Chapter 1.2
Boundary Schwarz's lemmas

In the first chapter we learned to appreciate the importance of Schwarz's lemma. Unfortunately, its original form has a shortcoming: it cannot be directly applied to get information about boundary behaviors. Julia first and Wolff then overcame this flaw, proving the lemmas known under their names which are the main argument of this chapter.

Their idea is quite simple. The Schwarz-Pick lemma says that a holomorphic function $f \in \mathrm{Hol}(\Delta, \Delta)$ sends Poincaré disks into Poincaré disks. Then to get information about a boundary point $\tau \in \partial\Delta$, choose a sequence of Poincaré disks with centers converging to τ and constant euclidean radius, apply Schwarz's lemma to each one of them and take the limit. Thus it turns out that the right geometrical object to consider is the horocycle: an euclidean disk internally tangent to a point of $\partial\Delta$. In fact, Julia's and Wolff's lemma say that, under suitable hypotheses, a holomorphic function $f \in \mathrm{Hol}(\Delta, \Delta)$ sends horocycles into horocycles in a very controllable way.

In this chapter we shall discuss two applications of Julia's and Wolff's lemmas, leaving the main one — iteration theory — to the next. The first application is one of our *leitmotive:* the behavior of the angular derivative. Let $f \in \mathrm{Hol}(\Delta, \Delta)$, take $\sigma \in \partial\Delta$, and assume, for sake of simplicity, that $f(z) \to \tau \in \partial\Delta$ as $z \to \sigma$. Then we would like to know something about the behavior of the derivative f' near σ. A natural approach is to study the incremental ratio $\big(f(z) - \tau\big)/(z - \sigma)$, and Julia's lemma is the ideal tool for this investigation. It turns out that the non-tangential limit of the incremental ratio at σ exists, possibly equal to infinity; moreover, if it is finite, it coincides with the non-tangential limit of the derivative at σ. This will allow us to give quite an interesting criterion for the existence of the non-tangential limit of f' at a boundary point.

The second application concerns the structure of the automorphism group of hyperbolic Riemann surfaces. Mixing Wolff's and Julia's lemma, we show that a function $f \in \mathrm{Hol}(\Delta, \Delta)$ can commute with a hyperbolic automorphism γ of Δ iff f itself is a hyperbolic automorphism with the same fixed points as γ — a first extension of Proposition 1.1.13. From this we shall infer several properties of $\mathrm{Aut}(X)$; for instance, we shall prove that the automorphism group of a compact hyperbolic Riemann surface is finite.

So, after having set the stage in the previous chapter, let's begin the real play.

1.2.1 *Julia's lemma and angular derivatives*

In this section we shall introduce our main characters, the horocycles: a sort of boundary Poincaré disks. Using them we shall state, prove and discuss Julia's lemma and its consequences concerning the angular derivative.

Fix $\tau \in \partial\Delta$. From a geometrical point of view, the limit of Poincaré disks of constant euclidean radius and center z for $z \to \tau$ should be an euclidean disk tangent to the boundary of Δ in τ. Then we are led to the following definition: the *horocycle* $E(\tau, R)$ of *center* $\tau \in \partial\Delta$ and *radius* $R > 0$ is the euclidean disk of radius $R/(R+1)$ tangent to $\partial\Delta$ in τ. Analytically, the definition is:

$$E(\tau, R) = \left\{ z \in \Delta \;\middle|\; \frac{|\tau - z|^2}{1 - |z|^2} < R \right\}.$$

$E(\tau, R)$ is the limit of Poincaré disks in the sense made precise by the following proposition:

Proposition 1.2.1: *Let $B_\nu = B_\omega(z_\nu, R_\nu)$ be a sequence of Poincaré disks such that $z_\nu \to \tau \in \partial\Delta$ and*

$$\lim_{\nu\to\infty} \frac{1 - |z_\nu|}{1 - \tanh R_\nu} = R \neq 0, \infty. \tag{1.2.1}$$

Then
- (i) *if $z \in B_\nu$ for infinitely many ν, then $z \in \overline{E(\tau, R)}$;*
- (ii) *if $z \in E(\tau, R)$, then $z \in B_\nu$ for all sufficiently large ν.*

Proof: We observe that $z \in B_\nu$ is equivalent to

$$\frac{|1 - \overline{z_\nu} z|^2}{1 - |z|^2} < \frac{1 - |z_\nu|^2}{1 - (\tanh R_\nu)^2} = \frac{1 + |z_\nu|}{1 + \tanh R_\nu} \cdot \frac{1 - |z_\nu|}{1 - \tanh R_\nu}. \tag{1.2.2}$$

If $z \in B_\nu$ for infinitely many ν, we may take the limit in (1.2.2) and, using (1.2.1), we obtain $z \in \overline{E(\tau, R)}$.

Conversely, if $z \in E(\tau, R)$, then

$$\lim_{\nu\to\infty} \frac{|1 - \overline{z_\nu} z|^2}{1 - |z|^2} < R = \lim_{\nu\to\infty} \frac{1 - |z_\nu|^2}{1 - (\tanh R_\nu)^2},$$

and (1.2.2) must hold for all sufficiently large ν, **q.e.d.**

The geometrical meaning of (1.2.1) is that the euclidean radius of $B_\omega(z_\nu, R_\nu)$ tends to the euclidean radius of $E(\tau, R)$; cf. (1.1.11).

There is another way to see the horocycles as limits (and we shall use this approach in several variables):

Proposition 1.2.2: *Let $\tau \in \partial\Delta$ and $R > 0$. Then*

$$E(\tau, R) = \left\{ z \in \Delta \;\middle|\; \lim_{w\to\tau} [\omega(z, w) - \omega(0, w)] < \tfrac{1}{2} \log R \right\}. \tag{1.2.3}$$

Proof: For any $z \in \Delta$ let γ_z be an automorphism of Δ such that $\gamma_z(z) = 0$. We have

$$\omega(z, w) - \omega(0, w) = \omega\big(0, \gamma_z(w)\big) - \omega(0, w) = \tfrac{1}{2} \log\left(\frac{1 + |\gamma_z(w)|}{1 - |\gamma_z(w)|} \cdot \frac{1 - |w|}{1 + |w|} \right).$$

Now (1.1.4) yields

$$\frac{1 - |w|^2}{1 - |\gamma_z(w)|^2} = \frac{|1 - \overline{w} z|^2}{1 - |z|^2};$$

therefore

$$\lim_{w\to\tau} [\omega(z, w) - \omega(0, w)] = \tfrac{1}{2} \log \frac{|\tau - z|^2}{1 - |z|^2},$$

q.e.d.

We have already met the horocycles, though in disguise: in Lemma 1.1.16, where we proved that a parabolic automorphism of Δ with fixed point $\tau \in \partial\Delta$ sends any horocycle centered at τ into itself. More generally, an automorphism of Δ sends horocycles in horocycles:

Proposition 1.2.3: *Let γ be an automorphism of Δ, and choose $\tau \in \partial\Delta$ and $R > 0$. Let $z_0 = \gamma^{-1}(0)$, and $\alpha = (1 - |z_0|^2)/(|\tau - z_0|^2)$. Then $\gamma\bigl(E(\tau, R)\bigr) = E\bigl(\gamma(\tau), \alpha R\bigr)$.*

Proof: For every $z \in \Delta$ we have

$$|\gamma(\tau) - \gamma(z)|^2 = \frac{|\tau - z|^2(1 - |z_0|^2)^2}{|1 - \overline{z_0}z|^2|1 - \overline{z_0}\tau|^2}.$$

Hence (1.1.4) yields

$$\frac{|\gamma(\tau) - \gamma(z)|^2}{1 - |\gamma(z)|^2} = \frac{|\tau - z|^2}{1 - |z|^2} \cdot \frac{1 - |z_0|^2}{|\tau - z_0|^2},$$

and the assertion follows, **q.e.d.**

As an easy application of this result we now prove that if $\gamma \in \mathrm{Aut}(\Delta)$ is a parabolic automorphism of fixed point $\tau \in \partial\Delta$ then $\tau - \gamma(0)$ is orthogonal to $\gamma(0)$. Indeed, we already know (Lemma 1.1.16) that $\gamma^{-1}\bigl(E(\tau, R)\bigr) = E(\tau, R)$ for every $R > 0$. Therefore, by Proposition 1.2.3, $1 - |\gamma(0)|^2 = |\tau - \gamma(0)|^2$, that is $|\gamma(0)|^2 = \mathrm{Re}\bigl(\overline{\tau}\gamma(0)\bigr)$, and the assertion follows.

Later on it will often be useful to transfer the problem under consideration back and forth from Δ to the upper half-plane H^+; thus we shall need the description of horocycles in H^+. Since

$$\forall z \in \Delta\ \forall \tau \in \partial\Delta \qquad \frac{|\tau - z|^2}{1 - |z|^2} = \left[\mathrm{Re}\,\frac{\tau + z}{\tau - z}\right]^{-1},$$

an easy computation shows that the horocycles in H^+ of center $a \in \partial H^+ = \mathbf{R} \cup \{\infty\}$ and radius $R > 0$ are given by

$$E(a, R) = \left\{ w \in H^+ \;\middle|\; \frac{1 + a^2}{|w - a|^2}\,\mathrm{Im}\,w > \frac{1}{R} \right\} \tag{1.2.4}$$

if $a \in \mathbf{R}$, and

$$E(\infty, R) = \left\{ z \in H^+ \;\middle|\; \mathrm{Im}\,z > \frac{1}{R} \right\}, \tag{1.2.5}$$

if centered at infinity.

Now we proceed toward the first boundary version of Schwarz's lemma: Julia's lemma.

