauchy's problem

$$\begin{cases} k(0) = k_0 \\ \dot{k}(t) = F(k(t)) - c(t) \quad t \geq 0 \end{cases} \tag{32}$$

in the unknown k, where $F : \mathbb{R} \to \mathbb{R}$ has the following properties:

$$\begin{cases} F \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \\ F' > 0 \text{ in } \mathbb{R} \\ F(0) = 0 \\ \lim_{x \to +\infty} F(x) = +\infty \\ F \text{ is concave over } [0, \underline{k}] \cup \left[\overline{k}, +\infty\right) \quad \text{for some } 0 < \underline{k} < \overline{k} \\ F \text{ is convex over } \left[\underline{k}, \overline{k}\right] \\ \lim_{x \to +\infty} F'(x) > 0 \end{cases}$$

Moreover, we set $L := \lim_{x \to +\infty} F'(x)$.

Definition 24. Let $k_0 \geq 0$.

The set of *admissible consumption strategies* with initial capital k_0 is

$$\Lambda(k_0) := \left\{ c \in \mathcal{L}^1_{loc}([0, +\infty), \mathbb{R}) / c \geq 0 \text{ almost everywhere, } k(\cdot; k_0, c) \geq 0 \right\}$$

The *intertemporal utility functional* $U(\cdot; k_0)$:$\Lambda(k_0) \to \mathbb{R}$ is

$$U(c; k_0) := \int_0^{+\infty} e^{-\rho t} u(c(t)) \, \mathrm{d}t \ \ \forall c \in \Lambda(k_0)$$

where $\rho > 0$, and the function $u : [0, +\infty) \to \mathbb{R}$, representing instantaneous utlity, satisfies:

$$\begin{cases} u \in \mathcal{C}^2\left((0,+\infty),\mathbb{R}\right) \cap \mathcal{C}^0\left([0,+\infty),\mathbb{R}\right) \\ u(0)=0,\ \lim\limits_{x\to+\infty} u(x)=+\infty \\ u \text{ is strictly increasing and strictly concave} \\ \lim\limits_{x\to0^+} u'(x)=+\infty,\ \lim\limits_{x\to+\infty} u'(x)=0 \\ \exists\epsilon_0>0 : \lim\limits_{t\to+\infty} e^{\epsilon_0 t}e^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)=0 \end{cases} \tag{33}$$

The *value function* $V : [0,+\infty) \to \mathbb{R}$ is

$$V(k_0) := \sup_{c\in\Lambda(k_0)} U(c;k_0)\ \ \forall k_0 \geq 0$$

Remark 25. The last condition in (33) implies:

$$\int_0^{+\infty} e^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t < +\infty$$
$$\int_0^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t < +\infty$$

as

$$te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right) = e^{-\frac{\epsilon_0}{2}t}\cdot\omega(t)$$

where $\omega(t) = te^{-\frac{\epsilon_0}{2}t}\cdot e^{\epsilon_0 t}e^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right) \to 0$ for $t \to +\infty$.

10 Preliminary results

Remark 26. Set

$$\overline{M} := \max_{[0,+\infty)} F' = \max \left\{ F'(0), F'\left(\bar{k}\right) \right\}.$$

Recalling that F is strictly increasing with $F(0) = 0$, we see that, for any $x, y \in [0, +\infty)$:

$$\begin{aligned} &|F(x) - F(y)| \leq \overline{M}\,|x - y| \\ &F(x) \leq \overline{M} x \end{aligned}$$

In particular F is Lipschitz-continuous, and so is the dynamics

$$(x, c) \to F(x) - c \quad \forall (x, c) \in [0, +\infty) \times \mathbb{R}$$

uniformly respect to c.

This implies that the Cauchy's problem (32) admits a unique global solution (that is to say, defined on $[0, +\infty)$).

Indeed the mapping

$$\mathcal{F}(k)(t) := k_0 + \int_0^t F(k(s))\,\mathrm{d}s - \int_0^t c(s)\,\mathrm{d}s$$

is a contraction on the space $X := \left(\mathcal{C}^0\left(\left[0, \frac{1}{1+\overline{M}}\right]\right), \|\cdot\|_\infty\right)$, and so admits a unique fixed point $k(\cdot; k_0, c)$. Considering the mapping

$$\mathcal{F}(k)(t) := k\left(\frac{1}{1+\overline{M}}; k_0, c\right) + \int_{\frac{1}{1+\overline{M}}}^t F(k(s))\,\mathrm{d}s - \int_{\frac{1}{1+\overline{M}}}^t c(s)\,\mathrm{d}s$$

on the space $X' := \left(\mathcal{C}^0\left(\left[\frac{1}{1+\overline{M}}, \frac{2}{1+\overline{M}}\right]\right), \|\cdot\|_\infty\right)$, one can extend $k(\cdot; k_0, c)$ to the interval $\left[\frac{1}{1+\overline{M}}, \frac{2}{1+\overline{M}}\right]$, and so on.

Lemma 27. *Let k_1, k_2 two solutions of* (32)*, both defined in $J \subseteq \mathbb{R}$. Then the function*

$$h(t) := \begin{cases} \dfrac{F(k_1(t)) - F(k_2(t))}{k_1(t) - k_2(t)} & \text{if } k_1(t) \neq k_2(t) \\ F'(k_1(t)) & \text{if } k_1(t) = k_2(t) \end{cases}$$

is continuous over J.

Proof. The continuity of h in the points where $k_1 \neq k_2$ is obvious, because k_1, k_2 and F' are continuous in J.

Suppose that $k_1(t_0) = k_2(t_0)$ for some $t_0 \in J$. If $k_1 = k_2$ in a neighbourhood of t_0, then

$$\lim_{t \to t_0} h(t) = \lim_{t \to t_0} F'(k_1(t)) = F'(k_1(t_0)) = h(t_0)$$

by the continuity of F' and k_1.

If there not exists any neighbourhood of t_0 in which $k_1 = k_2$, take $(t_n)_{n \geq 0} \subseteq J$ such that $t_n \to t_0$ and $k_1(t_n) \neq k_2(t_n)$ for every $n \in \mathbb{N}$. Then by Lagrange's theorem

$$\lim_{t \to t_0} h(t) = \lim_{n \to +\infty} h(t_n) = \lim_{n \to +\infty} \frac{F(k_1(t_n)) - F(k_2(t_n))}{k_1(t_n) - k_2(t_n)} = \lim_{n \to +\infty} F'(\xi_n)$$

and this limit is equal to $F'(k_1(t_0)) = h(t_0)$, because ξ_n is intermediate between $k_1(t_n)$ and $k_2(t_n)$ for any $n \in \mathbb{N}$, and by the continuity of the orbits k_1 and k_2:

$$\lim_{n \to +\infty} k_1(t_n) = \lim_{n \to +\infty} k_2(t_n) = k_1(t_0).$$

$\square$

Corollary 28 (**Comparison principle for the orbits**)**.** *Let* $k_1, k_2 \geq 0$, $c_1, c_2 \in \mathcal{L}^1_{loc}\left([0,+\infty), \mathbb{R}\right)$, $T_0 \geq 0$ *and* $T_1 \in (T_0, +\infty]$. *Assume*

$$\begin{aligned} &c_1 \leq c_2 \text{ almost everywhere in } [T_0, T_1] \\ &k\left(T_0; k_1, c_1\right) \geq k\left(T_0; k_2, c_2\right). \end{aligned}$$

Then $k\left(t; k_1, c_1\right) \geq k\left(t; k_2, c_2\right)$ for every $t \in [T_0, T_1]$.

Proof. Set $\kappa_1 := k\left(\cdot; k_1, c_1\right)$, $\kappa_2 := k\left(\cdot; k_2, c_2\right)$ and $h : [T_0, T_1] \to \mathbb{R}$ as defined in Lemma 27. Hence by definition of h and by (32) , we have for every $t \in [T_0, T_1]$:

$$\begin{aligned} &F\left(\kappa_1(t)\right) - F\left(\kappa_2(t)\right) = h(t)\left[\kappa_1(t) - \kappa_2(t)\right], \\ &\dot{\kappa_1}(t) - \dot{\kappa_2}(t) = F\left(\kappa_1(t)\right) - F\left(\kappa_2(t)\right) + c_2(t) - c_1(t) \end{aligned}$$

which implies

$$\dot{\kappa_1}(t) - \dot{\kappa_2}(t) = h(t)\left[\kappa_1(t) - \kappa_2(t)\right] + c_2(t) - c_1(t) \qquad \forall t \in [T_0, T_1].$$

Since h is measurable by Lemma 27, we can multiply both sides by the continuous function $t \to \exp\left(-\int_{T_0}^t h(s)\,\mathrm{d}s\right)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[\kappa_1(t) - \kappa_2(t)\right] e^{-\int_{T_0}^t h(s)ds}\right\} = \left[c_2(t) - c_1(t)\right] e^{-\int_{T_0}^t h(s)ds} \quad \forall t \in [T_0, T_1]$$

which implies, integrating between T_0 and any $t \in [T_0, T_1]$:

$$\left[\kappa_1(t) - \kappa_2(t)\right] e^{-\int_{T_0}^t h} = \kappa_1(T_0) - \kappa_2(T_0) + \int_{T_0}^t \left[c_2(s) - c_1(s)\right] e^{-\int_{T_0}^s h} \geq 0$$

where the last inequality holds by the assumptions over c_1, c_2, κ_1 and κ_2, and obviously implies the thesis. $\square$

Remark 29. The above argument also proves that the implication

$$k\left(T_0;k_1,c_1\right)>k\left(T_0;k_2,c_2\right)\implies\forall t\in\left[T_0,T_1\right]:k\left(t;k_1,c_1\right)>k\left(t;k_2,c_2\right)$$

holds when $c_1\leq c_2$ almost everywhere in $\left[T_0,T_1\right]$.

Lemma 30. *There exists a function* $g:(0,+\infty)\to(0,+\infty)$ *which is convex, decreasing and such that*

$$g\left(x\right)\leq u'\left(x\right)\quad\forall x>0.$$

Proof. Let

$$\begin{aligned}\Sigma_{u'}&:=\left\{(x,y)\in(0,+\infty)^2/y\geq u'\left(x\right)\right\}\\K_{u'}&:=\bigcap\left\{K\in\mathcal{P}\left(\mathbb{R}^2\right)/K=\overline{K},\ K\text{ is convex, }K\supseteq\Sigma_{u'}\right\}.\end{aligned}$$

In particular $K_{u'}$ is a closed-convex superset of $\Sigma_{u'}$. Observe that, for any $x>0$, the function $H_x\left(y\right):=\left(x,y\right)$ belongs to $\mathcal{C}^0\left(\mathbb{R},\mathbb{R}^2\right)$, so any set of the form

$$\left\{y\geq0/\left(x,y\right)\in K_{u'}\right\}=H_x^{-1}\left(K_{u'}\right)\bigcap\left[0,+\infty\right)$$

is closed in $\mathbb{R}$, and consequently it has a minimum element. Now define

$$\forall x>0:g\left(x\right):=\min\left\{y\geq0/\left(x,y\right)\in K_{u'}\right\}.$$

i) This definition implies that for every $\left(x,y\right)\in K_{u'}$, $g\left(x\right)\leq y$; hence

$$g\left(x\right)\leq u'\left(x\right)\quad\forall x>0$$

because for any $x>0$, $\left(x,u'\left(x\right)\right)\in\Sigma_{u'}\subseteq K_{u'}$.

ii) In the second place, g is convex in $(0,+\infty)$. Let $x_0,x_1>0$ and $\lambda\in(0,1)$.

By definition of g, $(x_0, g(x_0)), (x_1, g(x_1)) \in K_{u'}$, which is a convex set. Hence

$$(1-\lambda)(x_0, g(x_0)) + \lambda(x_1, g(x_1)) \in K_{u'}.$$

By the first property in i), this implies

$$g((1-\lambda)x_0 + \lambda x_1) \leq (1-\lambda) g(x_0) + \lambda g(x_1).$$

iii) g is decreasing. Indeed, take $0 < x_0 < x_1$. By ii) and by definition of convexity, for every $n \in \mathbb{N}$:

$$g(n(x_1 - x_0) + x_0) \quad \geq \quad n[g(x_1) - g(x_0)] + g(x_0).$$

Hence by the assumptions over u and by i):

$$\begin{aligned} 0 &= \lim_{n\to+\infty} u'(n(x_1 - x_0) + x_0) \geq \limsup_{n\to+\infty} g(n(x_1 - x_0) + x_0) \\ &\geq \lim_{n\to+\infty} n[g(x_1) - g(x_0)] + g(x_0) \end{aligned}$$

which implies $g(x_1) \leq g(x_0)$.
iv) Observe that the definition of g does not exclude that $g(x) = 0$ for some $x > 0$. Indeed we show that $g > 0$ in $(0, +\infty)$.
Fix $x > 0$, and consider the closed-convex aproximation of $\Sigma_{u'}$

$$K_x := \left\{(t, y) \in [0, x] \times [0, +\infty) / y \geq \frac{\min_{[0,x]} u'}{x}(x - t)\right\} \bigcup [x, +\infty) \times [0, +\infty).$$

By construction $K_{u'} \subseteq K_x$ which implies $(t, g(t)) \in K_x$ for any $t > 0$. In particular, for every $t \in (0, x)$:

$$g(t) \geq \frac{\min_{[0,x]} u'}{x}(x - t) > 0$$

because $u' > 0$. This is precisely the fact that allows us to repeat this construction for every $x > 0$, which ensures that $g > 0$ in $(0, +\infty)$. $\square$

Remark 31. The function h defined in Lemma 27 satisfies

$$|h| \leq \overline{M}.$$

where $\overline{M}$ is defined as in Remark 26.

Remark 32. Let $k_0 \geq 0$ and $c \in \Lambda(k_0)$. Then, for every $t \geq 0$:

$$\begin{aligned} k(t; k_0, c) &\leq k_0 e^{\overline{M}t} \\ \int_0^t c(s)\,\mathrm{d}s &\leq k_0 e^{\overline{M}t} \end{aligned}$$

Indeed, by Remark 26 and remembering that $c \geq 0$, we have, for every $t \geq 0$, $\dot{k}(t; k_0, c) \leq \overline{M} k(t; k_0, c)$ - which implies by Corollary 28:

$$k(t; k_0, c) \leq k_0 e^{\overline{M}t} \quad \forall t \geq 0.$$

Now integrating both sides of the state equation, again by Remark 26 and by the fact that $k(\cdot; k_0, c) \geq 0$ we see that, for every $t \geq 0$:

$$\begin{aligned} \int_0^t c(s)\,\mathrm{d}s &= k_0 - k(t; k_0, c) + \int_0^t F(k(s; k_0, c))\,\mathrm{d}s \\ &\leq k_0 + \overline{M} \int_0^t k(s; k_0, c)\,\mathrm{d}s \\ &\leq k_0 + \overline{M} k_0 \int_0^t e^{\overline{M}s}\mathrm{d}s = k_0 e^{\overline{M}t}. \end{aligned}$$

Lemma 33. *There exists a function* $N : (0,+\infty)^2 \to (0,+\infty)$*, increasing in both variables, such that:*

for every $(k_0, T) \in (0,+\infty)^2$ *and every* $c \in \Lambda(k_0)$*, there exists a control function* $c^T \in \Lambda(k_0)$ *satisfying*

$$U\left(c^T; k_0\right) \geq U\left(c; k_0\right)$$
$$c^T = c \wedge N\left(k_0, T\right) \text{ almost everywhere in } [0, T]$$

In particular, c^T *is bounded above, in* $[0,T]$*, by a quantity which does not depend on the original control c, but only on* T *and on the initial status* k_0*.*

Proof. Let g be the function defined in Lemma 30 and $\beta := \frac{\log\left(1+\overline{M}\right)}{\overline{M}}$. Define, for every $(k_0, T) \in (0,+\infty)^2$:

$$\alpha\left(k_0, T\right) := \beta e^{-\rho(T+\beta)} g\left[k_0\left(\frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T}\right)\right]$$
$$N\left(k_0, T\right) := \inf\left\{\tilde{N} > 0 / \forall N \geq \tilde{N} : u'\left(N\right) < \alpha\left(k_0, T\right)\right\}.$$

In the first place, $N\left(k_0, T\right) \neq +\infty$, because $\alpha\left(k_0, T\right) > 0$ for every $k_0 > 0$, $T > 0$ and $\lim_{N \to +\infty} u'\left(N\right) = 0$.

In the second place, $u'\left((0,+\infty)\right) = (0,+\infty)$, which implies $N\left(k_0, T\right) > 0$: otherwise, since $\left(u'\right)^{-1}(\alpha\left(k_0, T\right)) > 0$, there wolud exist $N > 0$ such that

$$N < \left(u'\right)^{-1}\left(\alpha\left(k_0, T\right)\right)$$
$$u'\left(N\right) < \alpha\left(k_0, T\right)$$

which is absurd because u' is decreasing; hence the quantity $u'\left(N\left(k_0, T\right)\right)$ is well defined. Moreover by the continuity of u',

$$u'\left(N\left(k_0, T\right)\right) \leq \alpha\left(k_0, T\right). \tag{34}$$

The function $N(\cdot,\cdot)$ is also increasing in both variables, because $\alpha(\cdot,\cdot)$ is decreasing in both variables and u' is decreasing.

