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The complex geodesics of non-compact hermitian symmetric spaces

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ABSTRACT

A complex geodesic in a complex manifold X is a holomorphic map $\varphi: D \to X$, where D is the unit disk in $\mathbb{C}$, which is an isometry between the Poincaré distance on D and the Kobayashi distance on X. In this paper we give a complete description of all complex geodesics in non-compact hermitian symmetric spaces. The proof relies on the fact that every non-compact hermitian symmetric spaces can be realized as the unit ball for a suitable norm in a complex vector space; then Vesentini's results on complex geodesics in balls combined with the structure theory of hermitian symmetric spaces allow us to provide the desired list. We end the paper providing *ad hoc* descriptions and proofs for the classical domains in E. Cartan's realization.

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0. Introduction

A *complex geodesic* in a complex manifold X is a holomorphic map $\varphi\colon D \to X$ from the unit disk $D \subset \mathbb{C}$ into X which is an isometry between the Poincaré distance of D and the Kobayashi distance of X (for definition and properties of the Kobayashi distance we refer to [14] and [2]). Originally introduced by Vesentini in [26] to study the automorphism group of the unit ball of $L^1(M,\mu)$, where (M,μ) is a measure space, they are a biholomorphic invariant attached to the manifold, and so their understanding may be useful for the study of the complex geometry of the manifold. For instance, after the work of Lempert [16, 17] the complex geodesics have become an important tool in the theory of bounded convex domains of $\mathbb{C}^n$; see, e.g., [5], [18] and [3]. Furthermore, there are important connections with the complex Monge-Ampère equation, connections leading to fruitful investigations of circular domains and, more generally, of manifolds of circular type; see [22], [23], [19] and [4]. Other important works on complex geodesics are [7], [8], [25], [26], [27] and [28]; see [2, chapter 2.6] for a more complete introduction to the theory.

An important problem is how to compute explicitely all complex geodesics passing through a given point in a complex manifold. As far as we know, up to now this problem has been solved only for the standard hermitian unit ball of $\mathbb{C}^n$ (Vesentini [27]) and for domains of the form

$$\{z \in \mathbb{C}^n \mid |z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1\}$$

(Poletskiĭ [24] for $p_1 = \cdots = p_n > 1$, Gentili [9] for $p_1 = \cdots = p_n = 1$, Blank *et al.* [6] for $n = 2$ and $p_1 = 2$, Jarnicky, Pflug and Zeinstra [13] for the general case).

In this paper we shall describe all complex geodesics in non-compact hermitian symmetric spaces. The complete statement (Theorem 1.8) is slightly technical, but the flavor of the result is easily conveyed by an example. Let $B(n)$ denote the *Siegel disk* of rank n, i.e., the domain of all symmetric complex $n \times n$ matrices of operator norm less than 1; $B(n)$ is a typical non-compact hermitian symmetric space. Since it is homogeneous, to list all complex geodesics it suffices to describe the complex geodesics φ such that $\varphi(0) = 0$. Then we shall prove that a holomorphic map $\varphi\colon D \to B(n)$ with $\varphi(0) = 0$ is a complex geodesic iff it is of the form

$$\varphi(\zeta) = U \begin{pmatrix} \zeta I_d & 0 \\ 0 & \tilde{\varphi}(\zeta) \end{pmatrix} {}^tU,$$

where $1 \le d \le n$, I_d is the $d \times d$ identity matrix, $U \in \mathrm{U}(n)$ and $\tilde{\varphi}\colon D \to B(n-d)$ is a holomorphic map with $\tilde{\varphi}(0) = 0$ and $\|\tilde{\varphi}(\zeta)\| < |\zeta|$ for all $\zeta \in D \setminus \{0\}$.

The general statement is very similar. Up to an automorphism, a complex geodesic in a non-compact hermitian symmetric space splits in two parts: a diagonal one, and a second one which is almost arbitrary and orthogonal to the first one.

Since the proof of our main theorem relies on the machinery of symmetric spaces (and thus it may be not easily comprehensible for people not used to that language), at the end of the paper we shall give a short discussion of complex geodesics in the classical domains, providing *ad hoc* elementary descriptions and proofs, with the hope of making less mysterious the arguments used to prove our main result.

After the completion of this work I became aware that the main result (Theorem 1.8) may be obtained as a consequence of the boundary structure of bounded symmetric domains in $\mathbb{C}^n$ (see, e.g., [20, Theorem 6.3]). Anyway, the present proof is more direct and elementary.

1. Bounded symmetric domains

Let X_0 be a non-compact hermitian symmetric space. We want to describe all complex geodesics $\varphi\colon D \to X_0$. Since X_0 is homogeneous, it suffices to describe all complex geodesics φ such that $\varphi(0) = x_0$, where $x_0 \in X_0$ is a fixed base point.

Let us recall a few facts from the theory of hermitian symmetric spaces; for all unproved assertions, we refer to [11], [29] and [1]. X_0 is a homogeneous space, and so it can be written as $X_0 = G_0/K_0$, where G_0 is a non-compact connected simply connected semisimple Lie group, and K_0, the identity component of the isotropy group of x_0, is a compact connected Lie subgroup of G_0. Let $\mathfrak{g}_0$ (respectively, $\mathfrak{k}_0$) be the Lie algebra of G_0 (respectively, K_0), and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{m}_0$ the splitting induced by the symmetry σ at x_0; $\mathfrak{m}_0$ can be naturally identified with $T_{x_0}X_0$, and thus it is endowed with a complex structure J. Let $\mathfrak{g}$ (respectively, $\mathfrak{k}$, $\mathfrak{m}$) be the complexification of $\mathfrak{g}_0$ (respectively, $\mathfrak{k}_0$, $\mathfrak{m}_0$), $\mathfrak{m}_c = i\mathfrak{m}_0$, $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \mathfrak{m}_c$ a compact real form of $\mathfrak{g}$, and τ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_c$ (so that $\tau|_{\mathfrak{k}_0} = \mathrm{id}_{\mathfrak{k}_0}$ and $\tau|_{\mathfrak{m}_0} = -\mathrm{id}_{\mathfrak{m}_0}$). If $\langle\,,\rangle$ is the Killing form of $\mathfrak{g}$, set

$$\forall u, v \in \mathfrak{g} \qquad \langle u, v\rangle_\tau = -\langle u, \tau v\rangle.$$