Let $f\colon \Delta \to \Delta$ be holomorphic. If there does not exist a sequence $\{z_\nu\} \subset \Delta$ converging toward the boundary such that $|f(z_\nu)| \to 1$, this means that $f(\Delta)$ is relatively compact in Δ; hence, by Corollary 1.1.34 f has a fixed point in Δ, and we can apply the standard Schwarz lemma. So we assume that there is a sequence $\{z_\nu\} \subset \Delta$ such that $|z_\nu| \to 1$ and $|f(z_\nu)| \to 1$. Looking at Proposition 1.2.1, it is evident that we need some information about the behavior of $(1 - |f(z_\nu)|)/(1 - |z_\nu|)$. One side is provided by

Lemma 1.2.4: *Let $f\colon \Delta \to \Delta$ be holomorphic. Then*

$$\forall z \in \Delta \qquad \frac{1-|f(z)|}{1-|z|} \ge \frac{1-|f(0)|}{1+|f(0)|} > 0. \tag{1.2.6}$$

Moreover, equality in (1.2.6) *holds at one point (and hence everywhere) iff $f(z) = e^{i\theta}z$ for a suitable $\theta \in \mathbf{R}$.*

Proof: By the Schwarz-Pick lemma for every $z \in \Delta$ we have

$$\omega\big(0, f(z)\big) \le \omega\big(0, f(0)\big) + \omega\big(f(0), f(z)\big) \le \omega\big(0, f(0)\big) + \omega(0, z),$$

that is

$$\forall z \in \Delta \qquad \frac{1+|f(z)|}{1-|f(z)|} \le \frac{1+|f(0)|}{1-|f(0)|} \cdot \frac{1+|z|}{1-|z|}. \tag{1.2.7}$$

Let $a_0 = (|f(0)| + |z|)/(1 + |f(0)||z|)$. Then the right-hand side of (1.2.7) is equal to $(1+a_0)/(1-a_0)$, and we get $|f(z)| \le a_0$, that is

$$\forall z \in \Delta \quad 1-|f(z)| \ge (1-|z|)\,\frac{1-|f(0)|}{1+|f(0)||z|} \ge (1-|z|)\,\frac{1-|f(0)|}{1+|f(0)|},$$

with equality at one point (and hence everywhere) iff $f(z) = e^{i\theta}z$ for some $\theta \in \mathbf{R}$, by Schwarz's lemma, **q.e.d.**

It is interesting to notice that (1.2.6) reduces to (1.1.1) when $f(0) = 0$; in short, Lemma 1.2.4 is just another incarnation of Schwarz's lemma.

And now we can state and prove *Julia's lemma:*

Theorem 1.2.5: *Let $f\colon \Delta \to \Delta$ be a holomorphic function, and take $\sigma \in \partial\Delta$ such that*

$$\liminf_{z\to\sigma} \frac{1-|f(z)|}{1-|z|} = \alpha < \infty. \tag{1.2.8}$$

Then there exists a unique $\tau \in \partial\Delta$ such that for every $z \in \Delta$

$$\frac{|\tau - f(z)|^2}{1-|f(z)|^2} \le \alpha \frac{|\sigma - z|^2}{1-|z|^2}, \tag{1.2.9}$$

that is

$$\forall R > 0 \qquad f\big(E(\sigma, R)\big) \subset E(\tau, \alpha R). \tag{1.2.10}$$

Moreover, equality in (1.2.9) *holds at one point (and hence everywhere) iff $f \in \mathrm{Aut}(\Delta)$.*

Proof: Schwarz's lemma yields

$$\forall z, w \in \Delta \qquad \left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \le \left|\frac{z-w}{1-\overline{w}z}\right|,$$

that is

$$\frac{|1-\overline{f(w)}f(z)|^2}{1-|f(z)|^2} \le \frac{1-|f(w)|^2}{1-|w|^2} \cdot \frac{|1-\overline{w}z|^2}{1-|z|^2}. \tag{1.2.11}$$

Now choose a sequence $\{z_\nu\} \subset \Delta$ such that $z_\nu \to \sigma \in \partial\Delta$ and

$$\lim_{\nu\to\infty} \frac{1-|f(z_\nu)|}{1-|z_\nu|} = \alpha;$$

in particular, up to a subsequence we can assume that $f(z_\nu) \to \tau \in \partial\Delta$ as $\nu \to \infty$. Then, setting $w = z_\nu$ in (1.2.11) and taking the limit as $\nu \to \infty$, we obtain (1.2.9).

The point τ is unique: if (1.2.10) holds for two distinct points τ_1, $\tau_2 \in \partial\Delta$, then we get a contradiction taking R so small that $E(\tau_1, \alpha R) \cap E(\tau_2, \alpha R) = \emptyset$.

The proof of the last statement is only a bit more involved. If $f \in \mathrm{Aut}(\Delta)$, then (1.2.9) is an equality for any $z \in \Delta$, as easily shown using (1.1.4).

For the converse, rewrite (1.2.9) as

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{\sigma+z}{\sigma-z} - \frac{\tau+f(z)}{\tau-f(z)}\right) \le 0.$$

If the equality holds at some point, the maximum principle for harmonic functions yields

$$\frac{\tau+f(z)}{\tau-f(z)} = \frac{1}{\alpha}\frac{\sigma+z}{\sigma-z} + ic$$

for some $c \in \mathbf{R}$, that is

$$f(z) = \sigma_0 \frac{z-z_0}{1-\overline{z_0}z}, \tag{1.2.12}$$

where

$$\sigma_0 = \tau\overline{\sigma}\frac{1+\alpha-ic\alpha}{1+\alpha+ic\alpha} \in \partial\Delta$$

and

$$z_0 = \sigma\frac{\alpha-ic\alpha-1}{\alpha-ic\alpha+1} \in \Delta,$$

and then $f \in \mathrm{Aut}(\Delta)$, **q.e.d.**

In particular, if the liminf in (1.2.8) is finite (it is positive by Lemma 1.2.4), then it is possible to associate a point $\tau \in \partial\Delta$ to f and σ in a very definite way. To better understand what is going on, let's look at the situation from a slightly different point of view.

Given $f\colon \Delta \to \Delta$ holomorphic and σ, $\tau \in \partial\Delta$, set

$$\beta_f(\sigma,\tau) = \sup_{z\in\Delta}\left\{ \frac{|\tau-f(z)|^2}{1-|f(z)|^2} \Big/ \frac{|\sigma-z|^2}{1-|z|^2} \right\} \tag{1.2.13}$$

and

$$\beta_f(\sigma) = \inf_{\tau\in\partial\Delta} \beta_f(\sigma,\tau). \tag{1.2.14}$$

$\beta_f(\sigma)$ is the *boundary dilatation coefficient* of f at σ. $\beta_f(\sigma)$ describes the behavior of horocycles at σ under the action of f; in particular, the same argument used to prove the uniqueness of the point $\tau \in \partial\Delta$ in Theorem 1.2.5 implies that for every $f \in \mathrm{Hol}(\Delta,\Delta)$ and $\sigma \in \partial\Delta$ there is at most one point $\tau \in \partial\Delta$ such that $\beta_f(\sigma,\tau)$ is finite.

As easily imagined, the boundary dilatation coefficient is a fancy way to express the liminf (1.2.8):

Proposition 1.2.6: *Take $f \in \mathrm{Hol}(\Delta,\Delta)$ and $\sigma \in \partial\Delta$. Then*

$$\liminf_{z\to\sigma} \frac{1-|f(z)|}{1-|z|} = \beta_f(\sigma).$$

Proof: For sake of brevity, set $\beta = \beta_f(\sigma)$ and

$$\alpha = \liminf_{z\to\sigma} \frac{1-|f(z)|}{1-|z|}.$$

Theorem 1.2.5 tells that $\beta \le \alpha$; so $\beta = +\infty$ implies $\alpha = +\infty$, and it remains to prove that $\alpha \le \beta$ when β is finite.

For every $\nu \in \mathbf{N}$ set $z_\nu = (\nu-1)\sigma/(\nu+1)$. Clearly $z_\nu \in \Delta$ and $z_\nu \to \sigma$ as $\nu \to +\infty$; moreover a quick computation shows that $z_\nu \in \partial E(\sigma, 1/\nu)$. By definition of β, then, there is $\tau \in \partial\Delta$ such that $f(z_\nu) \in \overline{E(\tau,\beta/\nu)}$. Now, the euclidean diameter of $E(\tau,R)$ is $2R/(1+R)$; therefore

$$|\tau - f(z_\nu)| \le \frac{2\beta}{\beta+\nu}.$$

Since $1-|z_\nu| = 2/(\nu+1)$, it follows that

$$\alpha \le \limsup_{\nu\to\infty} \frac{1-|f(z_\nu)|}{1-|z_\nu|} \le \lim_{\nu\to\infty} \beta\cdot\frac{\nu+1}{\nu+\beta} = \beta,$$

and we are done, **q.e.d.**

In particular, then, the boundary dilation coefficient is finite iff z does not approach the boundary too much faster than $f(z)$. To give an example of function f such that $\beta_f(1) = +\infty$, we transfer the stage to H^+.

Given $F \in \mathrm{Hol}(H^+,H^+)$ and $a, b \in \partial H^+ = \mathbf{R}\cup\{\infty\}$, the *boundary dilation coefficient* of F at a is given by

$$\beta_F^{H^+}(a,b) = \inf_{w\in H^+}\left\{\frac{1}{1+a^2}\,\frac{|w-a|^2}{\operatorname{Im} w}\Big/\frac{1}{1+b^2}\,\frac{|F(w)-b|^2}{\operatorname{Im} F(w)}\right\}$$

and

$$\beta_F^{H^+}(a) = \sup_{b\in\partial H^+} \beta_F^{H^+}(a,b),$$

where we put $|w-\infty|^2/(1+\infty^2)=1$ for all $w\in H^+$. Note that $\beta_F^{H^+}$ is not exactly the same thing as the boundary dilatation coefficient in Δ. The relation between them is given by

$$\beta_f(\sigma,\tau)=\left[\beta_F^{H^+}\big(\Psi(\sigma),\Psi(\tau)\big)\right]^{-1}, \tag{1.2.15}$$

where $f\in\mathrm{Hol}(\Delta,\Delta)$, $\sigma,\tau\in\partial\Delta$, $F=\Psi\circ f\circ\Psi^{-1}$ and $\Psi\colon\Delta\to H^+$ is the Cayley transform. In particular, $\beta_F^{H^+}$ is always finite (but it can be zero).