Indeed, for $k_0 \leq k_1$ and for a fixed $T > 0$, suppose that $N(k_1, T) < N(k_0, T)$. Then by definition of infimum we could choose $\tilde{N} \in [N(k_1, T), N(k_0, T))$ such that $u'\left(\tilde{N}\right) < \alpha(k_1, T)$, which implies

$$u'\left(\tilde{N}\right) < \alpha(k_0, T)$$

by the monotonicity of α. But since $\tilde{N} > 0$, $\tilde{N} < N(k_0, T)$ there also exists $N \geq \tilde{N}$ such that $u'(N) \geq \alpha(k_0, T)$ which implies, by the monotonicity of u',

$$u'\left(\tilde{N}\right) \geq \alpha(k_0, T),$$

a contradiction. With an analogous argument we prove that $N(\cdot,\cdot)$ is increasing in the second variable.

Now let $k_0, T > 0$ and $c \in \Lambda(k_0)$ as in the hypothesis. If $c \leq N(k_0, T)$ almost everywhere in $[0, T]$, then define $c^T := c$. If, on the contrary, $c > N(k_0, T)$ in a non-negligible subset of $[0, T]$, then define:

$$c^T(t) := \begin{cases} c(t) \wedge N(k_0, T) & \text{if } t \in [0, T] \\ c(t) + I_T & \text{if } t \in (T, T+\beta] \\ c(t) & \text{if } t > T+\beta \end{cases}$$

where $I_T := \int_0^T e^{-\rho t}(c(t) - c(t) \wedge N(k_0, T))\,\mathrm{d}t$. Observe that by Remark 32:

$$\begin{aligned} 0 < \ I_T &\leq \int_0^T (c(t) - c(t) \wedge N(k_0, T))\,\mathrm{d}t \\ &\leq \int_0^T c(t)\,\mathrm{d}t \\ &\leq k_0 e^{\overline{M}T} \end{aligned} \tag{35}$$

In order to prove the admissibility of such control function, we compare the orbit $k := k(\cdot; k_0, c)$ to the orbit $k^T := k(\cdot; k_0, c^T)$. In the first place, observe that by Corollary 28 and by definition of c^T:

$$k^T(t) \geq k(t) \quad \forall t \in [0, T] \tag{36}$$

Now by the state equation, we have:

$$\dot{k^T} - \dot{k} = F\left(k^T\right) - F(k) + c - c^T. \tag{37}$$

Set for every $t \geq 0$:

$$h(t) := \begin{cases} \frac{F\left(k^T(t)\right) - F(k(t))}{k^T(t) - k(t)} & \text{if } k^T(t) \neq k(t) \\ F'(k(t)) & \text{if } k^T(t) = k(t) \end{cases}$$

Hence by (37)

$$\dot{k^T}(t) - \dot{k}(t) = h(t)\left[k^T(t) - k(t)\right] + c(t) - c^T(t) \quad \forall t \geq 0.$$

By Lemma 27, the function h is continuous over its definition domain, so this is a typical linear equation with measurable coefficient of degree one, satisfied by $k^T - k$. Hence, multiplying both sides by the continuous function $t \to \exp\left(-\int_0^t h(s)\,\mathrm{d}s\right)$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[k^T(t) - k(t)\right] e^{-\int_0^t h(s)\mathrm{d}s}\right\} = \left[c(t) - c^T(t)\right] e^{-\int_0^t h(s)\mathrm{d}s} \quad \forall t \geq 0$$

which implies, integrating between 0 and any $t \geq 0$:

$$k^T(t) - k(t) \;=\; \int_0^t \left[c(s) - c^T(s)\right] e^{\int_s^t h}\mathrm{d}s \tag{38}$$

Now observe that

$$h \leq \overline{M} \text{ over } [0,+\infty)$$
$$h \geq 0 \text{ over } [0,T],$$

by (36) and the monotonicity of F. Set $t \in (T, T+\beta]$; then by (38):

$$\begin{aligned}
k^T(t) - k(t) &= \int_0^T [c(s) - c(s) \wedge N(k_0,T)]\, e^{\int_s^t h} \mathrm{d}s - I_T \cdot \int_T^t e^{\int_s^t h} \mathrm{d}s \\
&\geq \int_0^T [c(s) - c(s) \wedge N(k_0,T)]\, \mathrm{d}s - I_T \cdot \int_T^t e^{\overline{M}(t-s)} \mathrm{d}s \\
&\geq \int_0^T e^{-\rho s} [c(s) - c(s) \wedge N(k_0,T)]\, \mathrm{d}s - I_T \cdot \int_T^{T+\beta} e^{\overline{M}(T+\beta-s)} \mathrm{d}s \\
&= I_T \left(1 - \frac{e^{\overline{M}\beta} - 1}{\overline{M}}\right) = 0
\end{aligned} \tag{39}$$

This also implies, by Corollary 28 and by definition of c^T,

$$k^T(t) \geq k(t) \quad \forall t \geq T+\beta$$

Such inequality, together with (36) and (39), gives us the general inequality

$$k^T(t) \geq k(t) \geq 0 \quad \forall t \geq 0.$$

This implies, associated with the obvious fact that $c^T \geq 0$ almost everywhere in $[0,+\infty)$, that $c^T \in \Lambda(k_0)$.

Now we prove the "optimality" property of c^T respect to c. By the concavity of u, and setting $N := N(k_0, T)$ for simplicity of notation, we have:

$$\begin{aligned}
U(c;k_0) - U\left(c^T;k_0\right) &= \int_0^{+\infty} e^{-\rho t}\left[u(c(t)) - u\left(c^T(t)\right)\right] \mathrm{d}t \\
&= \int_{[0,T]\cap\{c\geq N\}} e^{-\rho t}\left[u(c(t)) - u(c(t)\wedge N)\right] \mathrm{d}t \\
&\quad + \int_T^{T+\beta} e^{-\rho t}\left[u(c(t)) - u(c(t)+I_T)\right] \mathrm{d}t \\
&\leq \int_{[0,T]\cap\{c\geq N\}} e^{-\rho t} u'(c(t)\wedge N)\left[c(t) - c(t)\wedge N\right] \mathrm{d}t \\
&\quad -I_T \int_T^{T+\beta} e^{-\rho t} u'(c(t)+I_T)\, \mathrm{d}t \\
&= u'(N) \int_0^T e^{-\rho t}\left[c(t) - c(t)\wedge N\right] \mathrm{d}t \\
&\quad -I_T \int_T^{T+\beta} e^{-\rho t} u'(c(t)+I_T)\, \mathrm{d}t \\
&= I_T\left[u'(N) - \int_T^{T+\beta} e^{-\rho t} u'(c(t)+I_T)\, \mathrm{d}t\right] \qquad (40)
\end{aligned}$$

Now we exhibit a certain lower bound wich is independent on the particular control function c. By Jensen inequality, by Lemma 30 and by (35), we have:

$$\begin{aligned}
\int_T^{T+\beta} e^{-\rho t} u'(c(t)+I_T)\, \mathrm{d}t &\geq \int_T^{T+\beta} e^{-\rho t} g(c(t)+I_T)\, \mathrm{d}t \\
&\geq e^{-\rho(T+\beta)} \int_T^{T+\beta} g(c(t)+I_T)\, \mathrm{d}t \\
&\geq \beta e^{-\rho(T+\beta)} g\left(\frac{1}{\beta}\int_T^{T+\beta}\left[c(t)+I_T\right] \mathrm{d}t\right) \\
&\geq \beta e^{-\rho(T+\beta)} g\left(\frac{1}{\beta}\int_0^{T+\beta} c(t)\, \mathrm{d}t + I_T\right) \\
&\geq \beta e^{-\rho(T+\beta)} g\left[k_0\left(\frac{e^{\overline{M}(T+\beta)}}{\beta} + e^{\overline{M}T}\right)\right] \\
&= \alpha(k_0, T).
\end{aligned}$$

Hence by (34) and (40):

$$\begin{aligned} U(c;k_0)-U\left(c^T;k_0\right) &\leq I_T\left[u'(N(k_0,T))-\int_T^{T+\beta} e^{-\rho t}u'(c(t)+I_T)\,\mathrm{d}t\right] \\ &\leq I_T\left[u'(N(k_0,T))-\alpha(k_0,T)\right]\leq 0. \end{aligned}$$

□

Lemma 34. *Let* $0<k_0<k_1$ *and* $c\in\Lambda(k_0)$*. Then there exists a control function* $\underline{c}^{k_1-k_0}\in\Lambda(k_1)$ *such that*

$$U\left(\underline{c}^{k_1-k_0};k_1\right)-U(c;k_0)\geq u'(N(k_0,k_1-k_0)+1)\int_0^{k_1-k_0} e^{-\rho t}dt$$

where N *is the function defined in* Lemma 33.

Proof. Fix k_0,k_1 and c as in the hypothesis and take $c^{k_1-k_0}$ as in Lemma 33 (where it is understood that $T=k_1-k_0$).Then define:

$$\underline{c}^{k_1-k_0}(t):=\begin{cases} c^{k_1-k_0}(t)+1 & \text{if } t\in[0,k_1-k_0) \\ c^{k_1-k_0}(t) & \text{if } t\geq k_1-k_0 \end{cases}$$

In the first place we prove that $\underline{c}^{k_1-k_0}\in\Lambda(k_1)$, showing that

$$\underline{k}:=k\left(\cdot;k_1;\underline{c}^{k_1-k_0}\right)>k\left(\cdot;k_0,c^{k_1-k_0}\right)=:k \tag{41}$$

over $(0,+\infty)$. Suppose by contradiction that this is not true, and take $\tau:=\inf\{t>0/\underline{k}(t)\leq k(t)\}$. Then by the continuity of the orbits, $\underline{k}(\tau)\leq k(\tau)$, which implies $\tau>0$. Considering the orbits as solutions to an integral equation we have:

$$k(\tau)=k_0+\int_0^\tau F(k(t))\,\mathrm{d}t-\int_0^\tau c^{k_1-k_0}(t)\,\mathrm{d}t$$

$$\underline{k}(\tau) = k_1 + \int_0^{\tau} F(\underline{k}(t))\,\mathrm{d}t - \int_0^{\tau} c^{k_1-k_0}(t)\,\mathrm{d}t - \min\{\tau, k_1 - k_0\}.$$

Hence

$$\begin{aligned} 0 \geq \underline{k}(\tau) - k(\tau) &= k_1 - k_0 + \int_0^{\tau} [F(\underline{k}(t)) - F(k(t))]\,\mathrm{d}t - \min\{\tau, k_1 - k_0\} \\ &\geq \int_0^{\tau} [F(\underline{k}(t)) - F(k(t))]\,\mathrm{d}t \end{aligned}$$

By the definition of τ and the strict monotonicity of F, this quantity must be strictly positive, which is absurd. Hence

$$\begin{aligned} &k\left(\cdot; k_1; \underline{c}^{k_1-k_0}\right) > k\left(\cdot; k_0, c^{k_1-k_0}\right) \geq 0 \text{ in } [0, +\infty) \\ &\underline{c}^{k_1-k_0} \geq c^{k_1-k_0} \geq 0 \text{ a.e. in } [0, +\infty) \end{aligned}$$

which implies $\underline{c}^{k_1-k_0} \in \Lambda(k_0)$.

In the second place, remembering the properties of $c^{k_1-k_0}$ given by Lemma 33, we have

$$\begin{aligned} U\left(\underline{c}^{k_1-k_0}; k_1\right) - U(c; k_0) &\geq U\left(\underline{c}^{k_1-k_0}; k_1\right) - U\left(c^{k_1-k_0}; k_0\right) \\ &= \int_0^{+\infty} e^{-\rho t}\left[u\left(\underline{c}^{k_1-k_0}(t)\right) - u\left(c^{k_1-k_0}(t)\right)\right]\mathrm{d}t \\ &\quad \int_0^{k_1-k_0} e^{-\rho t}\left[u\left(c^{k_1-k_0}(t) + 1\right) - u\left(c^{k_1-k_0}(t)\right)\right]\mathrm{d}t \\ &\geq \int_0^{k_1-k_0} e^{-\rho t} u'\left(c^{k_1-k_0}(t) + 1\right)\mathrm{d}t \\ &\geq u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t}\mathrm{d}t \end{aligned}$$

which concludes the proof. □

Remark 35. In the previous Lemma, the property (41) can also be proved with the "comparison technique", like we did for the admissibility of c^T in Lemma 33.

More generally, it can be proved that

$$k(\cdot; k_1, c_H) > k(\cdot; k_0, c)$$

where $k_1 > k_0 \geq 0$, $c \in \mathcal{L}^1_{loc}([0, +\infty), \mathbb{R})$ and

$$c_H(t) := \begin{cases} c(t) + H & \text{if } t \in [0, \delta_H) \\ c(t) & \text{if } t \geq \delta_H \end{cases}$$

and $\delta_H > 0$ satisfying $\delta_H \cdot H \leq k_1 - k_0$.

Indeed, set $k_H := k(\cdot; k_1, c_H)$ and $k := k(\cdot; k_0, c)$ and suppose by contradiction that $-\infty < \inf \{t > 0 / k_H(t) \leq k(t)\} =: \tau$. Then for a suitable, positive continuous function $h : [0, +\infty) \to \mathbb{R}$, the following equality holds:

$$k_H(\tau) - k(\tau) \quad = \quad e^{\int_0^\tau h} \left[k_1 - k_0 + \int_0^\tau (c(s) - c_H(s)) \, e^{-\int_0^s h} \mathrm{d}s \right].$$

Moreover $\tau \leq \delta_H$, because on the contrary by definition of infimum we would have $k_H > k$ in $[0, \delta_H]$; then remembering Remark 29 and the definition of c_H we would conclude that $k_H > k$ everywhere in $[0, +\infty)$, which contradicts $\tau > -\infty$. Then the above equality implies

$$k_H(\tau) - k(\tau) \; > \; k_1 - k_0 - H \int_0^\tau e^{-\int_0^s h} \mathrm{d}s \; > \; k_1 - k_0 - \tau H \; \geq \; k_1 - k_0 - \delta_H H \; \geq 0.$$

At the same time $k_H(\tau) \leq k(\tau)$ by the continuity of k_h and k and by definition of infimum (in fact the equality holds, again by continuity); hence we have reached the desired contradiction.

Now we state a simple characterisation of the admissible constant controls.

Proposition 36. *Let $k_0, c \geq 0$. Then*

i) $k(\cdot; k_0, F(k_0)) \equiv k_0$

ii) the function constantly equal to c is admissible at k_0 (which we write

$c \in \Lambda(k_0)$) *if, and only if*

$$c \in [0, F(k_0)] .$$

In particular the null function is admissible at any initial state $k_0 \geq 0$.

Proof. i) By the uniqueness of the orbit.

ii)($\Longleftarrow$) In the first place, observe that $F(k_0) \in \Lambda(k_0)$, by i). In the second place, assume $c \in [0, F(k_0))$ and set $k := k(\cdot; k_0, c)$. Hence

$$\dot{k}(0) = F(k_0) - c > 0$$

which means, by the continuity of $\dot{k}$, that we can find $\delta > 0$ such that k is strictly increasing in $[0, \delta]$. In particular $\dot{k}(\delta) = F(k(\delta)) - c > F(k_0) - c$ because F is strictly increasing too. By the fact that $\dot{k}(\delta) > 0$ we see that there exists $\hat{\delta} > \delta$ such that k is strictly increasing in $\left[0, \hat{\delta}\right]$ - and so on. Hence k is strictly increasing over $[0, +\infty)$ and in particular $k \geq 0$. This shows that $c \in \Lambda(k_0)$.

($\Longrightarrow$) Suppose that $c > F(k_0)$ and set again $k := k(\cdot; k_0, c)$. Then

$$\dot{k}(0) = F(k_0) - c < 0$$

so that we can find $\delta > 0$ such that k is strictly decreasing over $[0, \delta]$, and $\dot{k}(\delta) = F(k(\delta)) - c < F(k_0) - c < 0$. Hence one can arbitrarily extend the neighbourhood of 0 in which $\dot{k}$ is strictly less than the strictly negative constant $F(k_0) - c$, which implies that

$$\lim_{t \to +\infty} k(t) = -\infty.$$

Hence k cannot be everywhere-positive and $c \notin \Lambda(k_0)$. $\square$

Corollary 37. *The set sequence* $(\Lambda(k))_{k\geq 0}$ *is strictly increasing, that is:*

$$\Lambda(k_0) \subsetneq \Lambda(k_1)$$

for every $0 \leq k_0 < k_1$.