Since X_0 is non-compact, the Killing form restricted to $\mathfrak{k}_0$ is negative definite, and restricted to $\mathfrak{m}_0$ is positive definite; it follows that $\langle\,,\rangle_\tau$ is a positive definite hermitian product on $\mathfrak{g}$. In particular, we can introduce a first norm on $\mathfrak{g}$:

$$\forall u \in \mathfrak{g} \qquad |u| = (\langle u, u \rangle_\tau)^{1/2}.$$

A second norm is obtained pulling back the operator norm via the adjoint representation:

$$\forall u \in \mathfrak{g} \quad \|u\| = \|\mathrm{ad}(u)\| = \sup\{|[u, x]| \mid x \in \mathfrak{g}, |x| = 1\}.$$

The norm $\|\cdot\|$ can be used to realize X_0 as a bounded domain in the complex vector space $(\mathfrak{m}_0, J)$:

Theorem 1.1: (Harish-Chandra [10], Hermann [12]) *Let*

$$B = \{u \in \mathfrak{m}_0 \mid \|u\| < 1\}.$$

Then there is a biholomorphism $\chi: X_0 \to B$ such that $\chi(x_0) = 0$.

So our aim is to describe all complex geodesics $\varphi: D \to B$ such that $\varphi(0) = 0$. Vesentini has studied the complex geodesics through the origin in the unit ball for a norm on $\mathbb{C}^n$; his results are summarized in

Theorem 1.2: (Vesentini [26, 27, 28]) *Let $B = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the unit ball for a norm $\|\cdot\|$ on $\mathbb{C}^n$. Then for a holomorphic map $\varphi: D \to B$ with $\varphi(0) = 0$ the following assertions are equivalent:*

(i) *φ is a complex geodesic;*
(ii) *$\|\varphi(\zeta)\| = |\zeta|$ for all $\zeta \in D$;*
(iii) *there is $\zeta_0 \neq 0$ such that $\|\varphi(\zeta_0)\| = |\zeta_0|$.*

Note that if $u \in \partial B$ is a vector of norm 1, Theorem 1.2 shows that the map $\varphi(\zeta) = \zeta u$ is a complex geodesic. These maps are the only complex geodesics passing through the origin in the standard unit ball of $\mathbb{C}^n$:

Proposition 1.3: (Vesentini [27]) *Let B be the unit ball for the standard hermitian norm of $\mathbb{C}^n$. Then a holomorphic map $\varphi: D \to B$ with $\varphi(0) = 0$ is a complex geodesic iff it is of the form $\varphi(\zeta) = \zeta u$ for some $u \in \partial B$.*

In general, as we shall see in our case too, this is not true: there may be other complex geodesics passing through the origin in a unit ball.

Coming back to our problem, ~~our goal thus~~ is to describe all holomorphic maps $\varphi: D \to B$ with $\|\varphi(\zeta)\| = |\zeta|$ for all $\zeta \in D$. For this, we

need a sensible way to compute the norm $\|\cdot\|$ — whence more technical facts.

Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$, and let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. If $\alpha \in \Delta$, let $\mathfrak{g}^\alpha$ be the root space associated to α. For every $\alpha \in \Delta$, take $\tilde{h}_\alpha \in \mathfrak{h}$ such that $\alpha(h) = \langle h, \tilde{h}_\alpha \rangle$ for all $h \in \mathfrak{h}$; moreover, set $h_\alpha = 2\tilde{h}_\alpha/\alpha(\tilde{h}_\alpha) \in \mathfrak{h}$, so that $\alpha(h_\alpha) = 2$.

Let $\{e_\alpha \mid \alpha \in \Delta\}$ be a *Weyl basis* of $\mathfrak{g}$. The elements $e_\alpha \in \mathfrak{g}^\alpha$ have the following properties:

(a) first of all

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha,$$
$$\langle e_\alpha, e_{-\alpha} \rangle = 2/\alpha(\tilde{h}_\alpha), \quad \tau e_\alpha = -e_{-\alpha};$$

(b) if Δ_K (Δ_M) denotes the set of *compact (non-compact)* roots, i.e., the set of roots $\alpha \in \Delta$ such that $\mathfrak{g}^\alpha \subset \mathfrak{k}$ (respectively, $\mathfrak{g}^\alpha \subset \mathfrak{m}$), we have

$$e_\alpha - e_{-\alpha},\ i(e_\alpha + e_{-\alpha}) \in \mathfrak{k}_0 \qquad \text{if } \alpha \in \Delta_K,$$
$$e_\alpha + e_{-\alpha},\ i(e_\alpha - e_{-\alpha}) \in \mathfrak{m}_0 \qquad \text{if } \alpha \in \Delta_M;$$

(c) $[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$ for all $\alpha, \beta \in \Delta$, $\alpha + \beta \neq 0$, where $N_{\alpha,\beta} \in \mathbb{Z}$ is such that
 (i) $N_{\alpha,\beta} = 0$ iff $\alpha + \beta \notin \Delta$;
 (ii) $N_{-\alpha,-\beta} = N_{\beta,\alpha} = -N_{\alpha,\beta}$.