Set $F(w)=\log w$, where log is the principal branch of the logarithm in H^+. Furthermore, we shall denote by arg the principal argument function; so $\mathrm{Im}\log w=\arg(w)\in(0,\pi)$ for all $w\in H^+$. Then consideration of the points $w_\nu=i\nu$ with $\nu\in\mathbf{N}$ forces $\beta_F^{H^+}(\infty,b)=0$ for all $b\in\partial H^+$. In particular, a function $f\in\mathrm{Hol}(\Delta,\Delta)$ such that $\beta_f(1)=+\infty$ is exactly $f=\Psi^{-1}\circ F\circ\Psi$.

Thus the border is, as usual, between a linear approach and a logarithmic one. Coming back to Julia's lemma, we would like to remark one fact. Choose $f\in\mathrm{Hol}(\Delta,\Delta)$ and $\sigma\in\partial\Delta$ such that $\beta=\beta_f(\sigma)$ is finite, and let τ be the unique point of $\partial\Delta$ such that $\beta_f(\sigma,\tau)=\beta$. Now choose a sequence $\{z_\nu\}\subset\Delta$ converging to σ so that $(1-|f(z_\nu)|)/(1-|z_\nu|)$ admits a finite limit. Therefore $|f(z_\nu)|\to1$ as $\nu\to\infty$, and so every limit point of the sequence $\{f(z_\nu)\}$ must belong to $\partial\Delta$. Let $\tau_1\in\partial\Delta$ be one of these limit points; the same argument used to prove Julia's lemma then shows that $\beta_f(\sigma,\tau_1)$ is finite. But this means that $\tau_1=\tau$, and so $f(z_\nu)\to\tau$ as $\nu\to+\infty$.

The conclusion of this argument is that $\beta_f(\sigma)<+\infty$ must imply that f admits limit in some sense when z tends to σ. Actually, much more is true: even f' admits limit at σ, and this limit can be computed starting from $\beta_f(\sigma)$. This is the content of the Julia-Wolff-Carathéodory theorem that we shall now describe.

Take $\tau\in\partial\Delta$ and $M>0$. The *Stolz region* $K(\tau,M)$ of *vertex* τ and *amplitude* M is

$$K(\tau,M)=\left\{z\in\Delta\;\middle|\;\frac{|\tau-z|}{1-|z|}<M\right\}. \tag{1.2.16}$$

Note that $K(\tau,M)=\emptyset$ if $M\le1$, for $|\tau-z|\ge1-|z|$.

Figure 1.3 *A Stolz region.*

Geometrically, $K(\tau, M)$ is a sort of angle with vertex at τ (see Figure 1.3). Therefore we can use Stolz regions to characterize the non-tangential limit: a function $f\colon \Delta \to \widehat{\mathbf{C}}$ has non-tangential limit c at $\sigma \in \partial\Delta$ if $f(z) \to c$ as z tends to σ within $K(\sigma, M)$ for all $M > 1$, and we shall write $K\text{-}\lim\limits_{z\to\tau} f(z) = c$.

The non-tangential limit (or *angular limit*) is particularly well-suited to complex analysis. One reason may be the resemblance (not only apparent, as we shall see in the second part of this book) between Stolz regions and horocycles, which at least allows us to prove the announced *Julia-Wolff-Carathéodory theorem:*

Theorem 1.2.7: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and τ, $\sigma \in \partial\Delta$. Then*

$$K\text{-}\lim_{z\to\sigma} \frac{\tau - f(z)}{\sigma - z} = \tau\overline{\sigma}\beta_f(\sigma, \tau). \tag{1.2.17}$$

If it is finite, then f has non-tangential limit τ at σ and

$$K\text{-}\lim_{z\to\sigma} f'(z) = \tau\overline{\sigma}\beta_f(\sigma, \tau). \tag{1.2.18}$$

In particular, if $\tau = \sigma$ then the non-tangential limit of f' at σ is a positive real number.

Proof: If $z \in K(\sigma, M)$ then

$$\left| \frac{\tau - f(z)}{\sigma - z} \right| \geq \frac{1}{M} \frac{1 - |f(z)|}{1 - |z|}; \tag{1.2.19}$$

therefore (1.2.17) is proved if $\beta_f(\sigma) = +\infty$. Assume then $\beta = \beta_f(\sigma)$ finite, and let τ_0 be the unique point of $\partial\Delta$ such that $\beta_f(\sigma, \tau_0) = \beta$. If $\tau \neq \tau_0$, we have already saw that for no sequence $\{z_\nu\}$ converging non-tangentially to σ the sequence $\{f(z_\nu)\}$ can converge to τ with $|\tau - f(z_\nu)|/|\sigma - z_\nu|$ bounded, by (1.2.19); hence (1.2.17) is proved for $\tau \neq \tau_0$.

Now by definition,

$$\frac{1 - |z|^2}{|\sigma - z|^2} = \mathrm{Re}\, \frac{\sigma + z}{\sigma - z} \leq \beta\, \mathrm{Re}\, \frac{\tau_0 + f(z)}{\tau_0 - f(z)} = \beta \frac{1 - |f(z)|^2}{|\tau_0 - f(z)|^2},$$

with equality at one point (and hence everywhere) iff $f \in \mathrm{Aut}(\Delta)$. Therefore we can write

$$\beta \frac{\tau_0 + f(z)}{\tau_0 - f(z)} - \frac{\sigma + z}{\sigma - z} = \frac{\tau_0 + F(z)}{\tau_0 - F(z)} \tag{1.2.20}$$

for a suitable $F\colon \Delta \to \mathbf{C}$ holomorphic with $|F| \leq 1$. Now $|F| = 1$ at one point iff $f \in \mathrm{Aut}(\Delta)$, and in this case (1.2.17) and (1.2.18) are easily verified using Proposition 1.2.3. So we can assume $F \in \mathrm{Hol}(\Delta, \Delta)$; hence

$$\frac{1}{\beta_F(\sigma, \tau_0)} = \beta \cdot \inf_{z\in\Delta} \left\{ \mathrm{Re}\, \frac{\tau_0 + f(z)}{\tau_0 - f(z)} \middle/ \mathrm{Re}\, \frac{\sigma + z}{\sigma - z} \right\} - 1 = 0. \tag{1.2.21}$$

In particular, $(\sigma - z)/(\tau_0 - F(z))$ has non-tangential limit 0 at σ. Then it follows from (1.2.20) that f has non-tangential limit τ at σ, and that $(\tau - f)/(\sigma - z)$ has non-tangential limit $\overline{\sigma}\tau\beta$ at σ.

Now, F is given by

$$F(z) = \tau_0 \frac{\beta \dfrac{\sigma - z}{\tau_0 - f(z)}\bigl(\tau_0 + f(z)\bigr) - 2\sigma}{\beta \dfrac{\sigma - z}{\tau_0 - f(z)}\bigl(\tau_0 + f(z)\bigr) - 2z};$$

therefore F has non-tangential limit τ_0 at σ, and this, together with (1.2.21), yields $\beta_F(\sigma) = \infty$. In particular, by Proposition 1.2.6, $(1 - |z|)/(1 - |F(z)|) \to 0$ as $z \to \sigma$.

Differentiating (1.2.20) we obtain

$$\beta \frac{\tau_0 f'(z)}{\bigl(\tau_0 - f(z)\bigr)^2} - \frac{\sigma}{(\sigma - z)^2} = \frac{\tau_0 F'(z)}{\bigl(\tau_0 - F(z)\bigr)^2}.$$

We know, by the Schwarz-Pick lemma, that $|F'|/(1 - |F|^2) \leq 1/(1 - |z|^2)$. Therefore, if $z \in K(\sigma, M)$, we have

$$\begin{aligned}\left|\beta\tau_0\overline{\sigma} f'(z)\left(\frac{\sigma - z}{\tau_0 - f(z)}\right)^2 - 1\right| &= |F'(z)| \left|\frac{\sigma - z}{\tau_0 - F(z)}\right|^2 \leq M^2(1 - |z|)^2 \frac{|F'(z)|}{(1 - |F(z)|)^2} \\ &\leq 2M^2 \frac{1 - |z|}{1 - |F(z)|} \to 0\end{aligned}$$

as $z \to \sigma$, and hence f' has non-tangential limit $\tau_0\overline{\sigma}\beta$ at σ, **q.e.d.**

The non-tangential limit in (1.2.17) is called *angular derivative* of f at σ and, at least when $\beta_f(\sigma)$ is finite, is usually denoted by $f'(\sigma)$.

When $\beta_f(\sigma)$ is infinite, in general we cannot infer anything about the behavior of f'. For instance, for every $\lambda \in \mathbf{C}$ set $f_\lambda(z) = \lambda z^{k_\lambda}/k_\lambda$, where k_λ is the smallest integer greater than $|\lambda|$. Then $f_\lambda \in \mathrm{Hol}(\Delta, \Delta)$, $\beta_{f_\lambda}(1) = +\infty$ (for $|f_\lambda(1)| < 1$) and $f'(1) = \lambda$.

In this example, we obtained $\beta_f(1) = +\infty$ by taking a function f with $f(\Delta)$ strictly contained in Δ. If we rule out this possibility, the link between angular derivative and usual derivative is much tighter:

Proposition 1.2.8: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that for a suitable $\sigma \in \partial\Delta$*

$$\limsup_{r \to 1} |f(r\sigma)| = 1. \tag{1.2.22}$$

Then

$$\beta_f(\sigma) = \limsup_{r \to 1} |f'(r\sigma)|. \tag{1.2.23}$$

Proof: If the limsup in (1.2.23) is infinite, then $\{|f'(r\sigma)|\}$ cannot be bounded as $r \to 1$, and thus, by Theorem 1.2.7, $\beta_f(\sigma) = +\infty$. So assume it is finite — and thus $|f'(r\sigma)| < M$

for all $r \in (0,1)$ and a suitable $M < +\infty$ —; again by Theorem 1.2.7 it suffices to show that $\beta_f(\sigma)$ is also finite.