Proof. For every $c \in \Lambda(k_0)$, $k(\cdot; k_0, c) \leq k(\cdot; k_1, c)$ for Corollary 28, which implies the second orbit being positive, and so $c \in \Lambda(k_1)$.
On the other hand, by Proposition 36 and by the strict monotonicity of F, the constant control $\hat{c} \equiv F\left(\hat{k}\right)$ belongs to $\Lambda(k_1) \setminus \Lambda(k_0)$ for any $\hat{k} \in (k_0, k_1]$. □

11 Basic properties of the value function

Now we deal with the first problem one has to solve in order to develop the theory: the good definition of the value function. We start setting a result which is analogous to the one we cleared up in Remark 31, and which also follows from a certain sublinearity property of the production function F.
Remark 38. Set $M_0, \hat{M} \geq 0$ such that:

$$\begin{aligned}&\forall x \geq M_0 : F(x) \leq (L + \epsilon_0)\, x \\ &\hat{M} := \max_{[0,M_0]} F.\end{aligned}$$

(which is possible because $\lim_{x\to+\infty} \frac{F(x)}{x} = L$). Hence, for every $x \geq 0$:

$$F(x) \leq (L + \epsilon_0)\, x + \hat{M}$$

Remark 39. Since u is a concave function satisfying $u(0) = 0$, u is subadditive over $[0, +\infty)$ and satisfies:

$$\forall x > 0 : \forall K > 1 : u(Kx) \leq K u(x)$$

Lemma 40. *Let $k_0 \geq 0$, and $c \in \Lambda(k_0)$. Hence, setting $M(k_0) := 1 + \max\left\{(L+\epsilon_0)k_0, \hat{M}\right\}$:*

$$\begin{aligned} &i) \quad \forall t \geq 0 : \int_0^t c(s)\,ds \leq tM(k_0)\left[1 + e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{L+\epsilon_0} \\ &ii) \quad \lim_{t\to+\infty} e^{-\rho t}\int_0^t u(c(s))\,ds = 0 \\ &iii) \quad U(c;k_0) = \rho\int_0^{+\infty} e^{-\rho t}\int_0^t u(c(s))\,ds dt \leq \gamma(k_0) \end{aligned}$$

where $\gamma(k_0)$ is a finite quantity depending on k_0 and on the problem's data.

Proof. i) Set $\kappa := k(\cdot;k_0,c)$ and $M(k_0)$ as in the hypothesis. Observe that, by Remark 38, for every $x \geq 0$:

$$F(x) \leq (L+\epsilon_0)x + M(k_0).$$

Fix $t \geq 0$; by the state equation, we have for any $s \in [0,t]$

$$\kappa(s) \leq k_0 + sM(k_0) + (L+\epsilon_0)\int_0^s \kappa(\tau)\,\mathrm{d}\tau$$

which implies by Lemma 2:

$$\kappa(s) \leq [k_0 + sM(k_0)]\,e^{(L+\epsilon_0)s} \quad \forall s \in [0,t]$$

So

$$\begin{aligned}\int_0^t (L+\epsilon_0)\,\kappa(s)\,\mathrm{d}s &\le k_0(L+\epsilon_0)\int_0^t e^{(L+\epsilon_0)s}\mathrm{d}s + M(k_0)(L+\epsilon_0)\int_0^t s\cdot e^{(L+\epsilon_0)s}\mathrm{d}s\\ &= k_0e^{(L+\epsilon_0)t} - k_0 + tM(k_0)\,e^{(L+\epsilon_0)t} - \frac{M(k_0)}{(L+\epsilon_0)}e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L+\epsilon_0)}\\ &= tM(k_0)\,e^{(L+\epsilon_0)t} + \left[k_0 - \frac{M(k_0)}{(L+\epsilon_0)}\right]e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L+\epsilon_0)} - k_0\\ &\le tM(k_0)\,e^{(L+\epsilon_0)t} + \frac{M(k_0)}{(L+\epsilon_0)} - k_0\end{aligned}$$

Hence, again by the state equation, for every $t \ge 0$:

$$\begin{aligned}\int_0^t c(s)\,\mathrm{d}s &= k_0 - \kappa(t) + \int_0^t F(\kappa(s))\,\mathrm{d}s\\ &\le k_0 + tM(k_0) + \int_0^t (L+\epsilon_0)\,\kappa(s)\,\mathrm{d}s\\ &\le tM(k_0)\left[1+e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{(L+\epsilon_0)}.\end{aligned}$$

which proves the first assertion.

ii) In the second place, it follows by Jensen inequality, the monotonicity of u and Remark 39, that for every $t \ge 0$:

$$\begin{aligned}0 \le e^{-\rho t}\int_0^t u(c(s))\,\mathrm{d}s &\le te^{-\rho t}u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right)\\ &\le te^{-\rho t}u\left(M(k_0)\left[1+e^{(L+\epsilon_0)t}\right] + \frac{M(k_0)}{t(L+\epsilon_0)}\right)\\ &\le te^{-\rho t}\left\{u(M(k_0)) + M(k_0)\,u\left(e^{(L+\epsilon_0)t}\right) + \right.\\ &\qquad \left. u\left(\frac{M(k_0)}{t(L+\epsilon_0)}\right)\right\};\end{aligned}$$

observe that this quantity tends to 0 as $t \to +\infty$, particulary by the last condition assumed in (33) about u; so also the second claim is proven. Finally, integrating by parts, and using ii)

$$\begin{aligned}
U(c;k_0) &= \int_0^{+\infty} e^{-\rho t} u(c(t))\,\mathrm{d}t \\
&= \lim_{T\to+\infty}\left\{e^{-\rho T}\int_0^T u(c(s))\,\mathrm{d}s + \rho\int_0^T e^{-\rho t}\int_0^t u(c(s))\,\mathrm{d}s\mathrm{d}t\right\} \\
&= \rho\int_0^{+\infty} e^{-\rho t}\int_0^t u(c(s))\,\mathrm{d}s\mathrm{d}t \\
&\leq \rho\int_0^{+\infty} te^{-\rho t}\left\{u(M(k_0)) + M(k_0)\,u\left(e^{(L+\epsilon_0)t}\right) + \right. \\
&\quad \left. u\left(\frac{M(k_0)}{t(L+\epsilon_0)}\right)\right\}\mathrm{d}t \\
&\leq \rho u(M(k_0))\int_0^{+\infty} te^{-\rho t}\mathrm{d}t + \rho M(k_0)\int_0^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\quad +\rho u\left(\frac{M(k_0)}{L+\epsilon_0}\right)\left\{\int_0^1 e^{-\rho t}\mathrm{d}t + \int_1^{+\infty} te^{-\rho t}\mathrm{d}t\right\}
\end{aligned}$$

Now it is sufficient to observe that by Remark 25 this upper bound is finite and set it equal to $\gamma(k_0)$. □

Corollary 41. *The value function $V:[0,+\infty)\to\mathbb{R}$ is well-definite; that is, for every $k_0\geq 0$, $V(k_0)<+\infty$.*

Proof. Take $k_0\geq 0$ and set $\gamma(k_0)$ as in Lemma 40. Hence:

$$V(k_0) = \sup_{c\in\Lambda(k_0)} U(c;k_0) \leq \gamma(k_0) < +\infty.$$

□

Theorem 42 (**Asimptotic properties of the value function**). *The value function* $V : [0, +\infty) \to \mathbb{R}$ *satisfies:*

$$\begin{aligned} i) \quad & \lim_{k \to +\infty} V(k) = +\infty \\ ii) \quad & \lim_{k \to +\infty} \frac{V(k)}{k} = 0 \\ iii) \quad & \lim_{k \to 0} V(k) = V(0) = 0 \end{aligned}$$

Proof. i) For every $k_0 \geq 0$ the constant control $F(k_0)$ is admissible at k_0 by Proposition 36; hence

$$V(k_0) \geq U(F(k_0); k_0) = \frac{u(F(k_0))}{\rho} \to +\infty$$

as $k_0 \to +\infty$, by the assumptions over u and F.

ii) Set $\hat{M} > 0$ as in Remark 38 and $k_0 > 0$ such that:

$$k_0 > \frac{1}{L + \epsilon_0} \hat{M} \tag{42}$$

Hence, for every $x > 0$:

$$F(x) \leq (L + \epsilon_0)(x + k_0) \tag{43}$$

By reasons that will be clear later, suppose also that:

$$k_0 > \frac{1}{L + \epsilon_0}$$

Observe that the proof of Lemma 40, i) does not require $M(k_0) \geq 1$, but only $M(k_0) \geq \hat{M}$; hence (42) and (43) imply that the property in Lemma

40, i) holds for $M(k_0) = k_0(L+\epsilon_0)$ - which means that:

$$\forall t \geq 0 : \int_0^t c(s)\,\mathrm{d}s \leq k_0 + tk_0(L+\epsilon_0)\left[1+e^{(L+\epsilon_0)t}\right]. \tag{44}$$

In particular

$$\forall t \geq 1 : \frac{\int_0^t c(s)\,\mathrm{d}s}{t} \leq k_0 + k_0(L+\epsilon_0) + k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}. \tag{45}$$

Now set

$$J_c(\alpha,\beta) := \int_\alpha^\beta te^{-\rho t}u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right)\mathrm{d}t \tag{46}$$

and fix $N > 0$.

We provide three different estimates, over $J_c(0,1)$, $J_c(1,N)$ and $J_c(N,+\infty)$, using Remark 39.

In the first place, we have by (44):

$$\begin{aligned}
J_c(0,1) &\leq \int_0^1 te^{-\rho t}\frac{1}{t}u\left(\int_0^1 c(s)\,\mathrm{d}s\right)\mathrm{d}t \\
&\leq u\left[k_0\left(1+(L+\epsilon_0)\left(1+e^{(L+\epsilon_0)}\right)\right)\right]\frac{1-e^{-\rho}}{\rho} \\
&\leq u(k_0)\frac{1-e^{-\rho}}{\rho}\left[1+(L+\epsilon_0)\left(1+e^{(L+\epsilon_0)}\right)\right].
\end{aligned}$$

Moreover, by (45):

$$\begin{aligned}
J_c(1,N) &\leq \int_1^N te^{-\rho t}u\left(k_0+k_0(L+\epsilon_0)+k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\leq u(k_0+k_0(L+\epsilon_0))\int_1^N te^{-\rho t}\mathrm{d}t+u(k_0(L+\epsilon_0))\int_1^N te^{-\rho t}e^{(L+\epsilon_0)t}\mathrm{d}t \\
&\leq u[k_0(1+L+\epsilon_0)]\left(1+e^{(L+\epsilon_0)N}\right)\int_1^N te^{-\rho t}\mathrm{d}t \\
&= u[k_0(1+L+\epsilon_0)]\left(1+e^{(L+\epsilon_0)N}\right)\left(\frac{e^{-\rho}-Ne^{-\rho N}}{\rho}-\frac{e^{-\rho N}-e^{-\rho}}{\rho^2}\right) \\
&\leq u(k_0)(1+L+\epsilon_0)\left(1+e^{(L+\epsilon_0)N}\right)\left(\frac{e^{-\rho}}{\rho}+\frac{e^{-\rho}}{\rho^2}\right).
\end{aligned}$$

Finally, remembering that $k_0(L+\epsilon_0)>1$,

$$\begin{aligned}
J_c(N,+\infty) &\leq \int_N^{+\infty} te^{-\rho t}u\left(k_0+k_0(L+\epsilon_0)+k_0(L+\epsilon_0)e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&\leq u(k_0+k_0(L+\epsilon_0))\int_N^{+\infty} te^{-\rho t}\mathrm{d}t+k_0(L+\epsilon_0)\int_N^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t \\
&= u[k_0(1+L+\epsilon_0)]\frac{e^{-\rho N}(\rho N+1)}{\rho^2}+ \\
&\quad +k_0(L+\epsilon_0)\int_N^{+\infty} e^{-\frac{\epsilon_0}{2}t}\omega(t)\,\mathrm{d}t,
\end{aligned}$$

where ω is the infinitesimal function we defined in Remark 25. Hence, if N is choosen such that $\omega(t)\leq 1$ for $t\geq N$ (so that the choice of N depends only on L, ϵ_0, ρ), then we obtain:

$$\begin{aligned}
J_c(N,+\infty) &\leq u[k_0(1+L+\epsilon_0)]\frac{e^{-\rho N}(\rho N+1)}{\rho^2}+k_0(L+\epsilon_0)\int_N^{+\infty} e^{-\frac{\epsilon_0}{2}t}\mathrm{d}t \\
&\leq u(k_0)(1+L+\epsilon_0)\frac{e^{-\rho N}(\rho N+1)}{\rho^2}+\frac{2(L+\epsilon_0)}{\epsilon_0}k_0e^{-\frac{\epsilon_0}{2}N}.
\end{aligned}$$

Now we show that

$$\lim_{k\to+\infty}\frac{V(k)}{k}=0$$

Fix $\eta>0$, and take $N_\eta>0$ such that

$$\rho\frac{2(L+\epsilon_0)}{\epsilon_0}e^{-\frac{\epsilon_0}{2}N_\eta}<\eta.$$

We can assume that $\omega(t)\le 1$ for $t\ge N_\eta$ because the function

$$X\to\rho\frac{2(L+\epsilon_0)}{\epsilon_0}e^{-\frac{\epsilon_0}{2}X}$$

is decreasing; observe that this N_η still depends only on L, ϵ_0, ρ.

Hence for k_0 satisfying:

$$k_0>\max\left\{\frac{1}{L+\epsilon_0}\hat{M},\frac{1}{L+\epsilon_0}\right\}$$

and for *every* $c\in\Lambda(k_0)$, we have by (46) and Lemma 40, iii):

$$\begin{aligned}
U(c;k_0) &= \rho\int_0^{+\infty}e^{-\rho t}\int_0^t u(c(t))\,\mathrm{d}t\\
&\le \rho J_c(0,1)+\rho J_c(1,N_\eta)+\rho J_c(N_\eta,+\infty)\\
&\le u(k_0)\left(1-e^{-\rho}\right)\left[1+(L+\epsilon_0)\left(e^{(L+\epsilon_0)}+1\right)\right]+\\
&\quad u(k_0)(1+L+\epsilon_0)\left(1+e^{(L+\epsilon_0)N_\eta}\right)\left(e^{-\rho}+\frac{e^{-\rho}}{\rho}\right)+\\
&\quad u(k_0)(1+L+\epsilon_0)\frac{e^{-\rho N_\eta}(\rho N_\eta+1)}{\rho}+k_0\eta
\end{aligned}$$

Now observe that:

$$\lim_{k_0\to+\infty}\frac{u(k_0)}{k_0}=\lim_{k_0\to+\infty}u'(k_0)=0.$$

Hence for k_0 sufficiently large (say $k_0 > k^*$):

$$\begin{aligned}\frac{u(k_0)}{k_0} &< \eta\Bigg\{\left(1-e^{-\rho}\right)\left[1+(L+\epsilon_0)\left(e^{(L+\epsilon_0)}+1\right)\right]+ \\ &\quad (1+L+\epsilon_0)\left(1+e^{(L+\epsilon_0)N_\eta}\right)\left(e^{-\rho}+\frac{e^{-\rho}}{\rho}\right)+ \\ &\quad (1+L+\epsilon_0)\frac{e^{-\rho N_\eta}(\rho N_\eta+1)}{\rho}\Bigg\}^{-1}\end{aligned}$$

Observe that this is possible because the expression into the brackets does not depend on k_0. In fact, it depends on η and on the problem's data L, ϵ_0, ρ - and so does k^*.

This implies, for *every* $c \in \Lambda(k_0)$:

$$U(c;k_0) \leq 2k_0\eta$$

which gives, taking the sup over $\Lambda(k_0)$:

$$V(k_0) \leq 2k_0\eta.$$

Hence the assertion is proven, because the previous inequality holds for every

$$k_0 > \max\left\{\frac{1}{L+\epsilon_0}\hat{M}, \frac{1}{L+\epsilon_0}, k^*\right\}$$

which is a threshold depending only on η and on the problem's data.

iii) In the first place, we prove that

$$V(0) = 0.$$

Let $c \in \Lambda(0)$; by definition, $c \geq 0$ so that

$$\forall t \geq 0 : \dot{k}(t;0,c) \leq F(k(t;0,c))$$

Observe that F is precisely the function which defines the dynamics of $k(\cdot;0,0)$, hence by Corollary 28:

$$\forall t \geq 0 : k(t;0,c) \leq k(t;0,0) = 0$$

where the last equality holds by Lemma 36, i).

Hence $k(\cdot;0,c) \equiv 0$ which together with $F(0) = 0$ implies $c \equiv 0$. So $\Lambda(0) = \{0\}$, which implies

$$V(0) = U(0;0) = \int_0^{+\infty} e^{-\rho t} u(0)\,\mathrm{d}t = 0$$

Now we show that

$$\lim_{k \to 0} V(k) = 0.$$

In this case we have to study the behaviour of $V(k_0)$ when $k_0 \to 0$, so we use the sublinearity of $F(x)$ for $x \to +\infty$ and the concavity of F near 0.