Let $\mathfrak{h}_\mathbb{R} = \sum_{\alpha\in\Delta} \mathbb{R}h_\alpha$. Then $\mathfrak{h} \cap \mathfrak{k}_0 = i\mathfrak{h}_\mathbb{R}$; furthermore, there is $iz \in \mathfrak{h}_\mathbb{R}$ so that $\mathrm{ad}(iz)|\mathfrak{m}$ yields the complex structure J on $\mathfrak{m}_0$ inherited by the identification with $T_{x_0}X_0$. Let $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ be the decomposition induced by J ($\mathfrak{m}^\pm$ is the $\pm i$-eigenspace of J). Choose an ordering on Δ such that, denoting by Δ_M^+ (Δ_K^+) the set of non-compact (compact) positive roots, we have $\mathfrak{m}^+ = \bigoplus_{\alpha\in\Delta_M^+} \mathfrak{g}^\alpha$. Set

$$x_{\alpha,o} = (e_\alpha + e_{-\alpha})/2, \quad y_{\alpha,o} = i(e_\alpha - e_{-\alpha})/2 \in \mathfrak{m}_0, \qquad \text{for } \alpha \in \Delta_M^+;$$

then $\{x_{\alpha,o}, y_{\alpha,o} \mid \alpha \in \Delta_M^+\}$ is a $\mathbb{R}$-basis of $\mathfrak{m}_0$; furthermore, $Jx_{\alpha_o} = y_{\alpha,o}$ and $Jy_{\alpha,o} = -x_{\alpha,o}$.

Let $\mathfrak{b}_0$ be a maximal abelian subalgebra of $\mathfrak{m}_0$; the real dimension r of $\mathfrak{b}_0$ is called the *rank* of X_0. This is a well defined quantity since it does not depend on the choice of the subalgebra $\mathfrak{b}_0$. In fact, given any two maximal abelian subalgebras $\mathfrak{b}_0$, $\mathfrak{b}_0' \subset \mathfrak{m}_0$, there exists an element $k \in K_0$ such that

$$\mathrm{Ad}(k)\mathfrak{b}_0 = \mathfrak{b}_0'.$$

Moreover, for every maximal abelian subalgebra $\mathfrak{b}_0 \subset \mathfrak{m}_0$ one has

$$\mathrm{Ad}(K_0)\mathfrak{b}_0 = \mathfrak{m}_0,$$

that is every element of $\mathfrak{m}_0$ is contained in a maximal abelian subalgebra.

A particularly nice maximal abelian subalgebra of $\mathfrak{m}_0$ has been constructed by Harish-Chandra. Two roots α, β are said to be *strongly orthogonal* if $\alpha+\beta$, $\alpha-\beta \notin \Delta$. Let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be the maximal set of strongly orthogonal non-compact positive roots constructed in [10] — where r is the rank of the non-compact hermitian symmetric space — ordered in such a way that $\psi_1 < \cdots < \psi_r$. Then $\mathfrak{a}_0 = \bigoplus_{j=1}^r \mathbb{R}x_{\psi_j,o}$ is a maximal abelian subalgebra of $\mathfrak{m}_0$. Let

$$\mathfrak{a} = \bigoplus_{j=1}^{r} (\mathbb{R}x_{\psi_j,o} \oplus \mathbb{R}y_{\psi_j,o}) = \mathfrak{a}_0 \oplus J\mathfrak{a}_0;$$

then $\mathfrak{a}$ is a 2-dimensional *complex* subspace of $(\mathfrak{m}_0, J)$ such that

$$\mathrm{Ad}(K_0)\mathfrak{a} = \mathfrak{m}_0.$$

$\mathfrak{a}$ is the complexification of $\mathfrak{a}_0$ in $(\mathfrak{m}_0, J)$; in particular, writing ζu instead of $(\mathrm{Re}\,\zeta)u + (\mathrm{Im}\,\zeta)Ju$ for $\zeta \in \mathbb{C}$ and $u \in \mathfrak{m}_0$, a generic element of $\mathfrak{a}$ is of the form

$$u = \sum_{\psi\in\Psi} \lambda_\psi x_{\psi,o},$$

with $\lambda_\psi \in \mathbb{C}$ for all $\psi \in \Psi$.

This is more or less all what we need to compute the norm $\|\cdot\|$. Indeed, in [1] it is proved the following

Proposition 1.4: (i) $\|\mathrm{Ad}(k)u\| = \|u\|$ *for all* $u \in \mathfrak{m}_0$ *and* $k \in K_0$;
(ii) *if* $u = \sum_{\psi\in\Psi} \lambda_\psi x_{\psi,o}$ *belongs to* $\mathfrak{a}$, *then*

$$\|u\| = \max\{|\lambda_\psi| \mid \psi \in \Psi\}.$$

So to compute the norm of $u \in \mathfrak{m}_0$ it suffices to move it in $\mathfrak{a}_0$ using $\mathrm{Ad}(K_0)$, and then apply the previous proposition.

To prove our main theorem, we shall also use the first norm we introduced. We summarize here the computations we shall need:

Lemma 1.5: (i) *For all* $\alpha \in \Delta$ *we have*

$$|ih_\alpha|^2 = \langle h_\alpha, h_\alpha\rangle = 4/\alpha(\tilde h_\alpha);$$

(ii) *for all* $\psi \in \Psi$ *we have*

$$|x_{\psi,o}| = \frac{1}{2}|ih_\psi|;$$

(iii) *for all* $\alpha \in \Delta_M^+$ *we have*

$$|x_{\alpha,o}| = |y_{\alpha,o}|;$$

(iv) $\{x_{\alpha,o}, y_{\alpha,o} \mid \alpha \in \Delta_M^+\}$ *is a real* $\langle\,.\,\rangle_\tau$*-orthogonal basis of* $\mathfrak{m}_0$.