Now for all r_1, $r_2 \in (0,1)$ we have

$$|f(r_2\sigma) - f(r_1\sigma)| = \left| \int_{r_1}^{r_2} f'(r\sigma)\, dr \right| \leq M|r_2 - r_1|. \tag{1.2.24}$$

By (1.2.22) there are $\tau \in \partial\Delta$ and a sequence $\{r_\nu\} \subset (0,1)$ converging to 1 such that $f(r_\nu\sigma) \to \tau$ as $\nu \to +\infty$. Therefore in (1.2.24) we get

$$\forall r \in (0,1) \qquad |\tau - f(r\sigma)| \leq M(1-r).$$

Hence

$$\beta_f(\sigma) = \liminf_{z \to \sigma} \frac{1 - |f(z)|}{1 - |z|} \leq \liminf_{r \to 1} \frac{1 - |f(r\sigma)|}{1 - r} \leq \liminf_{r \to 1} \frac{|\tau - f(r\sigma)|}{1 - r} \leq M,$$

q.e.d.

Proposition 1.2.8 gives a more practical way to compute $\beta_f(\sigma)$ than Proposition 1.2.6; furthermore, it is the last step toward the following result, which is the *summa* of our work about the angular derivative:

Corollary 1.2.9: *Let $f\colon \Delta \to D(0,R)$ be a bounded holomorphic function such that for a suitable $\sigma \in \partial\Delta$*

$$\limsup_{r \to 1} |f(r\sigma)| = R$$

and

$$\limsup_{r \to 1} |f'(r\sigma)| < +\infty.$$

Then both f and f' have non-tangential limit at σ.

Proof: Clearly we can suppose $R = 1$. Then, using Proposition 1.2.8 to deduce the finiteness of $\beta_f(\sigma)$, we end the proof invoking Theorem 1.2.7, **q.e.d.**

So Julia's lemma gives us an effective way to deal with the boundary behavior of the derivative of a function $f \in \mathrm{Hol}(\Delta, \Delta)$. As an application, we shall now prove two statements giving bounds on the angular derivative which are somehow akin to the bound on the derivative at a fixed point given by Schwarz's lemma.

The first statement is very easy to prove:

Corollary 1.2.10: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f(0) = 0$, and $f(r\sigma) \to \tau \in \partial\Delta$ for a suitable $\sigma \in \partial\Delta$. Then*

$$\beta_f(\sigma) \geq 1,$$

with equality iff $f(z) = \tau\bar{\sigma}z$ for all $z \in \Delta$. In particular, if $\beta_f(\sigma)$ is finite then

$$K\text{-}\lim_{z \to \sigma} |f'(z)| \geq 1.$$

Proof: Lemma 1.2.4 immediately yields $\beta_f(\sigma) \geq 1$. If $\beta_f(\sigma) = 1$, we have, by definition, $f\big(E(\sigma, R)\big) \subset E(\tau, R)$ for all $R > 0$. Fix $R < 1$, and let z_0 be the point of $\partial E(\sigma, R)$ closest to 0. Then $f(z_0) \in \overline{E(\tau, R)}$; on the other hand, by Schwarz's lemma, $|f(z_0)| \leq |z_0|$. This implies that $|f(z_0)| = |z_0|$ and, again by Schwarz's lemma, $f(z) = \tau\bar{\sigma}z$ for all $z \in \Delta$.

Finally, the last assertion follows from Theorem 1.2.7, **q.e.d.**

The second one is more involved (and quite more intriguing):

Theorem 1.2.11: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$. Assume there are σ_1, σ_2, τ_1, $\tau_2 \in \partial\Delta$ with $\sigma_1 \neq \sigma_2$ such that $f(r\sigma_1) \to \tau_1$ and $f(r\sigma_2) \to \tau_2$ as $r \to 1$. Write $\sigma_2 = e^{i\varphi}\sigma_1$ and $\tau_2 = e^{i\psi}\tau_1$. Then*

$$\beta_f(\sigma_1)\beta_f(\sigma_2) \geq \left[\frac{\sin(\psi/2)}{\sin(\varphi/2)}\right]^2,$$

with equality iff $f \in \mathrm{Aut}(\Delta)$. In particular, if both $\beta_f(\sigma_1)$ and $\beta_f(\sigma_2)$ are finite,

$$K\text{-}\lim_{z\to\sigma_1} |f'(z)| \cdot K\text{-}\lim_{z\to\sigma_2} |f'(z)| \geq \left[\frac{\sin(\psi/2)}{\sin(\varphi/2)}\right]^2.$$

Proof: If either of $\beta_f(\sigma_j)$ is infinite ($j = 1, 2$), or if $\tau_1 = \tau_2$, there is nothing to prove. So assume both $\beta_f(\sigma_j)$ finite ($j = 1, 2$), and $\psi \neq 0$. Let γ_1, $\gamma_2 \in \mathrm{Aut}(\Delta)$ be constructed as in Corollary 1.1.3 in such a way that $\gamma_1(1) = \sigma_1$, $\gamma_1(-1) = \sigma_2$, $\gamma_2(1) = \tau_1$ and $\gamma_2(-1) = \tau_2$; an easy calculation shows that

$$|\gamma_1'(1) \cdot \gamma_1'(-1)| = [\sin(\varphi/2)]^2;$$
$$|\gamma_2'(1) \cdot \gamma_2'(-1)| = [\sin(\psi/2)]^2.$$

Let $g = \gamma_2^{-1} \circ f \circ \gamma_1$. Then (since by Theorem 1.2.7 f has non-tangential limit τ_j at σ_j for $j = 1, 2$) g has non-tangential limit 1 at 1, -1 at -1 and

$$\beta_g(1)\beta_g(-1) = \left[\frac{\sin(\varphi/2)}{\sin(\psi/2)}\right]^2 \beta_f(\sigma_1)\beta_f(\sigma_2).$$

Therefore it suffices to show that $\beta_g(1)\beta_g(-1) \geq 1$, with equality iff g is a hyperbolic automorphism of Δ fixing 1 and -1.

By definition of boundary dilatation coefficient, $g(0) \in \overline{E\big(1, \beta_g(1)\big)} \cap \overline{E\big(-1, \beta_g(-1)\big)}$. Now, $\overline{E(1, R_1)} \cap \overline{E(-1, R_2)} \neq \emptyset$ iff $R_1R_2 \geq 1$, and $R_1R_2 = 1$ iff the intersection is just one point. Therefore $\beta_g(1)\beta_g(-1) \geq 1$, and $\beta_g(1)\beta_g(-1) = 1$ implies that $g(0) \in \partial E\big(1, \beta_g(1)\big)$, and so $g \in \mathrm{Aut}(\Delta)$, by the uniqueness statement of Julia's lemma, **q.e.d.**

Theorem 1.2.7 has, of course, an upper half-plane version. For simplicity, we shall state it for $a = b = \infty$ only. For $0 < \varepsilon < 1$, set $K_\varepsilon = \{w \in H^+ \mid \mathrm{Im}\, w > \varepsilon|w|\}$. Then

Corollary 1.2.12: *Let $F \in \mathrm{Hol}(H^+, H^+)$. Then for all $\varepsilon \in (0, 1)$ we have*

$$\lim_{\substack{w\to\infty \\ w\in K_\varepsilon}} \frac{F(w)}{w} = \lim_{\substack{w\to\infty \\ w\in K_\varepsilon}} F'(w) = \beta_F^{H^+}(\infty, \infty) = \beta < +\infty.$$

Furthermore, $\mathrm{Im}\big(F(w) - \beta w\big) \geq 0$ *for all* $w \in H^+$.

Proof: Let $\Phi\colon H^+ \to \Delta$ be the inverse of the Cayley transform, and set $f = \Phi \circ F \circ \Phi^{-1}$. Then it is easy to check that

$$\frac{F(w)}{w} = \frac{1 - \Phi(w)}{1 - f\big(\Phi(w)\big)} \cdot \frac{1 + f\big(\Phi(w)\big)}{1 + \Phi(w)}$$

and $\Phi\big(K_\varepsilon \cap E(\infty, 1/\varepsilon)\big) \subset K(1, 1/\varepsilon)$. Then, by Theorem 1.2.7 and (1.2.15),

$$\lim_{\substack{w\to\infty\\ w\in K_\varepsilon}} \frac{F(w)}{w} = \frac{1}{\beta_f(1,1)} = \beta_F^{H^+}(\infty,\infty),$$

where $1/\infty = 0$, of course.

Next, the definition of $\beta = \beta_F^{H^+}(\infty,\infty)$ implies that F sends $E(\infty, R)$ into $E(\infty, R/\beta)$, and so $\operatorname{Im} f(z) \ge \beta \operatorname{Im} z$.

Finally, if $w_0 \in K_\varepsilon$ then $\overline{D(w_0, \varepsilon|w_0|/2)} \subset K_{\varepsilon/4}$. Set $C = \partial D(w_0, \varepsilon|w_0|/2)$. Since

$$F'(w_0) - \beta = \frac{1}{2\pi i} \int_C \frac{F(\zeta) - \beta\zeta}{(\zeta - w_0)^2}\, d\zeta,$$

we have

$$\begin{aligned} |F'(w_0) - \beta| &\le \frac{2}{\varepsilon|w_0|} \sup_{\zeta\in C} \big|F(\zeta) - \beta\zeta\big| \le 2\, \frac{|w_0| + \varepsilon|w_0|/2}{\varepsilon|w_0|} \sup_{\zeta\in C} \left|\frac{F(\zeta)}{\zeta} - \beta\right| \\ &\le \frac{2+\varepsilon}{\varepsilon} \sup_{\substack{|\zeta|\ge(1-\varepsilon/2)|w_0|\\ \zeta\in K_{\varepsilon/4}}} \left|\frac{F(\zeta)}{\zeta} - \beta\right|, \end{aligned}$$

and we are done, **q.e.d.**

In particular, then, the angular derivative at infinity is always finite — and we shall need this fact.