As a first step, we construct a linear function which is always above F with these two tools. Indeed we show that there is $m > 0$ such that the function

$$G(x) := \begin{cases} mx & \text{if } x \in \left[0, \bar{k}\right] \\ (L + \epsilon_0)\left(x - \bar{k}\right) + m\bar{k} & \text{if } x \geq \bar{k} \end{cases}$$

satisfies

$$\forall x \geq 0 : F(x) \leq G(x). \tag{47}$$

If $F'\left(\bar{k}\right) \leq L + \epsilon_0$ then it is enough to take

$$m > \max\left\{F'(0), F'\left(\bar{k}\right), \frac{F\left(\bar{k}\right)}{\bar{k}}\right\}$$

If $F'\left(\bar{k}\right) > L + \epsilon_0$ then observe that for every $x \geq \bar{k}$:

$$\frac{F(x) - F\left(\bar{k}\right)}{x - \bar{k}} \leq F'(x) \to L, \text{ for } x \to +\infty.$$

Hence there exists $M \geq \bar{k}$ such that, for every $x \geq M$,

$$F(x) \leq F\left(\bar{k}\right) + (L + \epsilon_0)\left(x - \bar{k}\right)$$

which implies that for every $x \geq \bar{k}$:

$$F(x) \leq (L + \epsilon_0)\left(x - \bar{k}\right) + F\left(\bar{k}\right) + \max_{\left[\bar{k}, M\right]} F.$$

Hence we replace the third condition over m with

$$m\bar{k} > F\left(\bar{k}\right) + \max_{\left[\bar{k}, M\right]} F.$$

Observe that condition $m > F'\left(\bar{k}\right)$ is necessary to ensure that $mx > F(x)$ for $x \in \left[\underline{k}, \bar{k}\right]$ (Lagrange's thorem proves that it is sufficient).

Suppose also, for reasons that will be clear later, that

$$m > 1. \tag{48}$$

Now take $k_0 > 0$, $c \in \Lambda(k_0)$ and consider the function $h : [0, +\infty) \to \mathbb{R}$

which is the unique solution to the Cauchy's problem

$$\begin{cases} h(0) = k_0 \\ \dot{h}(t) = G(h(t)) \quad t \geq 0 \end{cases}$$

Hence, by (47) and Corollary 28:

$$k := k(\cdot; k_0, c) \leq h.$$

So, setting

$$\bar{t} := \frac{1}{m} \log\left(\frac{\bar{k}}{k_0}\right)$$
$$\hat{k} := \bar{k}(m - L - \epsilon_0)$$

we get that, for every $t \in [0, \bar{t}]$:

$$h(t) = k_0 e^{mt}$$

and, for every $t \geq \bar{t}$:

$$\begin{aligned} h(t) &= e^{(L+\epsilon_0)t} \int_{\bar{t}}^{t} e^{-(L+\epsilon_0)s} \hat{k} \mathrm{d}s + \bar{k} e^{-(L+\epsilon_0)\bar{t}} \\ &= \frac{\hat{k} e^{-(L+\epsilon_0)\bar{t}}}{L+\epsilon_0} e^{(L+\epsilon_0)t} + \bar{k} e^{-(L+\epsilon_0)\bar{t}} - \frac{\hat{k}}{L+\epsilon_0} \\ &=: \omega_0(k_0) e^{(L+\epsilon_0)t} + \omega_1(k_0) - \frac{\hat{k}}{L+\epsilon_0} \end{aligned}$$

where

$$\begin{aligned}
\omega_0(k_0) &= \frac{\hat{k}}{L+\epsilon_0} e^{-(L+\epsilon_0)\bar{t}} = \frac{\hat{k}}{L+\epsilon_0} \left(\frac{k_0}{\bar{k}}\right)^{\frac{(L+\epsilon_0)}{m}} \\
\omega_1(k_0) &= \bar{k} e^{-(L+\epsilon_0)\bar{t}} = \bar{k} \left(\frac{k_0}{\bar{k}}\right)^{\frac{(L+\epsilon_0)}{m}}
\end{aligned}$$

Hence, by

$$\forall t \geq 0 : k(t) - k_0 = \int_0^t F(k(s))\,\mathrm{d}s - \int_0^t c(s)\,\mathrm{d}s$$

it follows that, for every $t \in [0, \bar{t}]$, remembering that h is increasing so that $\forall s \leq t : \ h(s) \leq \bar{k}$:

$$\begin{aligned}
\int_0^t c(s)\,\mathrm{d}s &\leq k_0 + \int_0^t F(k(s))\,\mathrm{d}s \leq k_0 + \int_0^t G(h(s))\,\mathrm{d}s \qquad (49) \\
&= k_0 + \int_0^t m k_0 e^{ms} \mathrm{d}s = k_0 e^{mt}
\end{aligned}$$

For every $t > \bar{t}$:

$$\begin{aligned}
\int_0^t c(s)\,\mathrm{d}s &\leq k_0 + \int_0^{\bar{t}} G(h(s))\,\mathrm{d}s + \int_{\bar{t}}^t G(h(s))\,\mathrm{d}s \\
&\leq k_0 e^{m\bar{t}} + \int_{\bar{t}}^t \left\{(L+\epsilon_0) h(s) + \hat{k}\right\} \mathrm{d}s \\
&\leq \bar{k} + (t-\bar{t})\hat{k} + (L+\epsilon_0) \int_{\bar{t}}^t \left\{\omega_0(k_0) e^{(L+\epsilon_0)s} + \omega_1(k_0) - \frac{\hat{k}}{L+\epsilon_0}\right\} \mathrm{d}s \\
&\leq \bar{k} + \omega_0(k_0) \left[e^{(L+\epsilon_0)t} - e^{(L+\epsilon_0)\bar{t}}\right] + (L+\epsilon_0)(t-\bar{t})\,\omega_1(k_0) \\
&\leq \bar{k} + \omega_0(k_0) e^{(L+\epsilon_0)t} - \frac{\hat{k}}{L+\epsilon_0} + (L+\epsilon_0)(t-\bar{t})\,\omega_1(k_0) \qquad (50)
\end{aligned}$$

where we have used $h(s) \geq \bar{k}$ for $s \in (\bar{t}, t)$ and the fact that $k_0 e^{m\bar{t}} = \bar{k}$.

Now observe that

$$\begin{aligned}
&\lim_{k_0 \to 0} \bar{t} = \lim_{k_0 \to 0} \frac{1}{m} \log\left(\frac{\bar{k}}{k_0}\right) = +\infty \\
&\lim_{k_0 \to 0} \omega_0(k_0) = \lim_{k_0 \to 0} \omega_1(k_0) = 0.
\end{aligned} \tag{51}$$

Hence if k_0 is small enough (say $k_0 < k^*$), we may assume

$$\begin{aligned}
&\bar{t} > 1 \\
&\omega_0(k_0) \le 1
\end{aligned}$$

so that (50) implies, for every $t > \bar{t}$:

$$\begin{aligned}
\frac{\int_0^t c(s)\,\mathrm{d}s}{t} &\le \bar{k} + \omega_0(k_0)\, e^{(L+\epsilon_0)t} + (L+\epsilon_0)\frac{(t-\bar{t})}{t}\omega_1(k_0) \\
&\le \bar{k} + \omega_0(k_0)\, e^{(L+\epsilon_0)t} + (L+\epsilon_0)\,\omega_1(k_0)
\end{aligned} \tag{52}$$

Hence, by Lemma 40, iii) , by Remark 39, and by (49), (52), the following inequality holds for every $k_0 < k^*$ and every $c \in \Lambda(k_0)$:

$$\begin{aligned}
U(c; k_0) &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u(c(s))\,\mathrm{d}s\mathrm{d}t \le \rho \int_0^{+\infty} t e^{-\rho t} u\left(\frac{\int_0^t c(s)\,\mathrm{d}s}{t}\right)\mathrm{d}t \\
&\le \rho \int_0^1 e^{-\rho t} u\left(\int_0^t c(s)\,\mathrm{d}s\right)\mathrm{d}t + \rho \int_1^{\bar{t}} t e^{-\rho t} u\left(\frac{k_0 e^{mt}}{t}\right)\mathrm{d}t + \\
&\quad + \rho \int_{\bar{t}}^{+\infty} t e^{-\rho t} u\left(\bar{k} + \omega_0(k_0)\, e^{(L+\epsilon_0)t} + (L+\epsilon_0)\,\omega_1(k_0)\right)\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\leq \rho\int_0^1 e^{-\rho t}u\left(k_0e^{mt}\right)\mathrm{d}t+\rho u\left(\frac{k_0e^{m\bar{t}}}{\bar{t}}\right)\int_1^{\bar{t}} te^{-\rho t}\mathrm{d}t+\\
&\quad+\rho u\left(\bar{k}\right)\int_{\bar{t}}^{+\infty} te^{-\rho t}\mathrm{d}t+\rho\int_{\bar{t}}^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t+\\
&\quad+\rho u\left(\left(L+\epsilon_0\right)\omega_1\left(k_0\right)\right)\int_{\bar{t}}^{+\infty} te^{-\rho t}\mathrm{d}t\\
&\leq \rho u\left(k_0e^m\right)\int_0^1 e^{-\rho t}\mathrm{d}t+\rho u\left(\frac{\bar{k}}{\bar{t}}\right)\frac{e^{-\rho}\left(1+\rho\right)}{\rho^2}+\\
&\quad+\rho\left\{u\left(\bar{k}\right)+u\left(\left(L+\epsilon_0\right)\omega_1\left(k_0\right)\right)\right\}\int_{\bar{t}}^{+\infty} te^{-\rho t}\mathrm{d}t+\\
&\quad+\rho\int_{\bar{t}}^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t
\end{aligned}$$

where we used also the fact that the function

$$t\to\frac{e^{mt}}{t}$$

is increasing for $t>1$, by condition (48). By Remark 25, conditions (51) and the fact that $\lim_{x\to 0}u\left(x\right)=0$, we obtain:

$$\omega_1\left(k_0\right),\ u\left(\frac{\bar{k}}{\bar{t}}\right),\ \int_{\bar{t}}^{+\infty} te^{-\rho t}\mathrm{d}t,\ \int_{\bar{t}}^{+\infty} te^{-\rho t}u\left(e^{(L+\epsilon_0)t}\right)\mathrm{d}t\to 0$$

as $k_0\to 0$; moreover, these quantities do not depend on c. Hence for any $\epsilon>0$, there exists $\delta\in\left(0,k^*\right]$ such that for every $k_0\in\left(0,\delta\right)$ and for *every* $c\in\Lambda\left(k_0\right)$:

$$U\left(c;k_0\right)\leq\epsilon,$$

which implies, taking the sup over $\Lambda\left(k_0\right)$,

$$V\left(k_0\right)\leq\epsilon$$

□

12 Existence of the optimal control

In this section we deal with the central topic of any problem consiting of a controlled differential system: the existence of an optimal control, which means in our case that, for any $k_0 \geq 0$, we can find $c^* \in \Lambda(k_0)$ such that

$$U(c^*; k_0) = \sup_{c \in \Lambda(k_0)} U(c; k_0) = V(k_0).$$

We preliminary observe that the peculiar features of our problem, particularly the absence of any boundedness conditions over the admissible controls, force us to make use of this tool in proving certain properties of the value functions which usually do not require such a settlement - and which we posticipate for this reason.

First observe that by Theorem 42, iii) if we set $c_0 :\equiv 0$, then $U(c_0, 0) = 0 = V(0)$ (because $u(0) = 0$); hence c_0 is optimal at 0.

Let $k_0 > 0$; this will be the initial state which we will refer to during the whole section - hence the meaning of this symbol will not change in this context. We split the construction in various steps; but first it is necessary to settle an important notion.

Definition 43. Let $T > 0$, $(f_n)_{n \in \mathbb{N}}, f$ functions in $L^1([0,T], \mathbb{R})$. We say that $(f_n)_{n \in \mathbb{N}}$ weakly converges to f in $L^1([0,T], \mathbb{R})$, and we write

$$f_n \rightharpoonup f \text{ in } L^1([0,T], \mathbb{R})$$

if, and only if, for every $g \in L^\infty([0,T], \mathbb{R})$:

$$\lim_{n \to \infty} \int_0^T g(s) f_n(s) \, \mathrm{d}s = \int_0^T g(s) f(s) \, \mathrm{d}s$$

Remark 44. Suppose that $(f_n)_{n \in \mathbb{N}}, f$ are functions in $\mathcal{L}^1_{loc}([0,+\infty), \mathbb{R})$ such that for *every* $N \in \mathbb{N}$ $f_n \rightharpoonup f$ in $L^1([0,N], \mathbb{R})$. If $T > 0$, $T \in \mathbb{R}$, then by the latter definition we have, for $g \in L^{\infty}([0,T],\mathbb{R})$:

$$\begin{aligned}\int_0^T g(s) f_n(s)\,\mathrm{d}s &= \int_0^{[T]+1} \chi_{[0,T]} g(s) f_n(s)\,\mathrm{d}s \\ &\to \int_0^{[T]+1} \chi_{[0,T]} g(s) f(s)\,\mathrm{d}s \\ &= \int_0^T g(s) f(s)\,\mathrm{d}s.\end{aligned}$$

Hence $f_n \rightharpoonup f$ in $L^1([0,T],\mathbb{R})$, for every $T>0, T\in\mathbb{R}$.

Step 1. The first step is to find a maximizing sequence of controls which are admissible at k_0 and a function $\gamma \in \mathcal{L}^1_{loc}([0,+\infty),\mathbb{R})$, such that the sequence weakly converges to γ in $L^1([0,T],\mathbb{R})$, for every $T>0$.

By definition of supremum, we can find a maximizing sequence; that is to say, there exist a sequence $(c_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ of admissible controls satisfying:

$$\lim_{n\to+\infty} U(c_n;k_0) = V(k_0).$$

In order to apply the tools we set up at the beginning of the chapter, we need the following result.

Lemma 45. *Let $T\in\mathbb{N}$ and $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^1_{loc}([0,+\infty),\mathbb{R})$, $M(T)>0$ such that*

$$\forall n \in \mathbb{N} : \|f_n\|_{\infty,[0,T]} \leq M(T).$$

Then there exist a subsequence $\left(\overline{f}_n\right)_{n\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a function $f \in L^1([0,T],\mathbb{R})$ such that

$$\overline{f}_n \rightharpoonup f \text{ in } L^1([0,T],\mathbb{R}).$$

Proof. For every $0 \leq t_0 < t_1 \leq T$:

$$\int_{t_0}^{t_1} |f_n(s)| \, \mathrm{d}s \leq \|f_n\|_{\infty,[0,T]} \cdot (t_1 - t_0) \leq M(T) \cdot (t_1 - t_0).$$

Hence, by the fact that the family $\{(t_0, t_1) \in \mathcal{P}([0,T]) / t_0, t_1 \in [0,T]\}$ generates the Borel σ- algebra in $[0,T]$, we deduce that the latter relation holds for every measurable set $E \subseteq [0,T]$; that is to say

$$\int_E |f_n(s)| \, \mathrm{d}s \leq M(T) \cdot \mu(E).$$

This implies that the densities $\{d_n / n \in \mathbb{N}\}$ given by:

$$\begin{array}{rcl} d_n : \mathcal{B}(\mathbb{R}) & \to & \mathbb{R} \\ E & \mapsto & \displaystyle\int_E f_n(s) \, \mathrm{d}s \end{array}$$

are absolutely equicontinuous. Indeed take $\epsilon > 0$; then for any measurable set $E \subseteq [0,T]$ such that $\mu(E) \leq \epsilon / M(T)$ we have:

$$\forall n \in \mathbb{N} : \left| \int_E f_n(s) \, \mathrm{d}s \right| \leq \int_E |f_n(s)| \, \mathrm{d}s \leq \epsilon.$$

So the thesis follows from the Dunford-Pettis criterion (see [7]). Observe that the third condition required by such theorem, that is to say, for any $\epsilon > 0$ there exists a compact set $K_\epsilon \subseteq [0,T]$ such that

$$\forall n \in \mathbb{N} : \int_{[0,T] \setminus K_\epsilon} f_n(s) \, \mathrm{d}s \leq \epsilon$$

is obviously satisfied. □

Now we apply Lemma 33 to $(c_n)_{n\in\mathbb{N}}$ in order to find a new sequence $(c_n^1)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$ such that, for every $n \in \mathbb{N}$:

$$\begin{aligned}
&U\left(c_n^1; k_0\right) \geq U\left(c_n; k_0\right) \\
&c_n^1 = c_n \wedge N\left(k_0, 1\right) \text{ a.e. in } [0,1].
\end{aligned}$$

In particular $(c_n^1)_{n\in\mathbb{N}} \subseteq \mathcal{L}^1_{loc}([0,+\infty), \mathbb{R})$ and $\|c_n^1\|_{\infty,[0,1]} \leq N(k_0, 1)$ for every $n \in \mathbb{N}$. Hence by Lemma 45, there exists a sequence $(\overline{c}_n^1)_{n\in\mathbb{N}}$ extracted from $(c_n^1)_{n\in\mathbb{N}}$ and a function $c^1 \in L^1([0,1], \mathbb{R})$ such that

$$\overline{c}_n^1 \rightharpoonup c^1 \text{ in } L^1([0,1], \mathbb{R}).$$

Now define, for every $n \in \mathbb{N}$:

$$c_n^2 := \left(\overline{c}_n^1\right)^2$$

where $\left(\overline{c}_n^1\right)^2$ is understood with the notation of Lemma 33.
Hence for every $n \in \mathbb{N}$:

$$\begin{aligned}
&U\left(c_n^2; k_0\right) \geq U\left(\overline{c}_n^1; k_0\right) \\
&c_n^2 = \overline{c}_n^1 \wedge N\left(k_0, 2\right) \text{ a.e. in } [0,2].
\end{aligned}$$