Proof: (i) It suffices to compute:

$$|ih_\alpha|^2 = \langle ih_\alpha, ih_\alpha\rangle_\tau = -\langle ih_\alpha, ih_\alpha\rangle = \langle h_\alpha, h_\alpha\rangle = \frac{4}{\alpha(\tilde h_\alpha)^2}\langle \tilde h_\alpha, \tilde h_\alpha\rangle = \frac{4}{\alpha(\tilde h_\alpha)}.$$

(ii) Let c_Ψ be the generalized Cayley transform introduced by Korànyi and Wolf [15]. $\mathrm{Ad}(c_\Psi)$ is a $\langle\,,\rangle$-isometry such that $\mathrm{Ad}(c_\Psi)x_{\psi,o} = \frac{1}{2}h_\psi$ for all $\psi \in \Psi$. Hence

$$|x_{\psi,o}|^2 = \langle x_{\psi,o}, x_{\psi,o}\rangle = \frac{1}{4}\langle h_\psi, h_\psi\rangle = \frac{1}{4}|ih_\psi|^2.$$

(iii) Since $[ih_\alpha, x_{\alpha,o}] = 2y_{\alpha,o}$, it is easy to check that

$$\mathrm{Ad}\Big(\exp\big(\tfrac{\pi}{4}ih_\alpha\big)\Big)x_{\alpha,o} = y_{\alpha,o},$$

and the assertion follows.

(iv) Since e_α is orthogonal to e_β for $\beta \neq \alpha, -\alpha$, it remains to check that $x_{\alpha,o}$ is orthogonal to $y_{\alpha,o}$, which is an easy computation. □

As mentioned in the introduction, a generic complex geodesic will be the sum of two pieces, living in different subspaces. We turn now to the definition of these subspaces.

For every $\alpha \in \Delta_M^+$ let $\mathfrak{g}_{\alpha,o}$ denote the 3-dimensional real subalgebra

$$\mathfrak{g}_{\alpha,o} = \mathbb{R}(ih_\alpha) \oplus \mathbb{R}x_{\alpha,o} \oplus \mathbb{R}y_{\alpha,o} \subset \mathfrak{g}_0;$$

$\mathfrak{g}_{\alpha_0}$ is σ-invariant, τ-invariant and ad(iz)-invariant. Moreover, it is clear that $\mathfrak{g}_{\alpha,o} \cap \mathfrak{m}_0 = \mathbb{C}x_{\alpha,o}$ is a complex subspace of $(\mathfrak{m}_0, J)$.

Now, let $\Gamma \subset \Psi$ be any subset. We shall denote by Δ_Γ^+ the set of all non-compact positive roots which are orthogonal to $\Psi \setminus \Gamma$. Put

$$\mathfrak{m}_{\Gamma,o} = \bigoplus_{\alpha\in\Delta_\Gamma^+} (\mathbb{R}x_{\alpha,o} \oplus \mathbb{R}y_{\alpha,o}) = \bigoplus_{\alpha\in\Delta_\Gamma^+} \mathbb{C}x_{\alpha,o} \subset \mathfrak{m}_0,$$

and

$$\mathfrak{a}_{\Gamma,o} = \mathfrak{m}_{\Gamma,o} \cap \mathfrak{a}_0 = \bigoplus_{\gamma\in\Gamma} \mathbb{R}x_{\gamma,o}.$$

Note that, since (by [10]) a root $\alpha \in \Delta_M^+$ belongs to $\Delta_{\Psi\setminus\Gamma}^+$ iff $\alpha - \gamma$ is not a root for all $\gamma \in \Gamma$ — and $\alpha + \gamma$ is never a root — we have

$$[\mathfrak{a}_{\Gamma,o}, \mathfrak{m}_{\Psi\setminus\Gamma,o}] = (0).$$

No non-compact positive root can be orthogonal to all elements of Ψ; so, by Lemma 1.5.(iv), we obtain the orthogonal decomposition

$$\mathfrak{m}_0 = \mathfrak{m}_{\Gamma,o} \oplus \mathfrak{m}_{\Psi\setminus\Gamma,o} \oplus \mathfrak{m}_{\Gamma,o}^*,$$

where $\mathfrak{m}_{\Gamma,o}^*$ is the span of $x_{\alpha,o}$, $y_{\alpha,o}$ with $\alpha \in \Delta_\Gamma^* = \Delta_M^+ \setminus (\Delta_\Gamma^+ \cup \Delta_{\Psi\setminus\Gamma}^+)$, the set of roots which are orthogonal neither to Γ nor to $\Psi \setminus \Gamma$. Accordingly, any (holomorphic) map $\varphi: D \to \mathfrak{m}_0$ can be decomposed as

$$\varphi = \varphi_\Gamma + \tilde\varphi + \varphi^*, \tag{1.1}$$

with $\varphi_\Gamma(D) \subset \mathfrak{m}_{\Gamma,o}$, $\tilde\varphi(D) \subset \mathfrak{m}_{\Psi\setminus\Gamma,o}$ and $\varphi^*(D) \subset \mathfrak{m}_{\Gamma,o}^*$. We shall prove that, up to the action of Ad(K_0), any complex geodesic can be decomposed as in (1.1) with $\varphi^* \equiv 0$ and $\varphi_\Gamma(\zeta) = \zeta \sum_{\gamma\in\Gamma} x_{\gamma,o}$, for some $\Gamma \subset \Psi$.

We still need a preliminary lemma, allowing us to compute the norm of $u + v$ with $u \in \mathfrak{a}_{\Gamma,o}$ and $v \in \mathfrak{m}_{\Psi\setminus\Gamma,o}$.

Lemma 1.6: *Let $\Gamma \subset \Psi$, and take $v_0 \in \mathfrak{m}_{\Psi\setminus\Gamma,o}$. Then there is $k_0 \in K_0$ such that* Ad(k_0) *is the identity on $\mathfrak{a}_{\Gamma,o}$ and* Ad$(k_0)v_0 \in \mathfrak{a}_{\Psi\setminus\Gamma,o}$.