We end this section with another proof of Theorem 1.2.7, which gives an appealing interpretation of the boundary dilatation coefficient. We shall use a bit of measure theory; the relevant facts are proved, e.g., in Rudin [1966]. We begin with the *Herglotz representation formula*:

Theorem 1.2.13: *Let $f\colon \Delta \to \mathbf{C}$ be holomorphic with non-negative real part. Then there exists a positive measure μ on $\partial\Delta$ such that*

$$f(z) = \int_{\partial\Delta} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) + i \operatorname{Im} f(0). \tag{1.2.25}$$

Proof: Let $\varphi = \operatorname{Re} f$, and set $\varphi_r(z) = \varphi(rz)$ for $0 < r < 1$. Every φ_r is harmonic in Δ, continuous in $\overline{\Delta}$ and non-negative; thus

$$\operatorname{Re} f(0) = \varphi_r(0) = \frac{1}{2\pi i} \int_{\partial\Delta} \varphi_r(\zeta)\, d\zeta. \tag{1.2.26}$$

Let $\Lambda\colon C^0(\partial\Delta)\to\mathbf{R}$ be the $\mathbf{R}$-linear functional given by

$$\Lambda u=\limsup_{r\to 1}\frac{1}{2\pi i}\int_{\partial\Delta} u(\zeta)\varphi_r(\zeta)\,d\zeta.$$

Λ is well defined for, by (1.2.26), $\{\varphi_r\}\subset L^1(\partial\Delta)$ is uniformly bounded. Since every φ_r is non-negative, Λ gives rise to a positive measure μ on $\partial\Delta$ by

$$\Lambda u=\int_{\partial\Delta} u(\zeta)\,d\mu(\zeta);$$

in particular, $\mu(\partial\Delta)=\operatorname{Re} f(0)$.

Let $P(z,\zeta)$ be the Poisson kernel. Then

$$\begin{aligned}\int_{\partial\Delta} P(z,\zeta)\,d\mu(\zeta)&=\limsup_{r\to 1}\frac{1}{2\pi i}\int_{\partial\Delta} P(z,\zeta)\varphi_r(\zeta)\,d\zeta\\ &=\lim_{r\to 1}\varphi_r(z)=\varphi(z)=\operatorname{Re} f(z).\end{aligned}\tag{1.2.27}$$

Since

$$P(z,\zeta)=\operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right),$$

(1.2.27) implies (1.2.25), **q.e.d.**

Note that if f is continuous up to the boundary, then μ is but $\operatorname{Re} f|_{\partial\Delta}\,d\zeta$, and thus absolutely continuous with respect to the standard Lebesgue measure on $\partial\Delta$.

Now let $f\in\operatorname{Hol}(\Delta,\Delta)$ and choose σ, $\tau\in\partial\Delta$, as in the statement of Theorem 1.2.7. Then Theorem 1.2.13 associates to $F=(\tau+f)/(\tau-f)$ a positive measure μ_τ on $\partial\Delta$. Let $c_{\sigma,\tau}\geq 0$ denote $\mu_\tau\bigl(\{\sigma\}\bigr)$, i.e., the part of μ_τ concentrated at the point σ, and denote the rest of the measure by μ_0. Note that if $f(z)$ stays away from τ as $z\to\sigma$, then $|F|$ is finite near σ, and so $c_{\sigma,\tau}=0$.

Now (1.2.25) becomes

$$\frac{\tau+f(z)}{\tau-f(z)}=c_{\sigma,\tau}\frac{\sigma+z}{\sigma-z}+\int_{\partial\Delta}\frac{\zeta+z}{\zeta-z}\,d\mu_0(\zeta)+i\operatorname{Im} F(0).\tag{1.2.28}$$

Let

$$I(z)=(\sigma-z)\int_{\partial\Delta}\frac{\zeta+z}{\zeta-z}\,d\mu_0(\zeta);$$

we claim that $I(z)\to 0$ as $z\to\sigma$ in a Stolz region. Indeed, fix $\varepsilon>0$ and choose $\delta>0$ so small that the μ_0-measure of the arc C of amplitude δ around σ is less than ε. Write

$$I(z)=I_0(z)+I_1(z)=(\sigma-z)\int_C\frac{\zeta+z}{\zeta-z}\,d\mu_0(\zeta)+(\sigma-z)\int_{\partial\Delta\setminus C}\frac{\zeta+z}{\zeta-z}\,d\mu_0(\zeta).$$

It is obvious that $I_1(z) \to 0$ as $z \to \sigma$. Moreover, if $z \in K(\sigma, M)$ for some $M > 1$, we have

$$|I_0(z)| \le 2\,\frac{|\sigma - z|}{1 - |z|} \int_C d\mu_0(\zeta) \le 2M\varepsilon,$$

and the claim is established.

Applying the conclusion to (1.2.28), we immediately find

$$K\text{-}\lim_{z\to\sigma} \frac{\tau - f(z)}{\sigma - z} = \frac{\tau\overline{\sigma}}{c_{\sigma,\tau}}.$$

To completely recover Theorem 1.2.7 we need to show that $\beta_f(\sigma,\tau) = 1/c_{\sigma,\tau}$. If we take the real part of (1.2.28) we obtain

$$\frac{1 - |f(z)|^2}{|\tau - f(z)|^2} = c_{\sigma,\tau}\frac{1 - |z|^2}{|\sigma - z|^2} + \int_{\partial\Delta} \frac{1 - |z|^2}{|\zeta - z|^2}\, d\mu_0(\zeta). \tag{1.2.29}$$

This immediately implies

$$\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \le \frac{1}{c_{\sigma,\tau}}\frac{|\sigma - z|^2}{1 - |z|^2},$$

that is $\beta_f(\sigma,\tau) \le 1/c_{\sigma,\tau}$. Now we can rewrite (1.2.29) as

$$\frac{1 - |f(z)|^2}{|\tau - f(z)|^2} \Bigg/ \frac{1 - |z|^2}{|\sigma - z|^2} = c_{\sigma,\tau} + \int_{\partial\Delta} \frac{|\sigma - z|^2}{|\zeta - z|^2}\, d\mu_0(\zeta).$$

It is easy to show as before that the integral on the right side tends to 0 as $z \to \sigma$ in a Stolz region, and this proves that

$$\beta_f(\sigma,\tau) = \mu_\tau\big(\{\sigma\}\big)^{-1},$$

that is the interpretation we were seeking.

1.2.2 *Wolff's lemma and hyperbolic Riemann surfaces*

We now discuss Wolff's lemma, the second boundary version of Schwarz's lemma, and some of its consequences, mainly on the structure of the automorphism group of hyperbolic Riemann surfaces.

The original Schwarz lemma said something about functions $f \in \mathrm{Hol}(\Delta,\Delta)$ with a fixed point. Since then, we dropped this hypothesis, getting more general statements, like the Schwarz-Pick lemma, or even Julia's lemma. But now, assume as hypothesis that f has *no* fixed points in Δ. A suspicious reader may wonder whether there are consequences; for instance, it could be possible to infer the existence of a point $\tau \in \partial\Delta$ such that f sends every horocycle centered in τ into itself, exactly as a function with a fixed point $z_0 \in \Delta$ sends every Poincaré disk centered in z_0 into itself. Well; this is exactly the content of *Wolff's lemma:*

Theorem 1.2.14: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be without fixed points. Then there is a unique $\tau \in \partial\Delta$ such that for all $z \in \Delta$*

$$\frac{|\tau - f(z)|^2}{1 - |f(z)|^2} \le \frac{|\tau - z|^2}{1 - |z|^2}, \tag{1.2.30}$$

that is

$$\forall R > 0 \qquad f\big(E(\tau, R)\big) \subset E(\tau, R). \tag{1.2.31}$$

Moreover, the equality in (1.2.30) *holds at one point (and hence everywhere) iff f is a parabolic automorphism of Δ leaving τ fixed.*

Proof: For the uniqueness, assume that (1.2.31) holds for two distinct points τ, $\tau_1 \in \partial\Delta$. Then we can construct two horocycles, one centered at τ and the other centered at τ_1, tangent to each other at a point of Δ. By (1.2.31) this point would be a fixed point of f, contradiction.

For the existence, pick a sequence $\{r_\nu\} \subset (0, 1)$ with $r_\nu \to 1$, and set $f_\nu = r_\nu f$. Then $f_\nu(\Delta)$ is relatively compact in Δ; by Corollary 1.1.34 each f_ν has a unique fixed point $w_\nu \in \Delta$. Up to a subsequence, we can assume $w_\nu \to \tau \in \overline{\Delta}$. If τ were in Δ, we would have

$$f(\tau) = \lim_{\nu\to\infty} f_\nu(w_\nu) = \lim_{\nu\to\infty} w_\nu = \tau,$$

which is impossible; therefore $\tau \in \partial\Delta$.

Now, by Schwarz's lemma

$$1 - \left| \frac{f_\nu(z) - w_\nu}{1 - \overline{w_\nu} f_\nu(z)} \right|^2 \ge 1 - \left| \frac{z - w_\nu}{1 - \overline{w_\nu} z} \right|^2,$$

or, equivalently,

$$\frac{|1 - \overline{w_\nu} f_\nu(z)|^2}{1 - |f_\nu(z)|^2} \le \frac{|1 - \overline{w_\nu} z|^2}{1 - |z|^2}.$$

Taking the limit as $\nu \to \infty$ we get (1.2.30), as we want.

Finally, assume the equality holds for some $z \in \Delta$. Then, exactly as in Theorem 1.2.5, we see that f must be an automorphism of Δ, of the form (1.2.12). Moreover, since $\alpha = 1$ and $\sigma = \tau$, it is easily checked that (1.2.12) describes all the parabolic automorphisms of Δ leaving τ fixed, **q.e.d.**

By the way, a function $f \in \mathrm{Hol}(\Delta, \Delta)$, $f \ne \mathrm{id}_\Delta$, satisfying (1.2.31) for a point $\tau \in \partial\Delta$ cannot have fixed points, as it is easily seen using the argument described at the end of the proof of Corollary 1.2.10.