Again by Lemma 45, we can exhibit a subsequence $(\overline{c}_n^2)_{n\in\mathbb{N}}$ of $(c_n^2)_{n\in\mathbb{N}}$ and a function $c^2 \in L^1([0,2], \mathbb{R})$ such that

$$\overline{c}_n^2 \rightharpoonup c^2 \text{ in } L^1([0,2], \mathbb{R}).$$

Following this pattern we are able to give a recursive definition of a family $\left\{\left(\left(c_n^T\right)_{n\in\mathbb{N}}, \left(\overline{c}_n^T\right)_{n\in\mathbb{N}}, c^T\right) / T \in \mathbb{N}\right\}$ and $\{i(T,n) \in [0,+\infty) / T, n \in \mathbb{N}\}$ satisfying, for every $T, n \in \mathbb{N}$:

$$
\begin{aligned}
& c_n^T \in \Lambda(k_0), \ \overline{c}_n^T = c_{n+i(T,n)}^T \\
& U\left(c_n^{T+1}; k_0\right) \geq U\left(\overline{c}_n^T; k_0\right) \\
& c_n^{T+1} = \overline{c}_n^T \wedge N(k_0, T+1) \ \text{ a.e. in } \ [0, T+1] \\
& \overline{c}_n^T \rightharpoonup c^T \ \text{in } L^1([0,T], \mathbb{R})
\end{aligned} \tag{53}
$$

Now we show that, for every $T \in \mathbb{N}$,

$$c^{T+1} = c^T \text{ almost everywhere in } \ [0, T]. \tag{54}$$

Assume the notation "$\tilde{\forall} s \in A : P(s)$" with the meaning " for almost every $s \in A$, $P(s)$ holds" . Hence:

$$
\begin{aligned}
\tilde{\forall} s \in [0,T] : \overline{c}_n^{T+1}(s) &= c_{n+i(T,n)}^{T+1}(s) \\
&= \overline{c}_{n+i(T,n)}^T(s) \wedge N(k_0, T+1) \\
&= \overline{c}_{n+i(T,n)}^T(s)
\end{aligned}
$$

where the last equality holds because, by the penultimate condition in (53) and by the monotonicity of the function $N(k_0, \cdot)$, for any $p \in \mathbb{N}$:

$$\left\|\overline{c}_p^T\right\|_{\infty,[0,T]} = \left\|c_{p+i(T,p)}^T\right\|_{\infty,[0,T]} \leq N(k_0, T) \leq N(k_0, T+1).$$

Hence the assertion in (54) follows from the essential uniqueness of the weak limit in $L^1([0,T], \mathbb{R})$.

Now we want to set up a diagonal procedure in order to exhibit a sequence $(\gamma_n)_{n \in \mathbb{N}} \subseteq \Lambda(k_0)$ and a function $\gamma \in \mathcal{L}^1_{loc}([0, +\infty), \mathbb{R})$ such that

$$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T], \mathbb{R}) \quad \forall T > 0.$$

Definition 46. i) $\gamma : [0,+\infty) \to \mathbb{R}$ is the function

$$\gamma(t) := c^{[t]+1}(t) \quad \forall t \geq 0$$

ii) The sequence $(\gamma_n)_{n\in\mathbb{N}}$ is defined as follows:

$$\begin{cases} \gamma_1 := \overline{c}_1^1 \\ \forall n \geq 2: \text{ if } \gamma_n = c_{j(n)}^n \text{ then } \gamma_{n+1} = \overline{c}_m^{n+1}, \\ \text{where } m := \min\left\{k \in \mathbb{N}/k > j(n) \text{ and } c_k^{n+1} \in \left(\overline{c}_p^{n+1}\right)_{p\in\mathbb{N}}\right\} \end{cases}$$

This diagonal procedure is resumed by the following scheme, in which the elements of the (weakly) convergent subsequences $(\overline{c}_n^m)_{n\in\mathbb{N}}$, $m \geq 1$ are emphasized by the square brackets.

$$\begin{array}{cccccccccccccc} c_1^1 & c_2^1 & \ldots & c_h^1 & \ldots & \left[\mathbf{c^1_{j(1)}}\right] & \ldots & c_i^1 & \ldots & c_k^2 & \ldots & c_{j(2)}^1 & \ldots & c_{j(3)}^1 \\ c_1^2 & c_2^2 & \ldots & [c_h^2] & \ldots & c_{j(1)}^2 & \ldots & c_i^2 & \ldots & c_k^2 & \ldots & \left[\mathbf{c^2_{j(2)}}\right] & \ldots & c_{j(3)}^2 \\ c_1^3 & c_2^3 & \ldots & c_h^3 & \ldots & c_{j(1)}^3 & \ldots & [c_i^3] & \ldots & [c_k^3] & \ldots & c_{j(2)}^3 & \ldots & \left[\mathbf{c^3_{j(3)}}\right] \\ c_1^4 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

Remark 47. Let $T \in \mathbb{N}$. Condition (54) implies that $\gamma = c^T$ almost everywhere in $[0,T]$. Hence it follows from (53) that:

$$\overline{c}_n^T \rightharpoonup \gamma \text{ in } L^1([0,T],\mathbb{R}).$$

We have shown that $\left(\overline{c}_n^{T+1}\right)_{n\in\mathbb{N}}$, restricted to $[0,T]$, almost coincides with a subsequence of $\left(\overline{c}_n^T\right)_{n\in\mathbb{N}}$; we want to prove an analogous result in relation to $(\gamma_n)_{n\in\mathbb{N}}$.

We have $\gamma_1 := \overline{c}_1^1 = c_{j(1)}^1$ (with $j(1) = 1 + i(1,1)$) , so by Definition 46, there exists $m_2 > j(1)$ such that $\gamma_2 = \overline{c}_{m_2}^2 = c_{j(m_2)}^2$, where $j(m_2) := m_2 + i(2,m_2)$.

Hence

$$\begin{aligned}\tilde{\forall} s \in [0,1] : \gamma_2(s) &= \overline{c}^2_{m_2}(s) = c^2_{j(m_2)}(s) \\ &= \overline{c}^1_{j(m_2)}(s) \wedge N(k_0,2) = \overline{c}^1_{j(m_2)}(s)\end{aligned}$$

where the last equality again holds because by construction $\left\|\overline{c}^1_p\right\|_{\infty.[0,1]} \leq N(k_0,1) \leq N(k_0,2)$ for any $p \in \mathbb{N}$.

Moreover for some $m_3 > j(m_2)$, $\gamma_3 = \overline{c}^3_{m_3}$; setting $j(m_3) := m_3 + i(3,m_3)$ and $j(j(m_3)) := j(m_3) + i(2, j(m_3))$, we have:

$$\begin{aligned}\tilde{\forall} s \in [0,1] : \gamma_3(s) &= \overline{c}^3_{m_3}(s) = c^3_{m_3+i(3,m_3)}(s) \\ &= c^3_{j(m_3)}(s) = \overline{c}^2_{j(m_3)}(s) \wedge N(k_0,3) \\ &= c^2_{j(j(m_3))}(s) \wedge N(k_0,3) \\ &= \overline{c}^1_{j(j(m_3))}(s) \wedge N(k_0,2) \wedge N(k_0,3) \\ &= \overline{c}^1_{j(j(m_3))}(s)\end{aligned}$$

as $N(k_0,1) \leq N(k_0,2) \leq N(k_0,3)$, and

$$\begin{aligned}\tilde{\forall} s \in [0,2] : \gamma_3(s) &= c^3_{j(m_3)}(s) = \overline{c}^2_{j(m_3)}(s) \wedge N(k_0,3) \\ &= \overline{c}^2_{j(m_3)}\end{aligned}$$

Hence, by the fact that $1 < j(m_2) < j(j(m_3))$, we see that $(\gamma_1, \gamma_2, \gamma_3)$ coincides with a subsequence of $(\overline{c}^1_n)_{n\in\mathbb{N}}$almost everywhere in $[0,1]$; it follws from $j(m_3) > m_2$ that (γ_2, γ_3) coincides with a subsequence of $(\overline{c}^2_n)_{n\in\mathbb{N}}$ almost everywhere in $[0,2]$. Obviously this reasoning can be repeated to prove by induction the following

Proposition 48. *Let $(\gamma_n)_{n\in\mathbb{N}}$, γ as in Definition 46. Then $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$,$\gamma \in \mathcal{L}^1_{loc}([0,+\infty), \mathbb{R})$ and*

$$\lim_{n\to+\infty} U(\gamma_n; k_0) = V(k_0).$$

Moreover, for every $T \in \mathbb{N}$, $(\gamma_n)_{n \geq T}$ *coincides almost everywhere in* $[0, T]$ *with a subsequence of* $\left(\overline{c}_n^T\right)_{n \in \mathbb{N}}$. *Consequently*

$$\gamma_n \rightharpoonup \gamma \text{ in } L^1([0,T], \mathbb{R}) \quad \forall T > 0, T \in \mathbb{R},$$
$$\|\gamma_n\|_{\infty,[0,T]} \leq N(k_0, T) \quad \forall T, n \in \mathbb{N}.$$

Proof. By Remark 47, for every $T \in \mathbb{N}$, $\gamma = c^T$ almost everywhere in $[0, T]$; hence $\gamma \in L^1([0,T], \mathbb{R})$, which implies $\gamma \in \mathcal{L}^1_{loc}([0, +\infty), \mathbb{R})$ because T is generic.

By Definition 46, $\gamma_1 = c^1_{j(1)}$ for some $j(1) \geq 1$; hence by induction we have, for every $n \in \mathbb{N}$, $\gamma_n = \overline{c}^n_{j(n)}$ for some $j(n) \geq n$; in particular, by the first condition in (53), $\gamma_n \in \Lambda(k_0)$. With $n \to j(n)$ defined this way, set $p(n) := j(n) + i(n, j(n))$; so remembering the other conditions in (53):

$$\begin{aligned} |U(\gamma_n; k_0) - V(k_0)| &= V(k_0) - U(\gamma_n; k_0) = V(k_0) - U\left(\overline{c}^n_{j(n)}; k_0\right) \\ &= V(k_0) - U\left(c^n_{p(n)}; k_0\right) \leq V(k_0) - U\left(\overline{c}^{n-1}_{p(n)}; k_0\right) \\ &= V(k_0) - U\left(c^{n-1}_{p(n)+i(n-1,p(n))}; k_0\right) \\ &\leq \ldots \quad \leq V(k_0) - U\left(c^1_{q(n)}; k_0\right) \\ &\leq V(k_0) - U\left(c_{q(n)}; k_0\right) = \left|U\left(c_{q(n)}; k_0\right) - V(k_0)\right|, \end{aligned}$$

for some $q(n) \geq p(n) \geq n$. Hence the first assertion follows from the fact that $\lim_{k \to +\infty} U(c_k; k_0) = V(k_0)$.

Now fix $T \in \mathbb{N}$. The argument developed after Remark 47 inductively shows that there exists a sequence of natural numbers $n \to k_T(n)$ such that

$$\forall n \geq T : \tilde{\forall} s \in [0,T] : \gamma_n(s) = \overline{c}^T_{n+k_T(n)}(s).$$

This implies by Remark 47 that $\gamma_n \rightharpoonup \gamma$ in $L^1([0,T], \mathbb{R})$.

As this holds for every $T \in \mathbb{N}$, it is a consequence of Remark 44 that it must hold for every real number $T > 0$. The last condition obviously holds by construction and by (53). □

The first step is then accomplished.

Step 2. The next step is to show that γ is admissible at k_0. For this purpose, it is enough to prove the following

Proposition 49. *Let $T > 0$. Hence $\gamma \geq 0$ almost everywhere in $[0, T]$, and, for every $t \in [0, T]$, $k(t; k_0, \gamma) \geq 0$.*

Proof. It is well known that the weak convergence of $(\gamma_n)_{n\in\mathbb{N}}$ to γ in $L^1([0, T], \mathbb{R})$, ensured by Proposition 48, implies that there exists a set $A \subseteq [0, T]$ such that $\mu([0, T] \setminus A) = 0$ and for every $t \in A$:

$$\liminf_{n\to+\infty} \gamma_n(t) \leq \gamma(t)$$

Moreover, $(\gamma_n)_{n\in\mathbb{N}} \subseteq \Lambda(k_0)$, hence any γ_n is almost everywhere non-negative in $[0, T]$; that is to say, for every $n \in \mathbb{N}$ there exists a set $I_n \subseteq [0, T]$ such that $\mu([0, T] \setminus I_n) = 0$ and $\gamma_n(t) \geq 0$ for $t \in I_n$. Hence $\gamma(t) \geq 0$ for every $t \in \bigcap_{n\in\mathbb{N}} I_n \bigcap A$, that is to say $\gamma \geq 0$ almost everywhere in $[0, T]$, because

$$\mu\left([0, T] \setminus \left(\bigcap_{n\in\mathbb{N}} I_n \bigcap A\right)\right) = \mu\left(\bigcup_{n\in\mathbb{N}} ([0, T] \setminus I_n) \bigcup ([0, T] \setminus A)\right) = 0.$$

Set $\kappa := k(\cdot; k_0, \gamma)$ and $\kappa_n := k(\cdot; k_0, \gamma_n)$; we show that, for every $t \in [0, T]$:

$$\limsup_{n\to+\infty} \kappa_n(t) \leq \kappa(t).$$

Then the second assertion will follow from the fact that $\kappa_n \geq 0$ in $[0, T]$ for any $n \in \mathbb{N}$, by the admissibility of the γ_n's.

Fix $n \in \mathbb{N}$. Subtracting the state equation for κ from the state equation for

κ_n, we obtain, for every $t \in [0,T]$:

$$\begin{aligned}\dot{\kappa_n}(t) - \dot{\kappa}(t) &= F(\kappa_n(t)) - F(\kappa(t)) - [\gamma_n(t) - \gamma(t)] \\ &\leq \overline{M}[\kappa_n(t) - \kappa(t)] - [\gamma_n(t) - \gamma(t)]\end{aligned}$$

which implies

$$[\dot{\kappa_n}(t) - \dot{\kappa}(t)]\, e^{-\overline{M}t} - e^{-\overline{M}t}\overline{M}[\kappa_n(t) - \kappa(t)] \leq e^{-\overline{M}t}[\gamma(t) - \gamma_n(t)]$$

that is to say:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[[\kappa_n(t) - \kappa(t)]\, e^{-\overline{M}t}\right] \leq e^{-\overline{M}t}[\gamma(t) - \gamma_n(t)].$$

Hence, for every fixed $t \in [0,T]$:

$$\begin{aligned}\kappa_n(t) - \kappa(t) &\leq \int_0^t e^{\overline{M}(t-s)}[\gamma(s) - \gamma_n(s)]\,\mathrm{d}s \\ &= \int_0^T \chi_{[0,t]}(s)\, e^{\overline{M}(t-s)}[\gamma(s) - \gamma_n(s)]\,\mathrm{d}s.\end{aligned}$$

The function

$$s \to \chi_{[0,t]}(s)\, e^{\overline{M}(t-s)}$$

is bounded in $[0,T]$ (by 1 and $e^{\overline{M}t}$), hence we can apply the weak convergence $\gamma_n \rightharpoonup \gamma$ in $L^1([0,T],\mathbb{R})$ to deduce that the quantity at the right-hand member of the above inequality tends to 0 as $n \to +\infty$. Hence

$$\limsup_{n\to+\infty} \kappa_n(t) \leq \kappa(t).$$

□

As a consequence, γ is almost everywhere non-negative in $[0,+\infty)$ and $k(\cdot;k_0,\gamma)$ is everywhere non-negative in $[0,+\infty)$ - which precisely means that $\gamma \in$

$\Lambda(k_0)$. Hence the second step is also ended.

Step 3. Now it is time to define the control which is optimal at k_0. In order to do this, we need to extract a subsequence from $(\gamma_n)_{n\in\mathbb{N}}$ because the weak convergence to γ over the intervals could not be enough to ensure that $\lim_{n\to+\infty} U(\gamma_n; k_0) = U(\gamma; k_0)$; we will also need the admissibility of γ. By the last assertion stated in Proposition 48, and by the monotonicity of u, we have:

$$\|u(\gamma_n)\|_{\infty,[0,1]} \leq u(N(k_0,1)) \quad \forall n \in \mathbb{N}.$$

Hence by Lemma 45, there exists a function $f^1 \in L^1([0,1], \mathbb{R})$ and a sequence $(u(\gamma_{1,n}))_{n\in\mathbb{N}}$ extracted from $(u(\gamma_n))_{n\in\mathbb{N}}$, such that

$$u(\gamma_{1,n}) \rightharpoonup f^1 \text{ in } L^1([0,1], \mathbb{R}).$$

Again by Proposition 48 and the monotonicity of u,

$$\|u(\gamma_{1,n})\|_{\infty,[0,2]} \leq u(N(k_0,2)) \quad \forall n \in \mathbb{N}$$

which implies by Lemma 45 the existence of $f^2 \in L^1([0,2], \mathbb{R})$ and of a sequence $(u(\gamma_{2,n}))_{n\in\mathbb{N}}$ extracted from $(u(\gamma_{1,n}))_{n\in\mathbb{N}}$ such that

$$u(\gamma_{2,n}) \rightharpoonup f^2 \text{ in } L^1([0,2], \mathbb{R});$$

in particular $f^2 = f^1$ almost everywhere in $[0,1]$ by the essential uniqueness of the weak limit.