Proof: Let

$$\mathfrak{c}_\mathfrak{k} = \{u \in \mathfrak{k}_0 \mid [u, \mathfrak{a}_{\Gamma,o}] = (0)\}$$

be the centralizer of $\mathfrak{a}_{\Gamma,o}$ in $\mathfrak{k}_0$, and set $C = \overline{\exp(\mathfrak{c}_\mathfrak{k})} \subset K_0$; C is a compact subgroup of K_0 such that $\mathrm{Ad}(k)|_{\mathfrak{a}_{\Gamma,o}} = \mathrm{id}_{\mathfrak{a}_{\Gamma,o}}$ for all $k \in C$.

Take $u_0 \in \mathfrak{a}_0$ such that $\mathfrak{a}_0$ is the centralizer of u_0 in $\mathfrak{m}_0$, i.e., such that

$$\mathfrak{a}_0 = \{u \in \mathfrak{m}_0 \mid [u, u_0] = 0\}, \tag{1.2}$$

and consider the continuous function $f: C \to \mathbb{R}$ given by

$$f(k) = \langle u_0, \mathrm{Ad}(k)v_0 \rangle.$$

Being C compact, f attains its absolute minimum in a point $k_0 \in C$. In particular,

$$\forall u \in \mathfrak{c}_\mathfrak{k} \quad 0 = \frac{d}{dt}\langle u_0, \mathrm{Ad}\big(\exp(tu)\big)\mathrm{Ad}(k_0)v_0 \rangle \bigg|_{t=0},$$

that is

$$\forall u \in \mathfrak{c}_\mathfrak{k} \qquad \big\langle [\mathrm{Ad}(k_0)v_0, u_0], u \big\rangle = 0. \tag{1.3}$$

Now, it is easy to check that $[\mathrm{Ad}(k_0)v_0, u_0] \in \mathfrak{c}_\mathfrak{k}$. Hence, being $\langle\,,\rangle$ negative definite on $\mathfrak{k}_0$, (1.3) forces $[\mathrm{Ad}(k_0)v_0, u_0] = 0$ and thus, by (1.2), Ad$(k_0)v_0 \in \mathfrak{a}_0$. Finally, v_0 was orthogonal to $\mathfrak{a}_{\Gamma,o}$; thus Ad$(k_0)v_0$ must be so, and therefore Ad$(k_0)v_0 \in \mathfrak{a}_{\Psi\setminus\Gamma,o}$. □

Corollary 1.7: *Let $\Gamma \subset \Psi$ and take $u \in \mathfrak{a}_{\Gamma,o}$ and $v \in \mathfrak{m}_{\Psi\setminus\Gamma,o}$. Then*

$$\|u + v\| = \max\{\|u\|, \|v\|\}.$$

Proof: Write $u = \sum_{\gamma\in\Gamma} \lambda_\gamma x_{\gamma,o}$. Take $k_0 \in K_0$ as in Lemma 1.6, so that Ad$(k_0)v = \sum_{\psi\in\Psi\setminus\Gamma} \lambda_\psi x_{\psi,o}$. Then, recalling Proposition 1.4, we get

$$\begin{aligned}\|u + v\| &= \|\mathrm{Ad}(k_0)(u + v)\| = \|u + \mathrm{Ad}(k_0)v\| \\ &= \Big\|\textstyle\sum_{\psi\in\Psi} \lambda_\psi x_{\psi,o}\Big\| = \max\{|\lambda_\psi| \mid \psi \in \Psi\} \\ &= \max\big\{\max\{|\lambda_\gamma| \mid \gamma \in \Gamma\}, \max\{|\lambda_\psi| \mid \psi \in \Psi \setminus \Gamma\}\big\} \\ &= \max\big\{\|u\|, \|\mathrm{Ad}(k_0)v\|\big\} = \max\big\{\|u\|, \|v\|\big\}.\end{aligned}$$

□

We are finally able to prove our main theorem:

Theorem 1.8: *The complex geodesics $\varphi: D \to B$ with $\varphi(0) = 0$ are all the maps of the form*

$$\varphi(\zeta) = \mathrm{Ad}(k)\left(\zeta \sum_{\gamma\in\Gamma} x_{\gamma,o} + \tilde\varphi(\zeta)\right), \tag{1.4}$$

where $k \in K_0$, $\Gamma \subset \Psi$ and $\tilde\varphi: D \to \mathfrak{m}_{\Psi\setminus\Gamma,o}$ is a holomorphic map with $\tilde\varphi(0) = 0$ and $\|\tilde\varphi(\zeta)\| < |\zeta|$ for all $\zeta \neq 0$.

Proof: We start by showing that all maps of the form (1.4) are complex geodesics. Take $\zeta_0 > 0$; then, by Proposition 1.4 and Corollary 1.7,

$$\|\varphi(\zeta_0)\| = \left\|\zeta_0 \sum_{\gamma\in\Gamma} x_{\gamma,o} + \tilde\varphi(\zeta_0)\right\| = \max\{|\zeta_0|, \|\tilde\varphi(\zeta_0)\|\} = |\zeta_0|,$$

and φ is a complex geodesic by Theorem 1.2.

Conversely, let φ be a complex geodesic with $\varphi(0) = 0$, and choose $\zeta_0 > 0$. Up to the adjoint action of K_0, we can assume that

$$\varphi(\zeta_0) = \zeta_0 \sum_{\gamma\in\Gamma} x_{\gamma,o} + \sum_{\psi\in\Psi\setminus\Gamma} \lambda_\psi x_{\psi,o},$$

for some $\Gamma \subset \Psi$ and $\lambda_\psi > 0$ with $\lambda_\psi < \zeta_0$ for all $\psi \in \Psi \setminus \Gamma$. Write

$$\varphi(\zeta) = \sum_{\alpha\in\Delta_M^+} \varphi_\alpha(\zeta) x_{\alpha,o};$$

we must show that $\varphi_\alpha \equiv 0$ for $\alpha \in (\Delta_\Gamma^+ \cup \Delta_\Gamma^*) \setminus \Gamma$, $\varphi_\gamma(\zeta) \equiv \zeta$ for all $\gamma \in \Gamma$ and, setting

$$\tilde\varphi(\zeta) = \sum_{\alpha\in\Delta_{\Psi\setminus\Gamma}^+} \varphi_\alpha(\zeta) x_{\alpha,o},$$

that $\|\tilde\varphi(\zeta)\| < |\zeta|$ for all $\zeta \neq 0$.