Theorem 1.2.14 defines a functional τ on $\mathrm{Hol}(\Delta, \Delta) \setminus \{\mathrm{id}_\Delta\}$, with values in $\overline{\Delta}$: if $f \in \mathrm{Hol}(\Delta, \Delta)$ has a fixed point (and $f \ne \mathrm{id}_\Delta$), let $\tau(f)$ be this fixed point; otherwise, let $\tau(f) \in \partial\Delta$ be the point constructed in Theorem 1.2.14. $\tau(f)$ is the *Wolff point* of f.

Using the Wolff point, we can give a unified version of Schwarz's and Wolff's lemmas:

Corollary 1.2.15: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$, and let $\tau \in \overline{\Delta}$ be its Wolff point. Then*

$$\forall z \in \Delta \qquad \frac{|1 - \overline{\tau} f(z)|^2}{1 - |f(z)|^2} \le \frac{|1 - \overline{\tau} z|^2}{1 - |z|^2}. \tag{1.2.32}$$

Proof: If f is fixed points free, (1.2.32) is exactly (1.2.30). On the other hand, if f has a fixed point in Δ (which is, by definition, τ), then (1.2.32) follows from (1.1.5), setting $w = \tau$, **q.e.d.**

Entering the Julia-Wolff-Carathéodory theorem into play we can see even better why Wolff's lemma should be considered a boundary Schwarz lemma:

Corollary 1.2.16: *Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be without fixed points. Then*

$$K\text{-}\lim_{z \to \tau(f)} f(z) = \tau(f),$$

f' has non-tangential limit $\beta = \beta_f\bigl(\tau(f)\bigr)$ at $\tau(f)$, and

$$\frac{1 - |f(0)|}{1 + |f(0)|} \le \beta \le 1.$$

Moreover, $\tau(f)$ is the unique point $\tau \in \partial\Delta$ such that $K\text{-}\lim\limits_{z \to \tau} f(z) = \tau$ and $K\text{-}\lim\limits_{z \to \tau} |f'(z)| \le 1$.

Proof: The assertion follows from Theorem 1.2.14, Theorem 1.2.7, Lemma 1.2.4 and Proposition 1.2.8, **q.e.d.**

It is very natural to conjecture that we may complete the statement of Corollary 1.2.16 with a sentence like "Moreover, $\beta = 1$ iff f is a parabolic automorphism of Δ leaving τ fixed." Unfortunately, this is false: an example for $\tau = 1$ is the function

$$f(z) = \frac{1 + 3z^2}{3 + z^2}.$$

Anyway, Corollary 1.2.16 remains the most effective way of computing the Wolff point of a fixed point free function $f \in \mathrm{Hol}(\Delta, \Delta)$.

For sake of reference, we now state Wolff's lemma in H^+:

Proposition 1.2.17: *Let $F \in \mathrm{Hol}(H^+, H^+)$ be without fixed points. Then either there exists a unique point $a \in \mathbf{R}$ such that*

$$\forall w \in H^+ \qquad \frac{\mathrm{Im}\, w}{|w - a|^2} \le \frac{\mathrm{Im}\, F(w)}{|F(w) - a|^2}, \tag{1.2.33}$$

that is $F\bigl(E(a, R)\bigr) \subset E(a, R)$ for all $R > 0$, or

$$\forall w \in H^+ \qquad \mathrm{Im}\, w \le \mathrm{Im}\, F(w), \tag{1.2.34}$$

that is $F\bigl(E(\infty, R)\bigr) \subset E(\infty, R)$ for all $R > 0$. Equality in (1.2.33) — or in (1.2.34) — holds at one point, and hence everywhere, iff F is a parabolic automorphism of H^+ with fixed point a — respectively ∞. Finally, if (1.2.34) holds then

$$F'(\infty) \ge 1,$$

where $F'(\infty)$ is the angular derivative of F at infinity.

Proof: It is just a translation (using (1.2.4), (1.2.5) and the Cayley transform) of Theorem 1.2.14 and Corollary 1.2.16, **q.e.d.**

The main applications of Wolff's lemma are in iteration theory, as we shall see in the next chapter. For the moment, we shall describe another consequence of Wolff's and Julia's lemmas, with remarkable corollaries.

In Proposition 1.1.13 we saw that two automorphisms of Δ commute iff they have the same fixed points. We shall now prove a first extension of that result:

Theorem 1.2.18: *Let $\gamma \in \mathrm{Aut}(\Delta)$ be hyperbolic, and $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that*

$$f \circ \gamma = \gamma \circ f. \tag{1.2.35}$$

Then either f is a hyperbolic automorphism of Δ with the same fixed points as γ, or $f = \mathrm{id}_\Delta$.

Proof: Assume $f \neq \mathrm{id}_\Delta$; in particular, f cannot have more than one fixed point. If f has a fixed point $z_0 \in \Delta$, then by (1.2.35)

$$f\big(\gamma(z_0)\big) = \gamma\big(f(z_0)\big) = \gamma(z_0),$$

that is $\gamma(z_0) = z_0$, impossible. Hence f is fixed point free and we can apply Wolff's lemma, coming up with a point $\tau \in \partial\Delta$ satisfying (1.2.31). Then $\gamma(\tau)$ still satisfies (1.2.31), by (1.2.35) and Proposition 1.2.3; therefore the uniqueness part of Wolff's lemma implies $\gamma(\tau) = \tau$.

Now transfer everything on H^+ so that τ goes into ∞, and the fixed point set of γ goes into $\{0, \infty\}$. Then $\gamma(z) = \lambda z$ for some positive $\lambda \neq 1$ and, up to replacing γ by γ^{-1} we can assume $\lambda > 1$. So f satisfies

$$\forall z \in H^+ \qquad f(\lambda z) = \lambda f(z). \tag{1.2.36}$$

By Corollary 1.2.12 and Proposition 1.2.17, there exists $\beta \geq 1$ such that for any $w_0 \in H^+$

$$\lim_{k \to \infty} \frac{f(\lambda^k w_0)}{\lambda^k w_0} = \beta.$$

Then (1.2.36) implies $\beta = f(w_0)/w_0$ and we conclude that $f(w) = \beta w$, **q.e.d.**

Using this theorem, we shall now go deeper into the study of the structure of the automorphism group of a hyperbolic Riemann surface. The first new fact is:

Theorem 1.2.19: *Let X be a hyperbolic Riemann surface with non-abelian fundamental group. Then id_X is isolated in $\mathrm{Hol}(X, X)$. In particular, $\mathrm{Aut}(X)$ is discrete.*

Proof: Let $\pi \colon \Delta \to X$ be the universal covering map, and realize the fundamental group of X as the group Γ of automorphisms of the covering, acting properly discontinuosly on Δ.

Assume, by contradiction, that there would exist a sequence $\{f_\nu\} \subset \mathrm{Hol}(X, X)$ converging to id_X. Let $\tilde{f}_\nu \in \mathrm{Hol}(\Delta, \Delta)$ be a lifting of f_ν; we may choose $\tilde{f}_\nu$ so that $\tilde{f}_\nu \to \mathrm{id}_\Delta$ in $\mathrm{Hol}(\Delta, \Delta)$. Indeed, choose $z_0 \in X$ and fix $\tilde{z}_0 \in \pi^{-1}(z_0)$. Since $f_\nu(z_0) \to z_0$, we can choose $\tilde{f}_\nu$ so that $\tilde{f}_\nu(\tilde{z}_0) \to \tilde{z}_0$ as $\nu \to +\infty$. In particular, $\{\tilde{f}_\nu\}$ has no compactly divergent subsequences. Let $\tilde f$ be a limit point of $\{\tilde f_\nu\}$. Clearly, $\tilde f(\tilde z_0) = \tilde z_0$ and $\pi \circ \tilde f = \pi$; therefore $\tilde f = \mathrm{id}_\Delta$ in a neighbourhood of $\tilde z_0$, and hence everywhere. So id_Δ is the unique limit point of $\{\tilde f_\nu\}$, i.e., $\tilde f_\nu \to \mathrm{id}_\Delta$ as $\nu \to \infty$.

Now, for all $\gamma \in \Gamma$ and $\nu \in \mathbf{N}$, there is $\alpha(\gamma, \nu) \in \Gamma$ such that

$$\tilde f_\nu \circ \gamma = \alpha(\gamma, \nu) \circ \tilde f_\nu. \tag{1.2.37}$$

Since Γ is non-abelian, by Corollary 1.1.15 and Proposition 1.1.13 it must contain at least one hyperbolic automorphism γ_1 and another element γ_2 with fixed point set different from the fixed point set of γ_1. Now, $\tilde f_\nu \to \mathrm{id}_\Delta$, and (1.2.37) implies $\alpha(\gamma,\nu) \to \gamma$; since Γ is properly discontinuous (and thus discrete) we should have $\alpha(\gamma,\nu) = \gamma$ for sufficiently large ν. Then Theorem 1.2.18 applied with $\gamma = \gamma_1$ and $f = \tilde f_\nu$ for large enough ν shows that $\tilde f_\nu$ is a hyperbolic automorphism with the same fixed point set as γ_1. Finally, a second application of Theorem 1.2.18 to $\gamma = \tilde f_\nu$ and $f = \gamma_2$ implies that γ_2 has the same fixed point set as γ_1, contradiction, **q.e.d.**

Theorem 1.2.19 has several interesting consequences. We begin with the classical *Klein-Poincaré theorem:*

Corollary 1.2.20: *Let X be a hyperbolic Riemann surface with non-abelian fundamental group. Then* $\mathrm{Aut}(X)$ *acts properly discontinuosly on X.*

Proof: By Theorem 1.2.19, $\mathrm{Aut}(X)$ is discrete. The assertion then follows from Proposition 1.1.48, **q.e.d.**

We also get a bound on the cardinality of $\mathrm{Aut}(X)$:

Corollary 1.2.21: *Let X be a hyperbolic Riemann surface with non-abelian fundamental group. Then* $\mathrm{Aut}(X)$ *is countable.*