Going on this way we see that there exists a family $\left\{\left(u(\gamma_{T,n})_{n\in\mathbb{N}}, f^T\right) / T \in \mathbb{N}\right\}$

satisfying, for every $T \in \mathbb{N}$:

$$\begin{aligned}
&\|u(\gamma_{T,n})\|_{\infty,[0,T]} \leq u(N(k_0,T)) \quad \forall n \in \mathbb{N} \\
&(u(\gamma_{T+1,n}))_{n\in\mathbb{N}} \text{ is extracted from } (u(\gamma_{T,n}))_{n\in\mathbb{N}} \\
&f^{T+1} = f^T \text{ almost everywhere in } [0,T] \\
&u(\gamma_{T,n}) \rightharpoonup f^T \text{ in } L^1([0,T],\mathbb{R}).
\end{aligned}$$

Hence, for every $T \in \mathbb{N}$, the sequence $(u(\gamma_{n,n}))_{n\geq T}$ is extracted from $(u(\gamma_{T,n}))_{n\in\mathbb{N}}$. If we define $f(t) := f^{[t]+1}(t)$, then $f = f^T$ almost everywhere in $[0,T]$. So

$$u(\gamma_{n,n}) \rightharpoonup f \text{ in } L^1([0,T],\mathbb{R}) \quad \forall T > 0. \tag{55}$$

by construction and by Remark 44. This implies that

$$0 \leq \liminf_{n\to+\infty} u(\gamma_{n,n}(t)) \leq f(t)$$

for almost every $t \in \mathbb{R}$; hence the function $c^* : [0,+\infty) \to \mathbb{R}$ defined by

$$c^*(t) := \begin{cases} u^{-1}(f(t)) & \text{if } f(t) \geq 0 \\ 0 & \text{if } f(t) < 0 \end{cases}$$

is almost everywhere non-negative. Moreover, again by the properties of the weak convergence, for any $T \in \mathbb{N}$ and for almost every $t \in [0,T]$:

$$f(t) \leq \limsup_{n\to+\infty} u(\gamma_{n,n}(t)) \leq u(N(k_0,T)).$$

This implies, together with the fact that u^{-1} is increasing, that c^* is bounded above by $N(k_0,T)$ almost everywhere in $[0,T]$. In particular, $c^* \in L^1([0,T],\mathbb{R})$; as this is true for every $T \in \mathbb{N}$,

$$c^* \in L^1_{loc}([0,+\infty),\mathbb{R}). \tag{56}$$

To complete the proof of the admissibility of c^*, we show that $c^* \leq \gamma$ almost everywhere in $[0, +\infty)$.

Fix $T > 0$ and let $t_0 \in [0, T]$ be a Lebesgue point for both f and γ in $[0, T]$; then take $t_1 \in (t_0, T)$. By the concavity of u and by Jensen inequality:

$$\frac{\int_{t_0}^{t_1} u\left(\gamma_{n,n}\left(s\right)\right) \mathrm{d}s}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma_{n,n}\left(s\right) \mathrm{d}s}{t_1 - t_0}\right) \tag{57}$$

Observe that $\left(\gamma_{n,n}\right)_{n\geq 1}$ is a subsequence of $\left(\gamma_{1,n}\right)_{n\in\mathbb{N}}$, which is in its turn extracted from $\left(\gamma_n\right)_{n\in\mathbb{N}}$. Hence $\gamma_{n,n} \rightharpoonup \gamma$ in $L^1\left(\left[0, T\right], \mathbb{R}\right)$, which implies $\lim_{n\to+\infty} \int_{t_0}^{t_1} \gamma_{n,n}\left(s\right) \mathrm{d}s = \int_{t_0}^{t_1} \gamma\left(s\right) \mathrm{d}s$. So taking the limit for $n \to +\infty$ in (57), by the continuity of u and by (55), we have:

$$\frac{\int_{t_0}^{t_1} f\left(s\right) \mathrm{d}s}{t_1 - t_0} \leq u\left(\frac{\int_{t_0}^{t_1} \gamma\left(s\right) \mathrm{d}s}{t_1 - t_0}\right).$$

As t_0 is a Lebesgue point for both f and γ in $[0, T]$, we can take the limit for $t_1 \to t_0$ in the previous inequality and get $f\left(t_0\right) \leq u\left(\gamma\left(t_0\right)\right)$. Set $L_f, L_\gamma \subseteq [0, T]$ the sets of the Lebesgue points of f and γ, respectively. By the Lebesgue Point Theorem (see [5]), $\mu\left(\left[0, T\right] \setminus \left(L_f \cap L_\gamma \cap \{f \geq 0\}\right)\right) = 0$, so by the monotonicity of u^{-1} we deduce

$$c^* \leq \gamma \text{ almost everywhere in } \left[0, T\right].$$

Because T is generic, by Corollary 28 $k\left(t; k_0, c^*\right) \geq k\left(t; k_0, \gamma\right)$ for every $t \in \mathbb{R}$. Hence by the admissibility of γ at k_0, $k\left(\cdot; k_0, c^*\right) \geq 0$. This implies, together with (56) and $c^* \geq 0$ almost everywhere in $[0, +\infty)$,

$$c^* \in \Lambda\left(k_0\right).$$

Then by Proposition 48, by the fact that $\left(\gamma_{n,n}\right)_{n\in\mathbb{N}}$ is extracted from $\left(\gamma_n\right)_{n\in\mathbb{N}}$, by Lemma 40, iii) , by (55) and by Fatou's Lemma:

$$\begin{aligned} V\left(k_0\right) &= \lim_{n\to+\infty} U\left(\gamma_n;k_0\right) = \lim_{n\to+\infty} U\left(\gamma_{n,n};k_0\right) \\ &= \lim_{n\to+\infty} \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u\left(\gamma_{n,n}\left(s\right)\right) \mathrm{d}s\mathrm{d}t \\ &\leq \rho \int_0^{+\infty} e^{-\rho t} \limsup_{n\to+\infty} \int_0^t u\left(\gamma_{n,n}\left(s\right)\right) \mathrm{d}s\mathrm{d}t \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t f\left(s\right) \mathrm{d}s\mathrm{d}t \\ &= \rho \int_0^{+\infty} e^{-\rho t} \int_0^t u\left(c^*\left(s\right)\right) \mathrm{d}s\mathrm{d}t = U\left(c^*;k_0\right) \end{aligned}$$

which concludes the proof.

13 Further properties of the value function

Now it is possible to set some regularity properties of the value function, with the help of optimal controls. The next theorem uses the monotonicity with respect to the first variable of the function defined in Lemma 33.

Theorem 50. *The value function* $V:[0,+\infty)\to\mathbb{R}$ *satisfies:*

i) V *is strictly increasing*

ii) For every $k_0>0$, *there exists* $C\left(k_0\right),\delta>0$ *such that for every* $h\in(-\delta,\delta)$:

$$\frac{V\left(k_0+h\right)-V\left(k_0\right)}{h}\geq C\left(k_0\right)$$

iii) V *is Lipschitz-continuous over every closed sub-interval of* $(0,+\infty)$.

Proof. i) Let $0<k_0<k_1$. Set $c\in\left(0,F\left(k_0\right)\right]$ and $c_0\equiv c$ in $[0,+\infty)$; hence by Lemma 36 and by Theorem 42,

$$V\left(0\right)=0<\frac{u\left(c\right)}{\rho}=U\left(c_0;k_0\right)\leq V\left(k_0\right).$$

In order to establish that $V(k_0) < V(k_1)$, take $c \in \Lambda(k_0)$ optimal at k_0 and define $\underline{c}^{k_1-k_0}$ as in Lemma 34. As

$$u'(N(k_0, k_1 - k_0) + 1) \int_0^{k_1-k_0} e^{-\rho t} \mathrm{d}t > 0$$

we have

$$V(k_0) = U(c; k_0) < U\left(\underline{c}^{k_1-k_0}; k_1\right) \leq V(k_1)$$

ii) We split the proof in two parts.

First, take $k_0, h > 0$, c optimal at k_0 and set $k_1 := k_0 + h$. Because $k_1 > k_0$ we can choose $\underline{c}^{k_1-k_0} = \underline{c}^h \in \Lambda(k_0 + h)$ as in Lemma 34. Hence

$$\begin{aligned} V(k_0 + h) - V(k_0) &\geq U\left(\underline{c}^h; k_0 + h\right) - U(c; k_0) \\ &\geq u'(N(k_0, h) + 1) \int_0^h e^{-\rho t} \mathrm{d}t \end{aligned}$$

Now, by the fact that $\lim_{h \to 0} \frac{1}{h} \int_0^h e^{-\rho t} \mathrm{d}t = 1$ and that $N(k_0, \cdot)$ is increasing, there exists $\delta > 0$ such that, for any $h \in (0, \delta)$:

$$\begin{aligned} \frac{V(k_0 + h) - V(k_0)}{h} &\geq u'(N(k_0, h) + 1) \frac{\int_0^h e^{-\rho t} \mathrm{d}t}{h} \\ &\geq \frac{u'(N(k_0, 1) + 1)}{2} =: C(k_0) \end{aligned}$$

In the second place, fix $k_0 > 0$, $h < 0$ and c optimal at $k_0 + h$. Then again take $\underline{c}^{k_0-(k_0+h)} = \underline{c}^{-h} \in \Lambda(k_0)$ as in Lemma 34. Hence

$$\begin{aligned} V(k_0 + h) - V(k_0) &\leq U(c; k_0 + h) - U\left(\underline{c}^{-h}; k_0\right) \\ &\leq -u'(N(k_0 + h, -h) + 1) \int_0^{-h} e^{-\rho t} \mathrm{d}t. \end{aligned}$$

We can assume that $-\frac{1}{h} \int_0^{-h} e^{-\rho t} \mathrm{d}t \geq \frac{1}{2}$ for $-\delta < h < 0$. Hence, by the monotonicity of $N(\cdot,\cdot)$ in both variables, for every $h \in (-\delta, 0)$:

$$\frac{V(k_0+h)-V(k_0)}{h} \geq \frac{u'(N(k_0+h,-h)+1)}{2} \geq \frac{u'(N(k_0,1)+1)}{2} = C(k_0).$$

iii) Let $0 < k_0 < k_1$. We want a reverse inequality for $V(k_1)-V(k_0)$, so take $c_1 \in \Lambda(k_1)$ optimal at k_1. In order to define the proper $c_0 \in \Lambda(k_0)$, observe that the orbit $k = k(\cdot; k_0, 0)$ (with control constantly equal to 0) satisfies

$$\dot{k} = F(k).$$

With an argument similar to the one used in Proposition 37 we can see that $\dot{k}(t) > F(k_0) > 0$ for every $t > 0$, and so $\lim_{t\to+\infty} k(t) = +\infty$.
Then by Darboux's property there exists $\bar{t} > 0$ such that $k(\bar{t}) = k_1$. Observe that, since k and F are strictly increasing functions, $\dot{k}$ must also be strictly increasing.
Hence appling Lagrange's thorem to k gives for some $\xi \in (0,\bar{t})$:

$$k_1 - k_0 \quad = \quad k(\bar{t}) - k(0) = \bar{t}\cdot\dot{k}(\xi) > \bar{t}\dot{k}(0) = \bar{t}F(k_0) \tag{58}$$

Now define

$$c_0(t) := \begin{cases} 0 & \text{if } t \in [0,\bar{t}] \\ c_1(t-\bar{t}) & \text{if } t > \bar{t} \end{cases}$$

It is easy to check that $c_0 \in \Lambda(k_0)$, because

$$\begin{aligned} k(t; k_0, c_0) &= k(t; k_0, 0) > 0 \quad \forall t \in [0,\bar{t}] \\ k(t+\bar{t}; k_0, c_0) &= k(t; k_1, c_1) \geq 0 \quad \forall t \geq 0 \end{aligned}$$

by the uniqueness of the orbit; as far as the second equality is concerned, observe that both orbits pass through $(0, k_1)$ and satisfy the differential equation controlled with c_1 for $t > 0$.

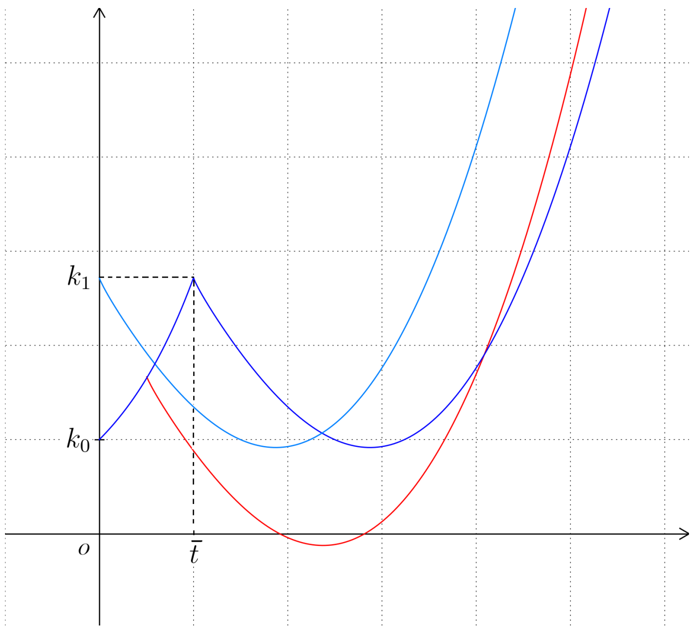
Hence, remembering that $u(0) = 0$:

$$\begin{aligned}
V(k_1) - V(k_0) &\leq U(c_1; k_1) - U(c_0; k_0) = \int_0^{+\infty} e^{-\rho t} \left[u(c_1(t)) - u(c_0(t))\right] \mathrm{d}t \\
&= \int_0^{+\infty} e^{-\rho t} u(c_1(t))\, \mathrm{d}t - \int_{\bar{t}}^{+\infty} e^{-\rho t} u(c_1(t - \bar{t}))\, \mathrm{d}t \\
&= \int_0^{+\infty} e^{-\rho t} u(c_1(t))\, \mathrm{d}t - \int_0^{+\infty} e^{-\rho(s+\bar{t})} u(c_1(s))\, \mathrm{d}s \\
&= \left(1 - e^{-\rho \bar{t}}\right) U(c_1; k_1) = \left(1 - e^{-\rho \bar{t}}\right) V(k_1) \\
&\leq \rho \bar{t} V(k_1) < \rho V(k_1) \frac{k_1 - k_0}{F(k_0)}
\end{aligned}$$

(in the last inequality we used (58)). By the monotonicity of V and F we have, for $a \leq k_0 < k_1 \leq b$:

$$V(k_1) - V(k_0) \leq \rho \frac{V(b)}{F(a)} (k_1 - k_0)$$

$\square$



The construction of Theorem 50.iii)

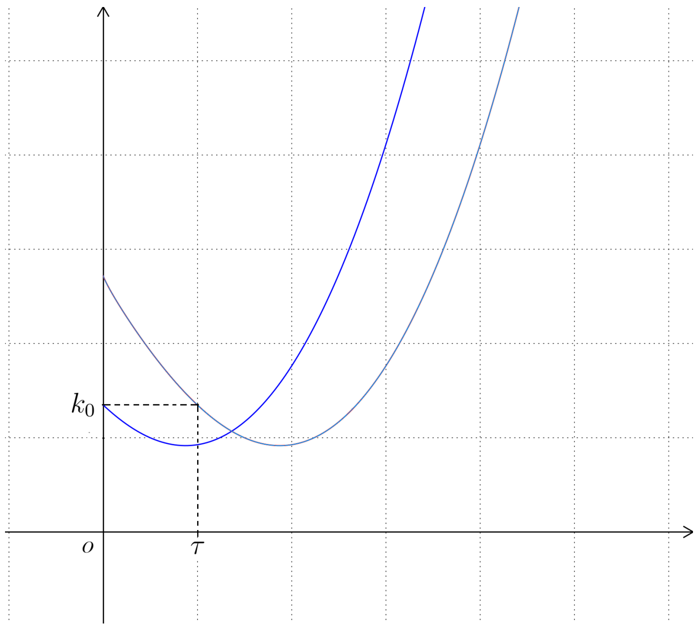
14 Dynamic Programming

In this section we study the properties of the value function as a solution to certain equations which arise naturally. Observe that we can translate an orbit by translating the control, according to the next remark.