Take $\gamma_0 \in \Gamma$. Since $\|\mathrm{ad}(\varphi(\zeta))\| < 1$, we must have

$$\|[\varphi(\zeta), ih_{\gamma_0}]\|^2 < |ih_{\gamma_0}|^2. \tag{1.5}$$

Put

$$C_{\gamma_0} = \{\alpha \in \Delta_M^+ \mid \alpha \neq \gamma_0 \text{ is not orthogonal to } \gamma_0\},$$

and write

$$\varphi(\zeta) = \xi_{\gamma_0}(\zeta) x_{\gamma_0,o} + \eta_{\gamma_0}(\zeta) y_{\gamma_0,o} + \sum_{\alpha\in C_{\gamma_0}} (\xi_\alpha(\zeta) x_{\alpha,o} + \eta_\alpha(\zeta) y_{\alpha,o}) + \varphi_0(\zeta),$$

where $\varphi_\alpha = \xi_\alpha + i\eta_\alpha$, and analogously for γ_0.

Now $[\varphi_0, ih_{\gamma_0}] \equiv 0$, because α is orthogonal to γ_0 iff $\alpha(h_{\gamma_0}) = 0$. Then

$$\begin{aligned}[\varphi(\zeta), ih_{\gamma_0}] = 2\big(&-\xi_{\gamma_0}(\zeta) y_{\gamma_0,o} + \eta_{\gamma_0}(\zeta) x_{\gamma_0,o}\big) \\ &+ \sum_{\alpha\in C_{\gamma_0}} (-\xi_\alpha(\zeta) y_{\alpha,o} + \eta_\alpha(\zeta) x_{\alpha,o}),\end{aligned} \tag{1.6}$$

where we used the fact (see, e.g., [21]) that $\alpha(h_{\gamma_0}) = 1$ for all $\alpha \in C_{\gamma_0}$. So Lemma 1.5 yields

$$\begin{aligned}\|[\varphi(\zeta), ih_{\gamma_0}]\|^2 &= 4|\varphi_{\gamma_0}(\zeta)|^2 |x_{\gamma_0,o}|^2 + \sum_{\alpha\in C_{\gamma_0}} |\varphi_\alpha(\zeta)|^2 |x_{\alpha,o}|^2 \\ &= |\varphi_{\gamma_0}(\zeta)|^2 |ih_{\gamma_0}|^2 + \sum_{\alpha\in C_{\gamma_0}} |\varphi_\alpha(\zeta)|^2 |x_{\alpha,o}|^2.\end{aligned} \tag{1.7}$$

The map $f(\zeta) = [\varphi(\zeta), ih_{\gamma_0}]$ is holomorphic and, by (1.5), its image is contained in a hermitian ball of radius $|ih_{\gamma_0}|$. By Schwarz's lemma, this implies

$$\forall \zeta \in D \qquad \|[\varphi(\zeta), ih_{\gamma_0}]\|^2 \le |ih_{\gamma_0}|^2 |\zeta|^2. \tag{1.8}$$

By assumption and (1.7), (1.8) is an equality for $\zeta = \zeta_0$. This means, by Theorem 1.2, that f is a complex geodesic in that hermitian ball, and hence, by Proposition 1.3 and (1.6), that

$$\varphi_{\gamma_0}(\zeta) \equiv \zeta \qquad \text{and} \qquad \varphi_\alpha \equiv 0 \text{ for all } \alpha \in C_{\gamma_0}.$$

We can repeat this argument for all $\gamma_0 \in \Gamma$. Since

$$(\Delta_\Gamma^+ \cup \Delta_\Gamma^*) \setminus \Gamma = \bigcup_{\gamma\in\Gamma} C_\gamma,$$

to finish the proof it only remains to show that $\|\tilde\varphi(\zeta)\| < |\zeta|$ for all $\zeta \neq 0$. But indeed we have already proved that φ can be expressed as in (1.4); thus,

$$\forall \zeta \in D \qquad |\zeta| = \|\varphi(\zeta)\| = \max\{|\zeta|, \|\tilde\varphi(\zeta)\|\},$$

again by Corollary 1.7. In particular, $\tilde\varphi(D)$ is contained in B. If we had $\|\tilde\varphi(\zeta_1)\| = |\zeta_1|$ for some $\zeta_1 \neq 0$, by Theorem 1.2 we would have $\|\tilde\varphi(\zeta)\| = |\zeta|$ for all $\zeta \in D$; but $\|\tilde\varphi(\zeta_0)\| < |\zeta_0|$ by assumption, contradiction, and we are done. □

2. The classical domains

We saw that any non-compact hermitian symmetric space can be realized as unit ball in a suitable complex vector space. It turns out that these domains can be explicitely described: if the space is irreducible, besides two exceptional domains, they belong to four infinite families, the so-called *classical domains* of E. Cartan.

In this section we shall give *ad hoc* (proofs and) descriptions of complex geodesics through the origin in classical domains, trying to make Theorem 1.8 less esoteric.