Proof: Assume, by contradiction, $\mathrm{Aut}(X)$ uncountable. Fix $z_0 \in X$, and define a function $\mu\colon \mathrm{Aut}(X) \to \mathbf{R}^+$ by

$$\mu(\gamma) = \omega_X\big(z_0, \gamma(z_0)\big).$$

Since $\mathrm{Aut}(X)$ is uncountable, we can find a sequence $\{\gamma_\nu\}$ of distinct elements of $\mathrm{Aut}(X)$ such that $\{\mu(\gamma_\nu)\}$ is bounded in $\mathbf{R}^+$. In particular, then, $\{\gamma_\nu\}$ cannot have compactly diverging subsequences; so, by Montel's theorem and Corollary 1.1.47, up to a subsequence we can suppose $\{\gamma_\nu\}$ converging to an element of $\mathrm{Aut}(X)$, and this is impossible by Theorem 1.2.19, **q.e.d.**

Corollary 1.2.22: *Let X be a compact hyperbolic Riemann surface. Then* $\mathrm{Aut}(X)$ *is finite.*

Proof: By Theorem 1.1.29, X has non-abelian fundamental group; thus (Theorem 1.2.19) $\mathrm{Aut}(X)$ is discrete. On the other hand, by Theorem 1.1.43 $\mathrm{Aut}(X)$ is compact (for X is compact), and hence is finite, **q.e.d.**

Theorem 1.2.19 does not apply to doubly connected domains; later on we shall need some sort of substitute, that now we shall describe.

Let $D \subset \mathbf{C}$ be a hyperbolic doubly connected domain different from Δ^*. By Theorem 1.1.29 there is a holomorphic covering map $\pi\colon H^+ \to D$ automorphic under a cyclic group Γ of hyperbolic automorphisms of H^+; we can assume that Γ is generated by $\gamma(z) = \lambda z$ for a suitable $\lambda > 1$; cf. (1.1.25). Then

Proposition 1.2.23: *Let $D \subset \mathbf{C}$ be a doubly connected hyperbolic domain not biholomorphic to Δ^*, and $f \in \mathrm{Hol}(D, D)$, $f \neq \mathrm{id}_D$. Then the following statements are equivalent:*
(i) *$f \notin \mathrm{Aut}(D)$;*
(ii) *$f_*\big(\pi_1(D)\big)$ is trivial;*
(iii) *there exists $\hat{f} \in \mathrm{Hol}(D, H^+)$ such that $f = \pi \circ \hat{f}$;*
(iv) *a lifting (and hence any lifting) $\tilde{f}$ is automorphic under Γ.*
Furthermore, if f is an automorphism, then either $\tilde{f}(w) = cw$ or $\tilde{f}(w) = -c/w$ for some $c > 0$.

Proof: We already know (Proposition 1.1.21) that (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv), and it is clear that (ii) $\Longrightarrow$ (i). To end the proof, let $f \in \mathrm{Hol}(D, D)$ be such that there exists a non-automorphic lifting $\tilde{f}$; then $\tilde{f}$ satisfies

$$\forall z \in H^+ \qquad \tilde{f}(\lambda z) = \lambda^n \tilde{f}(z),$$

for some $n \in \mathbf{Z}$. We claim that $|n| = 1$.

$n = 0$ is excluded, for $\tilde{f}$ is not automorphic. We know, by Corollary 1.2.12, that $\tilde{f}$ admits angular derivative $c < \infty$ given by

$$c = \lim_{y \to +\infty} \frac{\tilde{f}(iy)}{iy}.$$

Therefore for any $y_0 > 0$

$$c = \lim_{k \to \infty} \frac{\tilde{f}(\lambda^k i y_0)}{\lambda^k i y_0} = \frac{\tilde{f}(iy_0)}{iy_0} \lim_{k \to \infty} \lambda^{(n-1)k},$$

and hence $n \leq 1$.

Let $g = -1/\tilde{f}$. Then $g \in \mathrm{Hol}(H^+, H^+)$ and $g(\lambda z) = \lambda^{-n} g(z)$; arguing as before we find $n \geq -1$ and, summing up, $|n| = 1$.

If $n = 1$, then, by Theorem 1.2.18, $\tilde{f}(z) = cz$ for some $c > 0$; in particular, $\tilde{f} \in \mathrm{Aut}(H^+)$ and $f \in \mathrm{Aut}(D)$. If $n = -1$, we can apply Theorem 1.2.18 to $g = -1/\tilde{f}$; hence $\tilde{f}(z) = -c/z$ for some $c > 0$, $\tilde{f} \in \mathrm{Aut}(H^+)$ and $f \in \mathrm{Aut}(D)$, **q.e.d.**

In particular we infer

Corollary 1.2.24: *Let X be a hyperbolic Riemann surface not biholomorphic to Δ or Δ^*. Then $\mathrm{Aut}(X)$ is open and closed in $\mathrm{Hol}(X, X)$.*

Proof: If X is not doubly connected, the assertion follows from Theorem 1.2.19 and Corollary 1.1.47. If X is doubly connected (and different from Δ^*), $\mathrm{Hol}(X, X) \setminus \mathrm{Aut}(X, X)$ is closed in $\mathrm{Hol}(X, X)$ by Proposition 1.2.23 (a limit of automorphic functions is automorphic), and the assertion follows again from Corollary 1.1.47, **q.e.d.**

Corollary 1.2.24 does not hold for Δ^*. Take $f_\nu(z) = (1-1/\nu)z$; then $f_\nu \in \mathrm{Hol}(\Delta^*, \Delta^*)$ for all ν, each f_ν is not surjective and $f_\nu \to \mathrm{id}_{\Delta^*}$.

We saw that the automorphism group of a compact hyperbolic Riemann surface is finite. This is still true in another important case:

Theorem 1.2.25: *Let $D \subset \widehat{X}$ be a multiply connected domain of regular type. Assume D is not doubly connected. Then* $\mathrm{Aut}(D)$ *is finite.*

Proof: Suppose first that ∂D has no Jordan components; then $\partial D = \{x_1, \ldots, x_p\}$. Take $f \in \mathrm{Aut}(D)$. By the Big Picard Theorem 1.1.51, f extends to an automorphism $\hat{f}$ of $\widehat{X}$ sending ∂D into itself. There are three cases:

(a) $\widehat{X}$ hyperbolic. Then $\mathrm{Aut}(\widehat{X})$ is finite (by Corollary 1.2.22) and thus, *a fortiori*, $\mathrm{Aut}(D)$ is finite.

(b) $\widehat{X}$ is a torus. In this case, given two points z_0, $z_1 \in \widehat{X}$ there is only a finite number of automorphisms of $\widehat{X}$ sending z_0 in z_1 (cf. Proposition 1.1.32); it follows that $\mathrm{Aut}(D)$ must be finite.

(c) $\widehat{X} = \widehat{\mathbf{C}}$. In this case, $p \geq 3$. Now, an automorphism of $\widehat{\mathbf{C}}$ is completely determined by its action on 3 points; in particular, $\hat{f}$ is completely determined by its action on ∂D. In this way $\mathrm{Aut}(D)$ is identified with a subgroup of the permutation group on p elements, and hence is finite.

So assume ∂D has at least one Jordan component; in particular, $\overline{D} \neq \widehat{X}$. Let P denote the set of point components of ∂D. Then, again by the Big Picard Theorem 1.1.51, every $f \in \mathrm{Aut}(D)$ extends holomorphically to an automorphism $\hat{f}$ of $D \cup P$ sending P onto itself, where $D \cup P$ is still a domain of regular type. If $D \cup P$ is doubly connected, it is easily checked that the condition $\hat{f}(P) = P$ singles out a finite subgroup of $\mathrm{Aut}(D \cup P)$. Hence we are reduced to the case $P = \emptyset$ and D not doubly connected.

Assume, by contradiction, $\mathrm{Aut}(D)$ infinite, and let $\{\gamma_\nu\}$ be an infinite sequence of distinct elements of $\mathrm{Aut}(D)$. By Theorem 1.2.19, $\mathrm{Aut}(D)$ is discrete; hence, by Proposition 1.1.46 and Montel's Theorem 1.1.43, we can assume that $\{\gamma_\nu\}$ converges to a constant $\sigma \in \partial D$.

Let $\pi\colon \Delta \to D$ be the universal covering map of D, and realize its fundamental group as the automorphism group Γ of the covering. We associate to each γ_ν an automorphism $\tilde{\gamma}_\nu$ of Δ such that $\gamma_\nu \circ \pi = \pi \circ \tilde{\gamma}_\nu$. Moreover, by Theorem 1.1.57, we can associate to σ a point $\tau \in \partial\Delta$ such that $\tilde{\gamma}_\nu \to \tau$ as $\nu \to \infty$, and such that Γ is properly discontinuous at τ.

Let $\tilde{\gamma} \in \Gamma$ be different from id_Δ; then for any $\nu \in \mathbf{N}$ there is $\alpha(\tilde{\gamma}, \nu) \in \Gamma$ such that $\tilde{\gamma}_\nu \circ \tilde{\gamma} = \alpha(\tilde{\gamma}, \nu) \circ \tilde{\gamma}_\nu$. Let V be any neighbourhood of τ in $\overline{\Delta}$, and K any compact subset of V; since $\tilde{\gamma}_\nu \to \tau$, we have $\tilde{\gamma}_\nu(K) \subset V$ for any large enough ν. But we also have $\alpha(\tilde{\gamma}, \nu) \circ \tilde{\gamma}_\nu \to \tau$; hence $\alpha(\tilde{\gamma}, \nu)(V) \cap V \neq \emptyset$ for any large enough ν. Since Γ is properly discontinuous at τ, this implies that $\alpha(\tilde{\gamma}, \nu) = \mathrm{id}_\Delta$ for large enough ν. But then $\tilde{\gamma}_\nu \circ \tilde{\gamma} = \tilde{\gamma}_\nu$ implies $\tilde{\gamma} = \mathrm{id}_\Delta$, for $\tilde{\gamma}_\nu \in \mathrm{Aut}(\Delta)$, contradiction, **q.e.d.**

We conclude this section describing what happens in Theorem 1.2.19 if we assume γ elliptic or parabolic.

