Esercizi

- $$1+2\sum_{n=1}^{N}\cos nx = \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}}, \qquad x \neq 2k\pi \quad \forall k\in\mathbb{Z}.$$

Infatti, moltiplicando per $\sin\frac{x}{2}$, bisogna provare che

$$2\sum_{n=1}^{N}\cos nx \sin\frac{x}{2} = \sin\left(N+\frac{1}{2}\right)x - \sin\frac{x}{2},$$

e ciò segue dalle formule di prostaferesi; infatti, scegliendo $\alpha=\left(n+\frac{1}{2}\right)x$ e $\beta=\left(n-\frac{1}{2}\right)x$ si ha $\frac{\alpha+\beta}{2}=nx$ e $\frac{\alpha-\beta}{2}=x$, da cui

$$2\cos nx\sin\frac{x}{2} = \sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x,$$

e dunque

$$2\sum_{n=1}^{N}\cos nx\sin\frac{x}{2} = \sum_{n=1}^{N}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right] = \sin\left(N+\frac{1}{2}\right)x - \sin\frac{x}{2}.$$

- $$\sin x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}, \qquad \cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}.$$

Infatti

$$\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{1}{\cos^2\frac{x}{2}}} =$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x,$$

$$\frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}}{1+\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \cos x.$$

- $81^{2x-1} + 2 \cdot 9^{4x} + 711 = 81^{2x+1} + 3^{-2}$.

Risolviamo: deve essere

$$81^{2x}\left[\frac{1}{81} + 2 - 81\right] = \frac{1}{9} - 711,$$

cioè

$$3^{8x} = 81^{2x} = \frac{711 - \frac{1}{9}}{79 - \frac{1}{81}} = 9$$

ovvero

$$8x = \log_3 9 = 2,$$

ovvero $x = \frac{1}{4}$.

- $|\log_{10}|x|| = 100$

$\longrightarrow$ $x = 10^{100}, -10^{100}, 10^{-100}, -10^{-100}$.

- $\log_3(\log_4(x^2-5)) < 0$

Risolviamo: essendo $3>1$, si deve avere $0 < \log_4(x^2-5) < 1$; applicando $f(x) = e^{4x}$, si deve avere $1 < x^2-5 < 4$, ossia $6 < x^2 < 9$. Dunque $\sqrt{6} < x < 3$ oppure $-3 < x < -\sqrt{6}$.

- Risolvere $37^x = 0.58^{x^3}$.

$\rightarrow$ solo $x=0$.

- Risolvere $3^{x+1} \geq 5^{1-x}$.

$\longrightarrow$ $x \geq \log_{15} \frac{5}{3}$.

- Provare che $|a^x - 1| \leq a^{|x|} - 1 \quad \forall a \geq 1, \forall x \in \mathbb{R}$.

Se $x \geq 0$ vale l'uguaglianza; se $x<0$, $|a^x-1| = 1 - a^x = a^x[a^{|x|}-1] \leq a^{|x|}-1$.