Let $M_{p,q}(\mathbb{C})$ denote the space of $p \times q$ matrices with complex entries $(q \leq p)$, and let $\|\cdot\|$ denote the usual matrix (operator) norm. Then the first classical domain $B_1(p,q) \subset \mathbb{C}^{pq}$ is given by

$$B_1(p,q) = \{Z \in M_{p,q}(\mathbb{C}) \mid \|Z\| > 1\} = \{Z \in M_{p,q}(\mathbb{C}) \mid I_q - Z^*Z > 0\},$$

where $I_q \in M_{q,q}(\mathbb{C})$ is the identity matrix, $Z^* = {}^t\overline{Z}$ is the adjoint of Z and $A > 0$ means A is positive definite. $K_0 = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ acts on $B_1(p,q)$ by

$$Z \mapsto UZV.$$

Then the complex geodesics in $B_1(p,q)$ are given by

Proposition 2.1: *The complex geodesics $\varphi: D \to B_1(p,q)$ with $\varphi(0) = 0$ are all the maps of the form*

$$\varphi(\zeta) = U \begin{pmatrix} \zeta I_d & 0 \\ 0 & Z(\zeta) \end{pmatrix} V, \tag{2.1}$$

where $U \in \mathrm{U}(p)$, $V \in \mathrm{U}(q)$, $1 \leq d \leq q$, and $Z: D \to M_{p-d,q-d}(\mathbb{C})$ is a holomorphic map such that $Z(0) = 0$ and $\|Z(\zeta)\| < |\zeta|$ for all $\zeta \neq 0$.

Proof: It is clear that all the maps in (2.1) are complex geodesics. Conversely, let φ be a complex geodesic, and fix $\zeta_0 \in D$, $\zeta_0 > 0$. Up to the action of K_0, we can assume that

$$\varphi(\zeta_0) = \begin{pmatrix} \zeta_0 I_d & 0 \\ 0 & \Lambda \end{pmatrix}$$

for some $1 \leq d \leq q$, where Λ is a diagonal matrix with $\|\Lambda\| < |\zeta_0|$. Write

$$\varphi(\zeta) = \begin{pmatrix} Z_{11}(\zeta) & Z_{12}(\zeta) \\ Z_{21}(\zeta) & Z_{22}(\zeta) \end{pmatrix},$$

where $Z_{11}(\zeta) \in M_{d,d}(\mathbb{C})$, $Z_{12}(\zeta) \in M_{d,q-d}(\mathbb{C})$ and so on.

We have

$$\begin{aligned} \varphi(\zeta) \in D_1 &\Longrightarrow I_q - \varphi(\zeta)^*\varphi(\zeta) > 0 \\ &\Longrightarrow I_d - Z_{11}^*(\zeta)Z_{11}(\zeta) - Z_{21}^*(\zeta)Z_{21}(\zeta) > 0 \\ &\Longrightarrow \mathrm{tr}\big(I_d - Z_{11}^*(\zeta)Z_{11}(\zeta) - Z_{21}^*(\zeta)Z_{21}(\zeta)\big) > 0 \\ &\Longrightarrow \sum_{h=1}^{p}\sum_{k=1}^{d} |z_{hk}(\zeta)|^2 < d. \end{aligned}$$

So the image of the map $\zeta \mapsto \big(Z_{11}(\zeta), Z_{21}(\zeta)\big)$ is contained in the standard hermitian ball of radius $\sqrt{d}$ in $\mathbb{C}^{pd}$; in particular,

$$\sum_{h=1}^{p}\sum_{k=1}^{d} |z_{hk}(\zeta)|^2 \leq d|\zeta|^2.$$

But we have equality in ζ_0; therefore, by Theorem 1.2 and Proposition 1.3, this map is linear, that is $Z_{11}(\zeta) = \zeta I_d$ and $Z_{21} \equiv 0$.

In the same way one shows that $Z_{12} \equiv 0$; so it remains to prove that $\|Z_{22}(\zeta)\| < |\zeta|$ for all $\zeta \in D \setminus \{0\}$. Since φ is a complex geodesic and

$$\|\varphi(\zeta)\| = \max\{|\zeta|, \|Z_{22}(\zeta)\|\},$$

we immediately get $\|Z_{22}(\zeta)\| \leq |\zeta|$. If we had equality in one point, by Theorem 1.2 Z_{22} should be a complex geodesic in $B_1(p-d, q-d)$, and so we should have $\|Z_{22}(\zeta)\| = |\zeta|$ for all $\zeta \in D$. But we know that this is not true in ζ_0, and we are done. □

The second classical domain in the Cartan realization is given by

$$B_2(n) = \{Z \in M_{n,n}(\mathbb{C}) \mid \|Z\| < 1, {}^tZ = -Z\}.$$

The action of $K_0 = \mathrm{U}(n)$ on $B_2(n)$ is given by

$$Z \mapsto UZ\,{}^tU.$$

Define $J_d \in M_{2d,2d}(\mathbb{C})$ by

$$J_d = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

Then the complex geodesics in $B_2(n)$ are given by

Proposition 2.2: *The complex geodesics $\varphi: D \to B_2(n)$ with $\varphi(0) = 0$ are all the maps of the form*

$$\varphi(\zeta) = U \begin{pmatrix} \zeta J_d & 0 \\ 0 & Z(\zeta) \end{pmatrix} {}^tU, \tag{2.2}$$

where $U \in \mathrm{U}(n)$, $1 \le d \le [n/2]$ (the integer part of $n/2$), and moreover $Z: D \to M_{n-2d,n-2d}(\mathbb{C})$ is a holomorphic map such that $Z(0) = 0$, ${}^tZ = -Z$ and $\|Z(\zeta)\| < |\zeta|$ for all $\zeta \neq 0$.

Proof: It follows arguing exactly as in the proof of Proposition 2.1. □

The third classical domain in the Cartan realization is the Siegel disk

$$B_3(n) = \{Z \in M_{n,n}(\mathbb{C}) \mid \|Z\| < 1, {}^tZ = Z\}.$$

The action of $K_0 = \mathrm{U}(n)$ on $B_3(n)$ is again given by

$$Z \mapsto UZ\,{}^tU.$$

Then

Proposition 2.3: *The complex geodesics $\varphi: D \to B_3(n)$ with $\varphi(0) = 0$ are all the maps of the form*

$$\varphi(\zeta) = U \begin{pmatrix} \zeta I_d & 0 \\ 0 & Z(\zeta) \end{pmatrix} {}^tU, \tag{2.3}$$

where $U \in \mathrm{U}(n)$, $1 \le d \le n$, and $Z: D \to M_{n-2d,n-2d}(\mathbb{C})$ is a holomorphic map such that $Z(0) = 0$, ${}^tZ = Z$ and $\|Z(\zeta)\| < |\zeta|$ for all $\zeta \neq 0$.