Proposition 1.2.26: *Let γ be an elliptic automorphism of Δ with fixed point $z_0 \in \Delta$, and choose $\gamma_1 \in \mathrm{Aut}(\Delta)$ such that $\gamma_1(0) = z_0$. Let $f \in \mathrm{Hol}(\Delta, \Delta)$ be such that $f \circ \gamma = \gamma \circ f$. Then either*

(i) $(\gamma_1^{-1} \circ f \circ \gamma_1)(z) = az$ *for some* $a \in \overline{\Delta}$, *or*

(ii) $(\gamma_1^{-1} \circ f \circ \gamma_1)(z) = zg(z^n)$ *for some* $n \in \mathbf{N}$ *and* $g \in \mathrm{Hol}(\Delta, \Delta)$. *This latter possibility can occur only if γ is periodic of period n.*

Proof: We can assume, conjugating by γ_1 if necessary, $z_0 = 0$, and so $\gamma(z) = e^{i\theta}z$ for a suitable $\theta \in \mathbf{R}$. If f commutes with γ, it is obvious that $f(0) = 0$ and that f' is automorphic under the group generated by γ. If γ is not periodic, this implies that $f'(\tau z) = f'(z)$ for all $\tau \in \partial\Delta$; hence f' is constant, and we are in case (i).

If γ is periodic of period n, let $h(z) = f(z)/z$; by Schwarz's lemma, if we are not in case (i) then $h \in \mathrm{Hol}(\Delta, \Delta)$. Now, h is automorphic under the group generated by γ. If we expand h in Taylor series centered at 0, we deduce that $h^{(k)}(0) = 0$ if n does not divide k. This is equivalent to saying that $h(z) = g(z^n)$ for a suitable $g \in \mathrm{Hol}(\Delta, \Delta)$, **q.e.d.**

Finally, for the parabolic case we transfer everything on H^+, getting

Proposition 1.2.27: *Let γ be a parabolic automorphism of H^+ with fixed point $a \in \partial H^+$, and choose $\gamma_1 \in \mathrm{Aut}(H^+)$ such that $\gamma_1(\infty) = a$. Let $f \in \mathrm{Hol}(H^+, H^+)$ be such that $f \circ \gamma = \gamma \circ f$. Then either*

(i) $(\gamma_1^{-1} \circ f \circ \gamma_1)(w) = w + c$ *for some* $c \in \mathbf{R}$, *or*

(ii) $(\gamma_1^{-1} \circ f \circ \gamma_1)(w) = w + g\bigl(\exp(2\pi i w/b)\bigr)$ *for some* $g \in \mathrm{Hol}(\Delta^*, H^+)$, *where* $b \in \mathbf{R}^*$ *is such that* $(\gamma_1^{-1} \circ \gamma \circ \gamma_1)(w) = w + b$ *for all* $w \in H^+$.

Proof: We can assume, conjugating by γ_1 if necessary, $a = \infty$, and $\gamma(w) = w + b$. Arguing as in the proof of Theorem 1.2.19, we see that f has no fixed points in H^+, and that the Wolff point of f is exactly ∞. By Wolff's lemma, $\mathrm{Im}\, f(w) \geq \mathrm{Im}\, w$ for all $w \in H^+$, with equality iff $f(w) = w + c$ for some $c \in \mathbf{R}$. Excluding this case, the function $h(w) = f(w) - w$ sends H^+ into itself, and is automorphic under the group generated by γ. Hence, by Theorem 1.1.29 and (1.1.24), we can write $h(w) = g\bigl(\exp(2\pi i w/b)\bigr)$ for a suitable $g \in \mathrm{Hol}(\Delta^*, H^+)$, and we are done, **q.e.d.**

Notes

The horocycles are born with the non-euclidean geometry, in particular with Poincaré's model of hyperbolic geometry.

In euclidean geometry, a circumference can be defined as a trajectory orthogonal to a pencil of straight lines issuing from a given point. If we take as center of the pencil the point at infinity (i.e., if we have a pencil of lines parallel to a given one), we obtain another straight line. In hyperbolic geometry, a trajectory orthogonal to a pencil of hyperbolic lines (i.e., geodesics for the Poincaré metric) is a *cycle*: it is a Poincaré circle with the same center as the pencil. If we take a trajectory orthogonal to a pencil of hyperbolic lines passing through a given ideal point (i.e., a pencil of lines parallel to a given one) we obtain exactly a horocycle (Figure 1.4).

It is interesting to notice that in hyperbolic geometry there is a third kind of cycle: if we take a pencil of hyperbolic lines orthogonal to a given one, an associated orthogonal trajectory is called *hypercycle.* In euclidean geometry, horocycles and hypercycles are one and the same thing; in hyperbolic geometry they are distinct. In fact, geometrically, a hypercycle is a circular arc connecting two distinct points of $\partial\Delta$ (Figure 1.5).

Figure 1.4 *A horocycle.*

Figure 1.5 *A hypercycle.*

A classical exposition of the geometric theory of cycles in hyperbolic geometry is Julia [1930]; a modern treatment can be found in Gerretsen and Sansone [1969], Gans [1973] or Kelly and Matthews [1981].

A remark about notations. We used E to denote horocycles because the multidimensional version of horocycles in the unit ball of $\mathbf{C}^n$ will turn out to be ellipsoids, and we preferred using the same letter in both cases (and, moreover, the letter E was available at the time). Analogously, we used K for Stolz regions and non-tangential limits (following Rudin [1980]) because their multidimensional analogues have been introduced by Korányi.

Proposition 1.2.1 is classical; it can be found for instance in Julia [1930]. On the other hand Proposition 1.2.2 is the one-variable version of the lesser known characterization given by Yang [1978] of horospheres in the n-dimensional ball.

In connection with Proposition 1.2.3 it should be mentioned that every pair of horocycles is congruent under $\mathrm{Aut}(\Delta)$, as it is easily verified.

Julia's lemma (Theorem 1.2.5) was first proved by Julia [1920]; in Julia [1930] there is an analogous result for hypercycles. In the literature, Julia's lemma is often ancillary to Theorem 1.2.7, and, sometimes, it is the latter result which is referred to as Julia's lemma.

The Julia-Wolff-Carathéodory Theorem 1.2.7 was proved in several forms by several people (the most important are Wolff [1926d], Carathéodory [1929] and Landau and Valiron [1929]; Julia's name is due to the essential role played by Julia's lemma in Wolff's and Carathéodory's proofs), sometimes in the disk, sometimes in the upper half-plane. Our first proof is taken from Carathéodory [1929], whereas the second one is due to R. Nevanlinna [1929] (cf. Ahlfors [1930, 1973]). Consult Carathéodory [1960], Tsuji [1959] and Pommerenke [1975] for other applications and generalizations of the Julia-Wolff-Carathéodory theorem.

The discussion about the angular derivative following Theorem 1.2.7 is essentially taken from Carathéodory [1929]. Herzig [1940] has shown that the bound in Corollary 1.2.10 depends on the order of vanishing of f at 0; see also Unkelbach [1938, 1940].

Theorem 1.2.11 seems to be appeared for the first time (with different proofs) in Lewittes [1968] and in Behan [1973], but it was probably known before. For a related result see Lempert [1984]. Other estimates on the angular derivative in the same spirit as Corollary 1.2.10 and Theorem 1.2.11 are in Cowen and Pommerenke [1982].

With regard to Corollary 1.2.9, a sharp bound on the derivative of a bounded holomorphic function is due to Dieudonné [1931]. He proved that if $f \in \mathrm{Hol}(\Delta, \Delta)$ is such that $f(0) = 0$, then $|f'(z)| \leq 1$ if $|z| \leq \sqrt{2} - 1$, and

$$|f'(z)| \leq \frac{(1 + |z|^2)^2}{4|z|(1 - |z|^2)}$$

if $|z| > \sqrt{2} - 1$. A proof can be found, e.g., in Duren [1983].

Theorem 1.2.13 was first proved by Herglotz [1911] for nonnegative harmonic functions.

Wolff's lemma was partially stated by Julia [1920], but its true birth is in Wolff's paper [1926c] on iteration theory. It is less known than its akin due to Julia, but it is the cornerstone of iteration theory in hyperbolic domains. For a different application of Wolff's lemma, see Heins [1966, 1967].

Theorems 1.2.18 and 1.2.19, as well as the approach we followed in the rest of section 1.2.2, are from Heins [1941b]. Corollaries 1.2.20 and 1.2.21 has been proved by Poincaré [1885], following Klein's ideas. With regard to Corollary 1.2.21, it should be mentioned that every countable group arises as automorphism group of a Riemann surface; see Greenberg [1960].

Corollary 1.2.22 is due to Schwarz [1879] and Klein (see Poincaré [1885]). It is complemented by the renowned *Hurwitz's theorem* (Hurwitz [1893]; cf. also Bujalance, Etayo and Martínez [1987]): if X is a compact Riemann surface of genus $g \geq 2$, then the order of $\mathrm{Aut}(X)$ is at most $84(g - 1)$. For a modern proof see, e.g., Farkas and Kra [1980]. Furthermore, later on (Corollary 1.3.13) we shall show that in these hypotheses $\mathrm{Hol}(X, X)$ reduces to the union of $\mathrm{Aut}(X)$ with the set of constant maps.

Proposition 1.2.23 is in Heins [1941a].

Corollary 1.2.24 was already proved by H. Cartan [1932] for multiply connected domains D bounded by a finite number of disjoint Jordan curves. Indeed, he proved that $f \in \mathrm{Hol}(D, D)$ is an automorphism iff the induced homomorphism $f_*\colon \pi_1(D) \to \pi_1(D)$ is not nilpotent (see Corollary 1.3.21).

Finally, Theorem 1.2.25 is due to Kœbe [1918]; our proof is taken from Heins [1941b].