Remark 51 (Translation of the orbit). For every $k_0 \geq 0$ and every $c \in \mathcal{L}^1_{loc}((0,+\infty),\mathbb{R})$:

$$k(\cdot; k(\tau; k_0, c), c(\cdot + \tau)) = k(\cdot + \tau; k_0, c)$$

by the uniqueness of the orbit. In particular, if $c \in \Lambda(k_0)$ then $c(\cdot + \tau) \in \Lambda(k(\tau; k_0, c))$.



Translation of the orbit

The first step clearly consists in proving that the value function solves the Bellman Functional Equation; that is to say, we prove a suitable version of Dynamic Programming Principle.

Theorem 52 (**Bellman's Dynamic Programming Principle**). *For every $\tau > 0$, the value function $V : [0, +\infty) \to \mathbb{R}$ satisfies the following functional equation:*

$$\forall k_0 \geq 0 : \mathrm{v}\left(k_0\right) = \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u\left(c\left(t\right)\right) dt + e^{-\rho\tau} \mathrm{v}\left(k\left(\tau; k_0, c\right)\right) \right\} \tag{59}$$

in the unknown $\mathrm{v} : [0, +\infty) \to \mathbb{R}$.

Proof. Fix $\tau > 0$ and $k_0 \geq 0$, and set

$$\sigma\left(\tau, k_0\right) := \sup_{c \in \Lambda(k_0)} \left\{ \int_0^\tau e^{-\rho t} u\left(c\left(t\right)\right) \mathrm{d}t + e^{-\rho\tau} V\left(k\left(\tau; k_0, c\right)\right) \right\}.$$

We prove that

$$\sigma\left(\tau, k_0\right) = \sup_{c \in \Lambda(k_0)} U\left(c; k_0\right).$$

In the first place, we show that $\sigma\left(\tau, k_0\right)$ is an upper bound of $\left\{U\left(c; k_0\right) \,/\, c \in \Lambda\left(k_0\right)\right\}$. Fix $c \in \Lambda\left(k_0\right)$; then by Remark 51 $c\left(\cdot + \tau\right) \in \Lambda\left(k\left(\tau; k_0, c\right)\right)$; hence

$$\begin{aligned}
\sigma\left(\tau, k_0\right) &\geq \int_0^\tau e^{-\rho t} u\left(c\left(t\right)\right) \mathrm{d}t + e^{-\rho\tau} V\left(k\left(\tau; k_0, c\right)\right) \\
&\geq \int_0^\tau e^{-\rho t} u\left(c\left(t\right)\right) \mathrm{d}t + e^{-\rho\tau} U\left(c\left(\cdot + \tau\right); k\left(\tau; k_0, c\right)\right) \\
&= \int_0^\tau e^{-\rho t} u\left(c\left(t\right)\right) \mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u\left(c\left(t + \tau\right)\right) \mathrm{d}t \\
&= \int_0^\tau e^{-\rho t} u\left(c\left(t\right)\right) \mathrm{d}t + \int_\tau^{+\infty} e^{-\rho s} u\left(c\left(s\right)\right) \mathrm{d}t = U\left(c; k_0\right)
\end{aligned}$$

In the second place, fix $\epsilon > 0$, and take

$$0 < \epsilon' \leq \frac{2\epsilon}{\left(1 + e^{-\rho\tau}\right)}.$$

Hence there exists $\tilde{c}_\epsilon \in \Lambda(k_0)$ and $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$ such that

$$\begin{aligned}
\sigma(\tau, k_0) - \epsilon &\leq \sigma(\tau, k_0) - \frac{\epsilon'}{2}\left(1 + e^{-\rho\tau}\right) \\
&\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\, \mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, \tilde{c}_\epsilon)) - e^{-\rho\tau}\frac{\epsilon'}{2} \\
&\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\, \mathrm{d}t + e^{-\rho\tau} U\left(\tilde{\tilde{c}}_\epsilon; k(\tau; k_0, \tilde{c}_\epsilon)\right) \\
&= \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\, \mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u\left(\tilde{\tilde{c}}_\epsilon(t)\right) \mathrm{d}t
\end{aligned}$$

Now set

$$c_\epsilon(t) := \begin{cases} \tilde{c}_\epsilon(t) & \text{if } t \in [0, \tau] \\ \tilde{\tilde{c}}_\epsilon(t - \tau) & \text{if } t > \tau \end{cases}$$

Hence $c_\epsilon \in \mathcal{L}^1_{loc}((0, +\infty), \mathbb{R})$ and $\forall t > 0 : c_\epsilon(t + \tau) = \tilde{\tilde{c}}_\epsilon(t)$. So:

$$\begin{aligned}
\sigma(\tau, k_0) - \epsilon &\leq \int_0^\tau e^{-\rho t} u(\tilde{c}_\epsilon(t))\, \mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u\left(\tilde{\tilde{c}}_\epsilon(t)\right) \mathrm{d}t \\
&= \int_0^\tau e^{-\rho t} u(c_\epsilon(t))\, \mathrm{d}t + \int_0^{+\infty} e^{-\rho(t+\tau)} u(c_\epsilon(t + \tau))\, \mathrm{d}t \\
&= \int_0^{+\infty} e^{-\rho t} u(c_\epsilon(t))\, \mathrm{d}t \qquad (60)
\end{aligned}$$

Finally, it is easy to show that $c_\epsilon \in \Lambda(k_0)$. Observe that

$$\forall t \in [0, \tau] : k(t; k_0, c_\epsilon) = k(t; k_0, \tilde{c}_\epsilon)$$

because both the orbits satisfy the problem:

$$\begin{cases} \mathrm{h}(0) = k_0 \\ \dot{\mathrm{h}}(t) = F(\mathrm{h}(t)) - \tilde{c}_\epsilon(t) & \text{for } t \in [0, \tau] \end{cases}$$

in the unknown h. In particular $k(\tau; k_0, c_\epsilon) = k(\tau; k_0, \tilde{c}_\epsilon)$, so that $k(\cdot + \tau; k_0, c_\epsilon)$

and $k\left(\cdot; k\left(\tau; k_0, \tilde{c}_\epsilon\right), \tilde{\tilde{c}}_\epsilon\right)$ have the same initial value. Moreover, these two orbits satisfy:

$$\forall t > 0 : \dot{\mathrm{h}}(t) = F(\mathrm{h}(t)) - \tilde{\tilde{c}}_\epsilon(t)$$

in the unknown h, which implies

$$\forall t \geq 0 : k(t+\tau; k_0, c_\epsilon) = k\left(t; k(\tau; k_0, \tilde{c}_\epsilon), \tilde{\tilde{c}}_\epsilon\right)$$

Now it is enough to recall that by construction $\tilde{c}_\epsilon \in \Lambda(k_0)$ and $\tilde{\tilde{c}}_\epsilon \in \Lambda(k(\tau; k_0, \tilde{c}_\epsilon))$, so that $k(t; k_0, c_\epsilon) \geq 0$ for all $t \geq 0$. By (60) we can write

$$\sigma(\tau, k_0) - \epsilon \leq U(c_\epsilon; k_0)$$

and the assertion is proven. □

Equation (59) is called *Bellman Functional Equation.*

As in the finite-horizon case of Chapter 2, we have, in consequence of the previous theorem, that every control which is optimal respect to a state, is also optimal respect to every succesive optimal state.

Corollary 53. *Let $k_0 \geq 0$, $c^* \in \Lambda(k_0)$. Hence the follwing are equivalent:*

i) c^* is optimal at k_0

ii) For every $\tau > 0$:

$$V(k_0) = \int_0^\tau e^{-\rho t} u(c^*(t))\, dt + e^{-\rho\tau} V(k(\tau; k_0, c^*))$$

Moreover, i) or ii) imply that for every $\tau > 0$, $c^*(\cdot + \tau)$ is admissible and optimal at $k(\tau; k_0, c^*)$.

Proof. i) $\Rightarrow$ ii) Let us assume that c^* is admissible and optimal at $k_0 \geq 0$ and fix $\tau > 0$. Observe that $c^*(\cdot + \tau)$ is admissible at $k(\tau; k_0, c^*)$ by Remark

51. Hence, by Theorem 52:

$$
\begin{aligned}
V(k_0) &\geq \int_0^{\tau} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c^*)) \\
&\geq \int_0^{\tau} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho\tau} U(c^*(\cdot + \tau); k(\tau; k_0, c^*)) \\
&= \int_0^{+\infty} e^{-\rho t} u(c^*(t))\,\mathrm{d}t \\
&= U(c^*; k_0) = V(k_0)
\end{aligned}
\tag{61}
$$

where the last equality holds because of the optimality of c^*. In particular

$$
V(k_0) = \int_0^{\tau} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho\tau} V(k(\tau; k_0, c^*)). \tag{62}
$$

ii) $\Rightarrow$ i) Suppose that $c^* \in \Lambda(k_0)$ and (62) holds for every $\tau > 0$. For every $\epsilon > 0$ pick $\hat{c}_\epsilon \in \Lambda\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right)$ such that:

$$
V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon \leq U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right).
$$

Then define

$$
c_\epsilon(t) := \begin{cases} c^*(t) & \text{if } t \in \left[0, \frac{1}{\epsilon}\right] \\ \hat{c}_\epsilon\left(t - \frac{1}{\epsilon}\right) & \text{if } t > \frac{1}{\epsilon} \end{cases}
$$

By the same arguments we used in the proof of Theorem 52 , $c_\epsilon \in \Lambda(k_0)$ and $c_\epsilon\left(t + \frac{1}{\epsilon}\right) = \hat{c}_\epsilon(t)$ for every $t > 0$.

Hence, taking $\tau = 1/\epsilon$ in (62):

$$\begin{aligned} V(k_0) - \epsilon e^{-\rho/\epsilon} &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho/\epsilon}\left[V\left(k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) - \epsilon\right] \\ &\le \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + e^{-\rho/\epsilon} U\left(\hat{c}_\epsilon; k\left(\frac{1}{\epsilon}; k_0, c^*\right)\right) \\ &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + \int_0^{+\infty} e^{-\rho\left(t+\frac{1}{\epsilon}\right)} u\left(c_\epsilon\left(t + \frac{1}{\epsilon}\right)\right)\mathrm{d}t \\ &= \int_0^{1/\epsilon} e^{-\rho t} u(c^*(t))\,\mathrm{d}t + \int_{1/\epsilon}^{+\infty} e^{-\rho s} u(c_\epsilon(s))\,\mathrm{d}s \qquad (63) \end{aligned}$$

Observe that by Jensen inequality, for every $T \ge 1/\epsilon$:

$$\begin{aligned} \int_{1/\epsilon}^{T} e^{-\rho s} u(c_\epsilon(s))\,\mathrm{d}s &= \left[e^{-\rho s}\int_{1/\epsilon}^{s} u(c_\epsilon(\tau))\,\mathrm{d}\tau\right]_{s=1/\epsilon}^{s=T} + \rho\int_{1/\epsilon}^{T} e^{-\rho s}\int_{1/\epsilon}^{s} u(c_\epsilon(\tau))\,\mathrm{d}\tau\mathrm{d}s \\ &\le e^{-\rho T}\int_0^T u(c_\epsilon(\tau))\,\mathrm{d}\tau + \rho\int_{1/\epsilon}^{T} e^{-\rho s}\int_0^s u(c_\epsilon(\tau))\,\mathrm{d}\tau\mathrm{d}s \\ &\le e^{-\rho T}\int_0^T u(c_\epsilon(\tau))\,\mathrm{d}\tau + \rho\int_{1/\epsilon}^{T} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right)\mathrm{d}s \\ &\to \rho\int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right)\mathrm{d}s \quad \text{as } T \to +\infty, \qquad (64) \end{aligned}$$

by Lemma 40, ii) and by the admissibility of c_ϵ. By point i) of the same Lemma, for every $\epsilon < 1$ and every $s \ge 1/\epsilon$:

$$\begin{aligned} s e^{-\rho s} u\left(\frac{\int_0^s c_\epsilon(\tau)\,\mathrm{d}\tau}{s}\right) &\le s e^{-\rho s} u\left(M(k_0)\left[1 + e^{(L+\epsilon_0)s}\right] + \frac{M(k_0)}{s(L+\epsilon_0)}\right) \\ &\le s e^{-\rho s}\left\{u(M(k_0)) + M(k_0)\, u\left(e^{(L+\epsilon_0)s}\right) + u\left(\frac{M(k_0)}{L+\epsilon_0}\right)\right\} \end{aligned}$$

which implies, together with (64), for every $\epsilon < 1$:

$$\begin{aligned}
0 \leq \int_{1/\epsilon}^{+\infty} e^{-\rho s} u\left(c_{\epsilon}(s)\right) \mathrm{d}s &\leq \rho \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(\frac{\int_0^s c_{\epsilon}(\tau)\, \mathrm{d}\tau}{s}\right) \mathrm{d}s \\
&\leq \rho \left[u\left(M\left(k_0\right)\right) + u\left(\frac{M\left(k_0\right)}{L+\epsilon_0}\right)\right] \int_{1/\epsilon}^{+\infty} s e^{-\rho s} \mathrm{d}s + \\
&\quad \rho M\left(k_0\right) \int_{1/\epsilon}^{+\infty} s e^{-\rho s} u\left(e^{(L+\epsilon_0)s}\right) \mathrm{d}s.
\end{aligned}$$

By Remark 25 this quantity tends to 0 as $\epsilon \to 0$.

Hence, by (63):

$$V\left(k_0\right) - \epsilon e^{-\rho/\epsilon} \quad \leq \quad \int_0^{1/\epsilon} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t + o_{\epsilon \to 0}(1)$$

Passing to the limit for $\epsilon \to 0$ we find:

$$V\left(k_0\right) \leq \int_0^{+\infty} e^{-\rho t} u\left(c^*(t)\right) \mathrm{d}t = U\left(c^*; k_0\right)$$

which implies that c^* is optimal at k_0.

Finally, if i) holds, then by (61):

$$V\left(k\left(\tau; k_0, c^*\right)\right) = U\left(c^*(\cdot + \tau); k\left(\tau; k_0, c^*\right)\right)$$

□

Following the path traced in Chapter 2, a careful study of the behaviour of the difference quotients of the functions

$$t \to e^{-\rho t} V(k(t))$$

(for an orbit k) leads to the following definitions and theorems.

Definition 54. Let $f \in \mathcal{C}^0((0,+\infty),\mathbb{R})$; we say that $f \in \mathcal{C}^+((0,+\infty),\mathbb{R})$ if, and only if, for every $k_0 > 0$ there exist $\delta, C^+, C^- > 0$ such that

$$\frac{f(k_0+h) - f(k_0)}{h} \geq C^+ \quad \forall h \in (0,\delta)$$
$$\frac{f(k_0+h) - f(k_0)}{h} \geq C^- \quad \forall h \in (-\delta,0)$$

Remark 55. The value function V satisfies

$$V \in \mathcal{C}^+((0,+\infty),\mathbb{R})$$

by Theorem 50,(ii).

Moreover

$$\mathcal{C}^+((0,+\infty),\mathbb{R}) \cap \mathcal{C}^1((0,+\infty),\mathbb{R}) = \left\{f \in \mathcal{C}^1((0,+\infty),\mathbb{R}) / f' > 0\right\}$$

Definition 56. The function $H : [0,+\infty) \times (0,+\infty) \to \mathbb{R}$ defined by

$$H(k,p) := -\sup\{[F(k) - c] \cdot p + u(c) \ / \, c \in [0,+\infty)\}$$

is called *Hamiltonian.*

The equation

$$\rho \mathrm{v}(k) + H(k, \mathrm{v}'(k)) = 0 \quad \forall k > 0 \tag{65}$$

in the unknown $\mathrm{v} \in \mathcal{C}^+((0,+\infty),\mathbb{R}) \cap \mathcal{C}^1((0,+\infty),\mathbb{R})$ is called *Hamilton-Jacobi-Bellman equation* (HJB).

Remark 57. Definition 56 is well-posed. Indeed

$$-\sup_{c\in[0,+\infty)}\{[F(k) - c] \cdot p + u(c)\} > -\infty \iff p > 0.$$

If $p > 0$, by the fact that $\lim_{c\to+\infty} u'(c) = 0$, we can choose $c_p \geq 0$ such that

$$u'(c_p) \leq p$$

which implies, by the concavity of u:

$$\forall c \geq 0 : u(c) - cp \leq u(c) - u'(c_p)\, c \leq u(c_p) - u'(c_p)\, c_p$$

which implies

$$H(k,p) = -F(k)\, p - \sup_{c\in[0,+\infty)} \{u(c) - cp\} \geq -F(k)\, p - u(c_p) + u'(c_p)\, c_p > -\infty$$

If $p \leq 0$ then

$$H(k,p) = -F(k)\, p - \sup_{c\in[0,+\infty)} \{u(c) - cp\} \leq -F(k)\, p - \sup_{c\in[0,+\infty)} u(c) = -\infty$$

because $\lim_{c\to+\infty} u(c) = +\infty$.