Proof: Again, it follows arguing exactly as in the proof of Proposition 2.1. □

Finally, the fourth domain in the Cartan realization is given by

$$B_4(n) = \{z \in \mathbb{C}^n \mid (z,z) + \sqrt{(z,z)^2 - |(z,\bar{z})|^2} < 1\},$$

where $(\,,)$ is the standard hermitian product of $\mathbb{C}^n$. $B_4(n)$ is the unit ball for the norm

$$\|z\| = \left((z,z) + \sqrt{(z,z)^2 - |(z,\bar{z})|^2}\right)^{1/2}.$$

The action of $K_0 = \mathbf{S}^1 \times \mathrm{SO}(n)$ on $B_4(n)$ is given by

$$z \mapsto e^{i\theta}Uz.$$

Finally, the complex geodesics are given by

Proposition 2.4: *The complex geodesics $\varphi: D \to B_4(n)$ with $\varphi(0) = 0$ are all the maps of the form*

$$\varphi(\zeta) = e^{i\theta}U \begin{pmatrix} (\zeta + g(\zeta))/2 \\ i(\zeta - g(\zeta))/2 \\ 0 \end{pmatrix}, \tag{2.4}$$

where $\theta \in \mathbb{R}$, $U \in \mathrm{SO}(n)$ and $g: D \to D$ is a holomorphic function with $g(0) = 0$.

Proof: It is easy to check that all maps of the form (2.4) are complex geodesics. Conversely, let φ be a complex geodesic with $\varphi(0) = 0$. Up to an automorphism, we can find $\zeta_0 \ge \lambda \ge 0$ such that

$$\varphi(\zeta_0) = \tfrac{1}{2}{}^t(\zeta_0 + \lambda, i(\zeta_0 - \lambda), 0, \ldots, 0).$$

Write

$$\varphi(\zeta) = {}^t\Big(\tfrac{1}{2}(\varphi_1(\zeta) + \varphi_2(\zeta)), \tfrac{1}{2}i(\varphi_1(\zeta) - \varphi_2(\zeta)), \varphi_3(\zeta), \ldots, \varphi_n(\zeta)\Big),$$

which is always possible. Then

$$(\varphi,\varphi) = \frac{|\varphi_1|^2 + |\varphi_2|^2}{2} + \sum_{j=3}^n |\varphi_j|^2,$$

$$(\varphi,\bar{\varphi}) = \varphi_1\varphi_2 + \sum_{j=3}^n \varphi_j^2,$$

and so

$$(\varphi,\varphi)^2 - |(\varphi,\bar{\varphi})|^2$$

$$= \left(\frac{|\varphi_1|^2 - |\varphi_2|^2}{2}\right)^2 + \left[\left(\sum_{j=3}^{n}|\varphi_j|^2\right)^2 - \left|\sum_{j=3}^{n}\varphi_j^2\right|^2\right]$$

$$+ \left[(|\varphi_1|^2 + |\varphi_2|^2)\sum_{j=3}^{n}|\varphi_j|^2 - 2\,\mathrm{Re}\left(\varphi_1\varphi_2\sum_{j=3}^{n}\bar{\varphi}_j^2\right)\right].$$

Now

$$(|\varphi_1|^2 + |\varphi_2|^2)\sum_{j=3}^{n}|\varphi_j|^2 \geq 2\left|\varphi_1\varphi_2\sum_{j=3}^{n}\bar{\varphi}_j^2\right| \geq 2\,\mathrm{Re}\left(\varphi_1\varphi_2\sum_{j=3}^{n}\bar{\varphi}_j^2\right),$$

and

$$\left(\sum_{j=3}^{n}|\varphi_j|^2\right)^2 \geq \left|\sum_{j=3}^{n}\varphi_j^2\right|^2;$$

hence

$$\max\{|\varphi_1|^2, |\varphi_2|^2\} = \frac{|\varphi_1|^2 + |\varphi_2|^2}{2} + \sqrt{\frac{(|\varphi_1|^2 - |\varphi_2|^2)^2}{4}} \leq (\varphi,\varphi) + \sqrt{(\varphi,\varphi)^2 - |(\varphi,\bar{\varphi})|^2} = |\zeta|^2. \tag{2.5}$$

In particular, $\varphi_1(D)$, $\varphi_2(D) \subset D$; being $\varphi_1(\zeta_0) = \zeta_0$, Schwarz's lemma yields $\varphi_1(\zeta) \equiv \zeta$. But then (2.5) is an equality, and this may happen only if

$$\sum_{j=3}^{n}|\varphi_j|^2 \equiv 0,$$

and we are done. □

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Non-Abelian cohomology and field theory

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1. INTRODUCTION

This article deals with the trinity of topological field theory, Chern-Simons field theory and $U(n)$-target sigma-models.

Roughly speaking, this is a problem to clarify the relations between Chern classes, Chern-Simons classes and pull-back of cohomology generators of $U(n)$ by $U(n)$-valued maps ([11]). But since Chern-Simons classes come from the non-integrity of Chern-Simons forms, we need to consider fractional Chern classes which can not be expressed as characteristic classes of $U(n)$-bundles (cf. [14], [16]). So we need some extended objects, namely 2-dimensional non-abelian (NA) de Rham cocycles (with respect to $U(n)$), which are defined by using non-abelian cohomology. On the other hand, to clarify the relations between sigma-models and other objects, it is convenient

This paper is in final form and will not appear elsewhere.