Moreover by Remark 55, $\mathrm{v}'(k) > 0$ for every $k > 0$ in the previous definition.

Definition 58. A function $v \in \mathcal{C}^+((0,+\infty),\mathbb{R})$ is called a *viscosity subsolution* [*supersolution*] *of* (HJB) if, and only if:

for every $\varphi \in \mathcal{C}^1((0,+\infty),\mathbb{R})$ and for every local maximum [minimum] point $k_0 > 0$ of $v - \varphi$:

$$\begin{array}{rcl} \rho v(k_0) - \sup\{[F(k_0) - c]\cdot \varphi'(k_0) + u(c)\ /\ c \in [0,+\infty)\} & = & \\ \rho v(k_0) + H(k_0, \varphi'(k_0)) & \leq & 0 \\ & [\geq & 0] \end{array}$$

If v is both a viscosity subsolution of (HJB) and a viscosity supersolution of (HJB), then we say that v is a *viscosity solution of* (HJB).

Remark 59. The latter definition is well posed. Indeed, let $v \in \mathcal{C}^+((0,+\infty),\mathbb{R})$

and $\varphi \in \mathcal{C}^1((0,+\infty), \mathbb{R})$. If k_0 is a local maximum for $v - \varphi$ in $(0,+\infty)$, then for $h < 0$ big enough we have:

$$\begin{aligned}
&v(k_0) - v(k_0+h) \geq \varphi(k_0) - \varphi(k_0+h) \implies \\
&0 < C^- \leq \frac{v(k_0) - v(k_0+h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0+h)}{h}.
\end{aligned}$$

If k_0 is a local minimum for $v - \varphi$ in $(0,+\infty)$, then for $h > 0$ small enough we have:

$$\begin{aligned}
&v(k_0) - v(k_0+h) \leq \varphi(k_0) - \varphi(k_0+h) \implies \\
&0 < C^+ \leq \frac{v(k_0) - v(k_0+h)}{h} \leq \frac{\varphi(k_0) - \varphi(k_0+h)}{h}.
\end{aligned}$$

In both cases, we have $\varphi'(k_0) > 0$.

Lemma 60. *Let $k_0 > 0$ and $(c_T)_{T>0} \subseteq \Lambda(k_0)$ satifying:*

$$\|c_T\|_{\infty,[0,T]} \leq N(k_0, T) \quad \forall T > 0.$$

where N is the function defined in Lemma 33. *Hence*

$$\forall T \in [0,1] : \forall t \in [0,T] : |k(t; k_0, c_T) - k_0| \leq T e^{\bar{M}t} \left[F(k_0) + N(k_0, 1)\right].$$

In paricular $k(T; k_0, c_T) \to k_0$ as $T \to 0$.

Proof. Set k_0 and $(c_T)_{T>0}$ as in the hypothesis and fix $0 \leq T \leq 1$. Hence integrating both sides of the state equation we get, for every $t \in [0,T]$:

$$k(t; k_0, c_T) - k_0 \quad = \quad \int_0^t \left[F(k_0) - c_T(s)\right] \mathrm{d}s + \int_0^t \left[F(k(s; k_0, c_T)) - F(k_0)\right] \mathrm{d}s$$

which implies by Remark 31:

$$
\begin{aligned}
|k(t;k_0,c_T)-k_0| &\leq \int_0^t |F(k_0)-c_T(s)|\,\mathrm{d}s + \int_0^t |F(k(s;k_0,c_T))-F(k_0)|\,\mathrm{d}s \\
&\leq \int_0^T |F(k_0)-c_T(s)|\,\mathrm{d}s + \bar{M}\int_0^t |k(s;k_0,c_T)-k_0|\,\mathrm{d}s
\end{aligned}
$$

Hence by Gronwall's inequality and by the monotonicity of $N(k_0,\cdot)$, for every $T\in[0,1]$ and every $t\in[0,T]$:

$$
\begin{aligned}
|k(t;k_0,c_T)-k_0| &\leq e^{\bar{M}t}\int_0^T |F(k_0)-c_T(s)|\,\mathrm{d}s. \\
&\leq Te^{\bar{M}t}\left[F(k_0)+N(k_0,T)\right] \\
&\leq Te^{\bar{M}t}\left[F(k_0)+N(k_0,1)\right].
\end{aligned}
$$

□

Proposition 61. *The value function $V:[0,+\infty)\to\mathbb{R}$ is a viscosity solution of* (HJB)*.*
Consequently, if $V\in\mathcal{C}^1([0,+\infty),\mathbb{R})$, then V is strictly increasing and is a solution of (HJB) - (65) *in the classical sense.*

Proof. In the first place, we show that V is a viscosity supersolution of (HJB). Let $\varphi\in\mathcal{C}^1((0,+\infty),\mathbb{R})$ and $k_0>0$ be a local minumum point of $V-\varphi$, so that

$$V(k_0)-V\leq\varphi(k_0)-\varphi$$

in a proper neighbourhood of k_0. Now fix $c\in[0,+\infty)$ and set $k:=k(\cdot;k_0,c)$. As $k_0>0$, there exists $T_c>0$ such that $k>0$ in $[0,T_c]$. Hence the control

$$\tilde{c}(t):=\begin{cases} c & \text{if } t\in[0,T_c] \\ 0 & \text{if } t>T_c \end{cases}$$

is admissible at k_0. Then by Theorem 52, for every $\tau \in [0, T_c]$:

$$\begin{aligned} V(k_0) - V(k(\tau)) &\geq \int_0^{\tau} e^{-\rho t} u(\tilde{c}(t))\,\mathrm{d}t + V(k(\tau))\left[e^{-\rho\tau} - 1\right] \\ &= u(c) \int_0^{\tau} e^{-\rho t} \mathrm{d}t + V(k(\tau))\left[e^{-\rho\tau} - 1\right] \end{aligned}$$

which implies, by the continuity of k and for every $\tau > 0$ sufficiently small:

$$\frac{\varphi(k(0)) - \varphi(k(\tau))}{\tau} \geq u(c) \frac{\int_0^{\tau} e^{-\rho t} \mathrm{d}t}{\tau} + V(k(\tau)) \frac{\left[e^{-\rho\tau} - 1\right]}{\tau}.$$

Hence, letting $\tau \to 0$, using the continuity of V and k:

$$-\varphi'(k_0)[F(k_0) - c] \geq u(c) - \rho V(k_0)$$

which implies, taking the sup for $c \geq 0$:

$$\rho V(k_0) + H(k_0, \varphi'(k_0)) \geq 0$$

In the second place, we show that V is a viscosity subsolution of (HJB). Let $\varphi \in \mathcal{C}^1((0, +\infty), \mathbb{R})$ and $k_0 > 0$ be a local maximum point of $V - \varphi$, so that

$$V(k_0) - V \geq \varphi(k_0) - \varphi \tag{66}$$

in a proper neighbourhood $\mathcal{N}(k_0)$ of k_0.

Set $\epsilon > 0$ and define a familty of admissible controls $(c_{T,\epsilon})_{T>0} \subseteq \Lambda(k_0)$ such that, for every $T > 0$:

$$V(k_0) - T\epsilon \leq \int_0^T e^{-\rho t} u(c_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} V(k(T; k_0, c_{T,\epsilon})). \tag{67}$$

Now set $c^*_{T,\epsilon} \in \Lambda\left(k\left(T; k_0, c_{T,\epsilon}\right)\right)$ optimal at $k\left(T; k_0, c_{T,\epsilon}\right)$, and define

$$\hat{c}_{T,\epsilon}\left(t\right) := \begin{cases} c_{T,\epsilon}\left(t\right) & \text{if } t \in [0,T] \\ c^*_{T,\epsilon}\left(t-T\right) & \text{if } t > T \end{cases}$$

First observe that $\hat{c}_{T,\epsilon} \in \Lambda\left(k_0\right)$ because $\hat{c}_{T,\epsilon} \geq 0$ and

$$\begin{aligned} k\left(t; k_0, \hat{c}_{T,\epsilon}\right) &= k\left(t; k_0 c_{T,\epsilon}\right) \geq 0 \quad \forall t \in [0,T] \\ k\left(t+T; k_0, \hat{c}_{T,\epsilon}\right) &= k\left(t; k\left(T; k_0, c_{T,\epsilon}\right), c^*_{T,\epsilon}\right) \geq 0 \quad \forall t > T \end{aligned}$$

Moreover,

$$\begin{aligned} \int_T^{+\infty} e^{-\rho s} u\left(\hat{c}_{T,\epsilon}\left(s\right)\right) \mathrm{d}s &= e^{-\rho T} \int_T^{+\infty} e^{-\rho(s-T)} u\left(c^*_{T,\epsilon}\left(s-T\right)\right) \mathrm{d}s \\ &= e^{-\rho T} U\left(c^*_{T,\epsilon}; k\left(T; k_0, c_{T,\epsilon}\right)\right) \end{aligned}$$

Hence by (67) and the optimality of $c^*_{T,\epsilon}$:

$$\begin{aligned} V\left(k_0\right) - T\epsilon &\leq \int_0^T e^{-\rho t} u\left(c_{T,\epsilon}\left(t\right)\right) \mathrm{d}t + e^{-\rho T} V\left(k\left(T; k_0, c_{T,\epsilon}\right)\right) \\ &= \int_0^T e^{-\rho t} u\left(c_{T,\epsilon}\left(t\right)\right) \mathrm{d}t + e^{-\rho T} U\left(c^*_{T,\epsilon}; k\left(T; k_0, c_{T,\epsilon}\right)\right) \\ &= \int_0^T e^{-\rho t} u\left(\hat{c}_{T,\epsilon}\left(t\right)\right) \mathrm{d}t + \int_T^{+\infty} e^{-\rho s} u\left(\hat{c}_{T,\epsilon}\left(s\right)\right) \mathrm{d}s = U\left(\hat{c}_{T,\epsilon}; k_0\right) \end{aligned}$$

Now take $\left(\hat{c}_{T,\epsilon}\right)^T$ as in Lemma 33 and set $\bar{c}_{T,\epsilon} := \left(\hat{c}_{T,\epsilon}\right)^T$ for simplicity of notation (hence $\bar{c}_{T,\epsilon} \in \Lambda(k_0)$). We have:

$$\begin{aligned}
V(k_0) - T\epsilon &\leq U(\hat{c}_{T,\epsilon}; k_0) \leq U(\bar{c}_{T,\epsilon}; k_0) \\
&= \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} \int_T^{+\infty} e^{-\rho(s-T)} U(\bar{c}_{T,\epsilon}(s - T + T))\,\mathrm{d}s \\
&= \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} U(\bar{c}_{T,\epsilon}(\cdot + T); k(T; k_0, \bar{c}_{T,\epsilon})) \\
&\leq \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t + e^{-\rho T} V(k(T; k_0, \bar{c}_{T,\epsilon}))
\end{aligned}$$

where we have used Remark 51.

By Lemma 60 we have for $T > 0$ sufficiently small (say $T < \hat{T}$),

$$k(T; k_0, \bar{c}_{T,\epsilon}) \in \mathcal{N}(k_0).$$

Hence, setting $\bar{k}_{T,\epsilon} := k(\cdot; k_0, \bar{c}_{T,\epsilon})$, for every $T < \hat{T}$, we have by (66):

$$\begin{aligned}
\varphi(k_0) - \varphi\left(\bar{k}_{T,\epsilon}(T)\right) - e^{-\rho T} V\left(\bar{k}_{T,\epsilon}(T)\right) &\leq V(k_0) - V\left(\bar{k}_{T,\epsilon}(T)\right) - e^{-\rho T} V\left(\bar{k}_{T,\epsilon}(T)\right) \\
&\leq \int_0^T e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\,\mathrm{d}t - V\left(\bar{k}_{T,\epsilon}(T)\right) + T\epsilon
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_0^T -\left\{\varphi'\left(\bar{k}_{T,\epsilon}(t)\right)\left[F\left(\bar{k}_{T,\epsilon}(t)\right) - \bar{c}_{T,\epsilon}(t)\right] + e^{-\rho t} u(\bar{c}_{T,\epsilon}(t))\right\}\mathrm{d}t \\
&\leq V\left(\bar{k}_{T,\epsilon}(T)\right)\left[e^{-\rho T} - 1\right] + T\epsilon. \qquad (68)
\end{aligned}$$

Observe that the integral at the left hand member bigger than:

$$\begin{aligned}
&\int_0^T -\{[\varphi'(k_0)+\omega_1(t)][F(k_0)-\bar{c}_{T,\epsilon}(t)+\omega_2(t)]+u(\bar{c}_{T,\epsilon}(t))\}\,\mathrm{d}t = \\
&\int_0^T -\{\varphi'(k_0)[F(k_0)-\bar{c}_{T,\epsilon}(t)]+u(\bar{c}_{T,\epsilon}(t))\}\,\mathrm{d}t + \\
&+\int_0^T -\{\varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t+\omega_1(t)[\omega_2(t)+F(k_0)-\bar{c}_{T,\epsilon}(t)]\,\mathrm{d}t\} \qquad (69)
\end{aligned}$$

where ω_1, ω_2 are functions which are continuous in a neighbourhood of 0 and satisfy:

$$\omega_1(0)=\omega_2(0)=0.$$

This implies, for $T<1$:

$$\begin{aligned}
&\left|\int_0^T \varphi'(k_0)\,\omega_2(t)\,\mathrm{d}t+\int_0^T \omega_1(t)[\omega_2(t)+F(k_0)-\bar{c}_{T,\epsilon}(t)]\,\mathrm{d}t\right| \\
&\le |\varphi'(k_0)|\,o_1(T)+o_2(T)+\int_0^T |\omega_1(t)|\,[F(k_0)+\bar{c}_{T,\epsilon}(t)]\,\mathrm{d}t \\
&\le |\varphi'(k_0)|\,o_1(T)+o_2(T)+[F(k_0)+N(k_0,T)]\,o_3(T) \\
&\le |\varphi'(k_0)|\,o_1(T)+o_2(T)+[F(k_0)+N(k_0,1)]\,o_3(T)
\end{aligned}$$

where

$$\lim_{T\to 0}\frac{o_i(T)}{T}=0$$

for $i=1,2,3$. Observe that this is true even if the o_is depend on T, by Lemma 60. For instance,

$$\begin{aligned}
|o_1(T)| &= \left|\int_0^T \omega_2(t)\,\mathrm{d}t\right| = T\max_{[0,T]}|\omega_2| = T\,|\omega_2(\tau_T)| \\
&= T\left|F\left(\bar{k}_{T,\epsilon}(\tau_T)\right)-F(k_0)\right| \\
&\le \overline{M}T\left|\bar{k}_{T,\epsilon}(\tau_T)-k_0\right| \le \overline{M}T^2e^{\bar{M}\tau_T}\,[F(k_0)+N(k_0,1)]
\end{aligned}$$

Moreover, by the fact that $V \in \mathcal{C}^{+}([0,+\infty), \mathbb{R})$, we have for any $t \in [0,T]$:

$$\begin{aligned} -\{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\} &\geq -\sup_{c \geq 0}\{\varphi'(k_0)[F(k_0) - c] + u(c)\} \\ &= H(k_0, \varphi'(k_0)) > -\infty, \end{aligned}$$

by which we can write:

$$\int_0^T -\{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\}\,\mathrm{d}t \geq T \cdot H(k_0, \varphi'(k_0)).$$

Hence, by (68) and (69):

$$\begin{aligned} & V\left(\bar{k}_{T,\epsilon}(T)\right)\left[e^{-\rho T} - 1\right] + T\epsilon \\ & \geq -\int_0^T \{\varphi'(k_0)[F(k_0) - \bar{c}_{T,\epsilon}(t)] + u(\bar{c}_{T,\epsilon}(t))\}\,\mathrm{d}t + \\ & \ + \int_0^T -\{\varphi'(k_0)\omega_2(t)\,\mathrm{d}t + \omega_1(t)[\omega_2(t) + F(k_0) - \bar{c}_{T,\epsilon}(t)]\,\mathrm{d}t\} \\ & \geq T \cdot H(k_0, \varphi'(k_0)) + o_{T\to 0}(T) \end{aligned}$$

for any $0 < T < 1, \hat{T}$. Hence dividing by T, and then letting $T \to 0$, again by Lemma 60 and the continuity of V we obtain:

$$-\rho V(k_0) + \epsilon \geq H(k_0, \varphi'(k_0))$$

which proves the assertion since ϵ is arbitrary. □

References

[1] F. Gozzi, D. Fiaschi: Endogenous growth with convexo-concave technology (Draft) (2009)

[2] J.Yong, X. Zhou: Stochastic Controls - Hamiltonian Systems and HJB

Equations (Springer-Verlag, 1999)

[3] M. Bardi, I. Capuzzo Dolcetta: Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations (Birkhäuser, 1997)

[4] J. Zabczyk: Mathematical Control Theory - An Introduction (Birkhäuser, 1995)

[5] P. Acquistapace: Appunti di Analisi Funzionale (2011)

[6] L. Cesari: Optimization - Theory and Applications (Springer-Verlag, 1983)

[7] R. E. Edwards: Functional Analysis (Dover, 1995)